SOME RESULTS ON THE HOCHSCHILD COHOMOLOGY OF GROUP ALGEBRAS

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ABSTRACT. In this paper we investigate structure of the second cohomology of a discrete group G . First, for a G -set S we show that an isomorphism of vector spaces from $\mathcal{H}^n(\ell^1(G), \ell^\infty(S))$ onto $\bigoplus_{i \in I} \mathcal{H}^n(\ell^1(G), \ell^\infty(\mathcal{C}_i))$ exists, where $\{\mathcal{C}_i\}_{i \in I}$ is the set of orbits of S . Next we define the notion of pseudoderivation and apply it for the calculation of $\mathcal{H}_{b,2}^2(\ell^1(G), \ell^\infty(S))$.

1. INTRODUCTION

In this paper we investigate structure of the second bounded cohomology of a discrete group G with coefficients in $\ell^\infty(G)$. The theory of bounded cohomology of discrete groups is a part of the theory of the cohomology of Banach algebras which was developed by Johnson [6]. We also refer the reader to the book by Helemskii [4]. Johnson [6] used this theory to show that $\mathcal{H}^2(\ell^1(G), \mathbb{C}) \neq 0$ and so $\mathcal{H}^2(\ell^1(G), \ell^1(G)) \neq 0$, where G is the free group on two generators. Brooks [1] showed that for a large class of discrete groups $\mathcal{H}_b^2(G) \cong \mathcal{H}^2(\ell^1(G), \mathbb{C}) \neq 0$.

In [5] Ivanov and in [7] Matsumoto and Morita showed that $\mathcal{H}^2(\ell^1(G), \mathbb{C})$ is a Banach space for every discrete group G with trivial action on $\mathbb{C}$. In [8] the author showed that $\mathcal{H}^2(\ell^1(G), \ell^\infty(S))$ is a Banach space for every G -set S and in general for every locally compact group G he showed that $\mathcal{H}^2(L^1(G), L^\infty(G))$ is a Banach space. Grigorchuk [2] developed a method based on using the space of pseudocharacters and applied it to calculate $\mathcal{H}_b^2(G)$, where G is either one of a free group $\mathbb{F}_m$ of finite or countable rank m , a torus knot group or a surface group. Finally it is well known that $\mathcal{H}_b^n(G) = 0$ for all n if G is amenable.

In this article we want to investigate a generalization of Grigorchuk's result in the case where the coefficients are in $\ell^\infty(\mathcal{C}_i)$ instead of $\mathbb{R}$, where $\mathcal{C}_i$ is an orbit in G -set S .

2. SOME BASIC FACTS AND SOME CONVENTIONS

In this section we consider the relationship between bounded cohomology and Hochschild cohomology of groups. Let G be a locally compact group. A left Banach G -module X is a Banach space X which is a left G -module such that

- (i) there exists a constant $C > 0$ with $\|gx\| \leq C\|x\|$ for all $g \in G$, $x \in X$, and
- (ii) for any $y \in X$ the map $G \rightarrow X$ given by $g \mapsto gy$ is continuous.

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Similarly we define the notions of a right Banach G -module and a two-sided Banach G -module. Throughout, a Banach G -module means a two-sided Banach G -module. If X is a Banach G -module, then the dual space X' becomes a G -module by defining

$$(g\phi)(x) = \phi(xg) \quad \text{and} \quad (\phi g)(x) = \phi(gx) \quad (x \in X, \phi \in X', g \in G).$$

The dual space X' is a Banach G -module except that the maps $g \mapsto gy$ and $g \mapsto yg$ are possibly continuous only in the weak*-topology on X' . The dual space X' will be called a dual G -module. If G is discrete, then (ii) is trivially satisfied so that if X is a Banach G -module, then X' is a Banach G -module.

Let G be a locally compact group and let X be a G -module. Let $C^n(G, X)$ be the space of all functions from $G \times \cdots \times G$ to X and let $C^0(G, X) = X$. The linear mappings $\delta^n : C^{n-1}(G, X) \rightarrow C^n(G, X)$ are given by

$$\begin{aligned} \delta^n \phi(g_1, \dots, g_n) &= g_1 \phi(g_2, \dots, g_n) \\ &\quad + \sum_{j=1}^{n-1} (-1)^j \phi(g_1, \dots, g_j g_{j+1}, \dots, g_n) \\ &\quad + (-1)^n \phi(g_1, \dots, g_{n-1}) g_n, \end{aligned}$$

where $\phi \in C^{n-1}(G, X)$ and $g_1, \dots, g_n \in G$. It is easy to see that $\delta^{n+1} \delta^n = 0$ for every $n \in \mathbb{N} = \{1, 2, \dots\}$. The cohomology of G with coefficients in X is

$$H^n(G, X) = \ker \delta^{n+1} / \text{Im } \delta^n.$$

Now let X be a Banach G -module. Then the Banach space $C_b^n(G, X)$ is the space of all bounded functions from $G \times \cdots \times G$ to X with the $\|\cdot\|_\infty$ -norm. The restriction of the above map δ^n defines a map $\delta_b^n : C_b^{n-1}(G, X) \rightarrow C_b^n(G, X)$ for $n \in \mathbb{N}$. The bounded cohomology of G with coefficients in X is

$$\mathcal{H}_b^n(G, X) = \ker \delta_b^{n+1} / \text{Im } \delta_b^n.$$

Actually $\mathcal{H}_b^n(G, X)$ is isomorphic to $\mathcal{H}^n(L^1(G), X)$, the Hochschild cohomology of $L^1(G)$ with coefficients in X , because there is a one-to-one correspondence between Banach G -modules and Banach $L^1(G)$ -modules, and

$$\begin{aligned} C_b^n(G, X) &= \ell^\infty(G \times \cdots \times G, X) \\ &= \mathcal{L}^n(L^1(G), X), \end{aligned}$$

where $\mathcal{L}^n(L^1(G), X)$ is the space of bounded n -linear maps from $\ell^1(G) \times \cdots \times \ell^1(G)$ (n -times) into X .

Let X be a Banach G -module. We define a new G -module structure on X by

$$g \circ x = x, \quad x \circ g = g^{-1} x g$$

and denote the resulting module by X° . Then the isomorphism

$$\mathcal{H}^n(L^1(G), X') \cong \mathcal{H}^n(L^1(G), (X^\circ)')$$

holds [6]. Thus in cohomology theory it is enough to consider Banach G -modules with trivial action on one side.

If $X = \mathbb{C}$ we write $\mathcal{H}_b^n(G)$ instead of $\mathcal{H}_b^n(G, \mathbb{C})$. It is well known that $\mathcal{H}_b^1(G) = 0$ for all discrete groups G , because if $\phi \in C_b^1(G)$ such that $\delta\phi = 0$, then

$$\delta\phi(g_1, g_2) = \phi(g_1) - \phi(g_1 g_2) + \phi(g_2) = 0.$$

By induction $\phi(g^n) = n\phi(g)$ for every n . Therefore $\phi = 0$, that is, there do not exist nontrivial bounded homomorphisms from G into $\mathbb{C}$.

The space $C_b^n(G)$ with the norm $\|\cdot\|_\infty$ is a Banach space. Therefore in $\mathcal{H}_b^n(G)$ there is a seminorm $\|g\| = \inf_{f \in [g]} \|f\|_\infty$, where $[g]$ denotes the cohomology class corresponding to $g \in C_b^n(G)$. This seminorm is a norm if and only if $\text{Im } \delta_b^n$ is a closed subspace of $C_b^n(G)$, which means $\mathcal{H}_b^n(G)$ is a Banach space. It is unknown whether or not $\mathcal{H}_b^n(G)$ is a Banach space for every $n \geq 3$ and arbitrary locally compact group G . Meanwhile for the free group on two generators $G = \mathbb{F}_2$ and any discrete group G admitting a surjective homomorphism $f : G \rightarrow \mathbb{F}_2$, Soma [9] showed that $\mathcal{H}_b^3(G)$ is not a Banach space. He also showed that there exists a finitely generated discrete group G such that $\mathcal{H}_b^n(G)$ is not a Banach space for any $n \geq 5$.

Let G be a discrete group, and let S be a G -set with trivial action on the left-hand side. We define the relation $\sim$ on S by setting $x_1 \sim x_2$ if and only if $x_1, x_2 \in S$, and there is an element $g \in G$ such that $x_1 g = x_2$. The relation $\sim$ is an equivalence relation on S . So S is partitioned into disjoint equivalence classes with respect to the equivalence relation $\sim$. These equivalence classes are called the orbits of the action. For each $x \in S$, the equivalence class containing x is called the orbit of x , which is the subset $\{xg : g \in G\}$ of S .

The set $\text{Stab}_G(x) = \{g \in G : xg = x\}$, which is a subgroup of G , is called the stabiliser of x in G . If H is a subgroup of G , a right transversal of H in G is a subset T of G consisting of precisely one element from each right coset of H in G . Observe that $HT = G$ and $|T| = |G : H|$, where $|T|$ denotes the number of elements in T and $|G : H|$ is the index of H in G . Throughout by a transversal of H in G we mean a right transversal of H in G . Let $\{\mathcal{C}_i\}_{i \in I}$ be the set of orbits of S , which is a partition for S . Let $\mathcal{L}^n(\ell^1(G), \ell^\infty(\mathcal{C}_i))$ be the space of bounded n -linear functions from $\ell^1(G) \times \cdots \times \ell^1(G)$ (n -times) into $\ell^\infty(\mathcal{C}_i)$. (We identify $\mathcal{L}^0(\ell^1(G), \ell^\infty(\mathcal{C}_i))$ with $\ell^\infty(\mathcal{C}_i)$.) For each $n \in \mathbb{N}$ and each $i \in I$ the boundary map

$$\delta_i^n : \mathcal{L}^{n-1}(\ell^1(G), \ell^\infty(\mathcal{C}_i)) \rightarrow \mathcal{L}^n(\ell^1(G), \ell^\infty(\mathcal{C}_i))$$

is defined by

$$\begin{aligned} \delta_i^n \phi_i(g_1, \dots, g_n)[c] &= \phi_i(g_2, \dots, g_n)[cg_1] \\ &\quad + \sum_{j=1}^{n-1} (-1)^j \phi_i(g_1, \dots, g_j g_{j+1}, \dots, g_n)[c] \\ &\quad + (-1)^n \phi_i(g_1, \dots, g_{n-1})[c], \end{aligned}$$

where $\phi_i \in \mathcal{L}^{n-1}(\ell^1(G), \ell^\infty(\mathcal{C}_i))$, $g_1, \dots, g_n \in G$ and $c \in \mathcal{C}_i$. We have $\delta_i^{n+1} \delta_i^n = 0$, and we put $\mathcal{H}^n(\ell^1(G), \ell^\infty(\mathcal{C}_i)) = \ker \delta_i^{n+1} / \text{Im } \delta_i^n$ which is the cohomology group of $\ell^1(G)$ with coefficients in $\ell^\infty(\mathcal{C}_i)$. We will often drop the superscripts on δ_i^n .

3. THE COHOMOLOGY OF DISCRETE GROUPS

Let G be a discrete group, and let S be a G -set with trivial action on the left-hand side.

Lemma 3.1. *If $\mathcal{C}_i$ is an orbit in S , $c \in \mathcal{C}_i$ and if T is a transversal of $\text{Stab}_G(c)$ in G , then*

- (i) $\mathcal{C}_i = \{ct\}_{t \in T}$, and for every $s \in \mathcal{C}_i$ there is a unique $t \in T$ such that $s = ct$.
- (ii) For every g in G , $\text{Stab}_G(cg) = \text{Stab}_G(c)^g$, where $\text{Stab}_G(c)^g$ is equal to $\{g^{-1}xg : x \in \text{Stab}_G(c)\}$.
- (iii) For every $g \in G$, $g^{-1}T$ is a transversal for $\text{Stab}_G(cg)$.

Proof. (i) It is obvious that $\{ct\}_{t \in T} \subseteq \mathcal{C}_i$. Now if $k \in \mathcal{C}_i$, then there exists a $g \in G$ such that $k = cg$ but $g = ht$ for unique elements $h \in \text{Stab}_G(c)$ and $t \in T$, so $k = cht = ct$ and so $k \in \{ct\}_{t \in T}$. Now if $t_1, t_2 \in T$ such that $ct_1 = ct_2$, then $ct_1t_2^{-1} = c$, that means $t_1t_2^{-1} \in \text{Stab}_G(c)$, therefore $t_1 = t_2$.

(ii) Let $h \in \text{Stab}_G(c)$. Since $(cg)(g^{-1}hg) = chg = cg$, then $g^{-1}hg \in \text{Stab}_G(cg)$, that is, $\text{Stab}_G(c)^g \subseteq \text{Stab}_G(cg)$. Now let $k \in \text{Stab}_G(cg)$. Since $c(gkg^{-1}) = (cgk)g^{-1} = (cg)g^{-1} = c$, then $gkg^{-1} \in \text{Stab}_G(c)$, that is, $k \in g^{-1}\text{Stab}_G(c)g$, therefore $\text{Stab}_G(cg) \subseteq \text{Stab}_G(c)^g$.

(iii) If we consider any right coset $\text{Stab}_G(c)x$, then by hypothesis $|T \cap \text{Stab}_G(c)x| = 1$, and so $|g^{-1}T \cap g^{-1}\text{Stab}_G(c)gg^{-1}x| = |g^{-1}T \cap \text{Stab}_G(cg)x'| = 1$, where $x' = g^{-1}x$. $\square$

Lemma 3.2. *Let $\psi_i \in \mathcal{L}^1(\ell^1(G), \ell^\infty(\mathcal{C}_i))$. Then for every $c \in \mathcal{C}_i$*

- (i) $|\psi_i(g)[c]| \leq \|\delta_i \psi_i\|$ for all g in $\text{Stab}_G(c)$.
- (ii) $|\psi_i(h)[c] - \psi_i(gh)[c]| \leq 2\|\delta_i \psi_i\|$ for all g in $\text{Stab}_G(c)$ and h in G ,

where

$$\|\delta_i \psi_i\| = \sup_{\substack{g, h \in G \\ c \in \mathcal{C}_i}} |\delta_i \psi_i(g, h)[c]|.$$

Proof. (i) For every $g, h \in G$ we have

$$|\delta_i \psi_i(g, h)[c]| = |\psi_i(h)[cg] - \psi_i(gh)[c] + \psi_i(g)[c]| \leq \|\delta_i \psi_i\|.$$

Now if $g \in \text{Stab}_G(c)$, then $|\psi_i(h)[c] - \psi_i(gh)[c] + \psi_i(g)[c]| \leq \|\delta_i \psi_i\|$, so

$$\begin{aligned} |\psi_i(g)[c] - \psi_i(g^2)[c] + \psi_i(g)[c]| &\leq \|\delta_i \psi_i\|, \\ |\psi_i(h)[c] - \psi_i(g^2h)[c] + \psi_i(g^2)[c]| &\leq \|\delta_i \psi_i\|. \end{aligned}$$

Using the triangle inequality we obtain

$$|\psi_i(h)[c] - \psi_i(g^2h)[c] + 2\psi_i(g)[c]| \leq 2\|\delta_i \psi_i\|.$$

Thus for any $n > 0$ by induction we get

$$|\psi_i(h)[c] - \psi_i(g^n h)[c] + n\psi_i(g)[c]| \leq n\|\delta_i \psi_i\|$$

or

$$\left| \frac{1}{n}\psi_i(h)[c] - \frac{1}{n}\psi_i(g^n h)[c] + \psi_i(g)[c] \right| \leq \|\delta_i \psi_i\|.$$

Thus $|\psi_i(g)[c]| \leq \|\delta_i \psi_i\|$ for every $g \in \text{Stab}_G(c)$.

(ii) Apply (i) and the triangle inequality. $\square$

Remark 3.3. Fix $c \in \mathcal{C}_i$ and fix a transversal T for $\text{Stab}_G(c)$ in G , so by Lemma 3.1, $\mathcal{C}_i = \{ct\}_{t \in T}$. For $t_1 \in T$ let us consider the transversal $t_1^{-1}T$ for $\text{Stab}_G(ct_1)$ in G . For every $h \in G$ since $G = \text{Stab}_G(ct_1)(t_1^{-1}T)$, there exist unique elements $t_2 \in T$, $k \in \text{Stab}_G(c)$ such that $h = (t_1^{-1}kt_1)t_1^{-1}t_2$, where $t_1^{-1}kt_1 \in \text{Stab}_G(ct_1)$ and $t_1^{-1}t_2 \in t_1^{-1}T$. By Lemma 3.2(ii), since $t_1^{-1}kt_1 \in \text{Stab}_G(ct_1)$, we have

$$(3.1) \quad |\psi_i(h)[ct_1] - \psi_i(t_1^{-1}t_2)[ct_1]| \leq 2\|\delta_i \psi_i\|.$$

Theorem 3.4. *Let G be a discrete group. Then $\mathcal{H}^2(\ell^1(G), \ell^\infty(\mathcal{C}_i))$ is a Banach space, for every orbit $\mathcal{C}_i$ of G in S .*

Proof. We wish to show that there exists a constant M such that for every element of $\text{Im } \delta_i$ like $\delta_i \psi_i$ there exists a $\bar{\psi}_i \in \mathcal{L}^1(\ell^1(G), \ell^\infty(\mathcal{C}_i))$ such that $\delta_i \psi_i = \delta_i \bar{\psi}_i$ and $\|\bar{\psi}_i\| \leq M \|\delta_i \psi_i\|$.

Let $\psi_i \in \mathcal{L}^1(\ell^1(G), \ell^\infty(\mathcal{C}_i))$. We fix $c \in \mathcal{C}_i$ and fix a transversal T for $\text{Stab}_G(c)$ in G , so by Lemma 3.1 $\mathcal{C}_i = \{ct\}_{t \in T}$. For every $t_1 \in T$ and every $g \in G$ we can choose unique elements $t_2 \in T$, $k \in \text{Stab}_G(c)$ such that $g = (t_1^{-1}kt_1)t_1^{-1}t_2$. Now for every $t_1 \in T$ we define

$$\bar{\psi}_i(g)[ct_1] = \psi_i(g)[ct_1] + \psi_i(t_1)[c] - \psi_i(t_2)[c].$$

By the boundary map for every $t_0 \in T$ we have

$$(3.2) \quad \delta_i \bar{\psi}_i(g, h)[ct_0] = \bar{\psi}_i(h)[ct_0g] - \bar{\psi}_i(gh)[ct_0] + \bar{\psi}_i(g)[ct_0].$$

Next we calculate all terms on the right-hand side of the above identity:

$$(3.3) \quad \bar{\psi}_i(h)[ct_0g] = \bar{\psi}_i(h)[ct_1] = \psi_i(h)[ct_1] + \psi_i(t_1)[c] - \psi_i(t_2)[c],$$

where t_1 is a unique element of T such that $ct_0g = ct_1$ and $h = (t_1^{-1}kt_1)t_1^{-1}t_2$ for unique elements $t_2 \in T$, $k \in \text{Stab}_G(c)$.

Since $h = t_1^{-1}kt_2$, then $gh = t_0^{-1}(t_0gt_1^{-1}k)t_2$, but since $ct_0gt_1^{-1} = c$, then $t_0gt_1^{-1} \in \text{Stab}_G(c)$, so $t_0gt_1^{-1}k \in \text{Stab}_G(c)$ and so

$$(3.4) \quad \bar{\psi}_i(gh)[ct_0] = \psi_i(gh)[ct_0] + \psi_i(t_0)[c] - \psi_i(t_2)[c].$$

Finally since $g = t_0^{-1}(t_0gt_1^{-1})t_1$, then

$$(3.5) \quad \bar{\psi}_i(g)[ct_0] = \psi_i(g)[ct_0] + \psi_i(t_0)[c] - \psi_i(t_1)[c].$$

Thus substituting (3.3), (3.4) and (3.5) in equation (3.2) we obtain

$$\delta_i \bar{\psi}_i(g, h)[ct_0] = \delta_i \psi_i(g, h)[ct_0].$$

By inequality (3.1) we have

$$\begin{aligned} |\bar{\psi}_i(g)[ct_1]| &= |\psi_i(g)[ct_1] + \psi_i(t_1)[c] - \psi_i(t_2)[c]| \\ &\leq |\psi_i(g)[ct_1] - \psi_i(t_1^{-1}t_2)[ct_1]| + |\psi_i(t_1^{-1}t_2)[ct_1] + \psi_i(t_1)[c] - \psi_i(t_2)[c]| \\ &\leq 2 \|\delta_i \psi_i\| + \|\delta_i \psi_i\| \\ &\leq 3 \|\delta_i \psi_i\|. \end{aligned}$$

Therefore $|\bar{\psi}_i(g)[ct_1]| \leq 3 \|\delta_i \psi_i\|$ for every $g \in G$ and every $t_1 \in T$, and so we have $\|\bar{\psi}_i\| \leq 3 \|\delta_i \psi_i\|$ for every $i \in I$. $\square$

Let us consider the ℓ^∞ -direct sum $\bigoplus_{i \in I}^\infty \mathcal{L}^n(\ell^1(G), \ell^\infty(\mathcal{C}_i))$ which is the subset of the Cartesian product $\prod_{i \in I} \mathcal{L}^n(\ell^1(G), \ell^\infty(\mathcal{C}_i))$ consisting of all $\{\phi_i\}_{i \in I}$, where $\phi_i \in \mathcal{L}^n(\ell^1(G), \ell^\infty(\mathcal{C}_i))$ and $\|\{\phi_i\}\| = \sup_{i \in I} \|\phi_i\| < \infty$. The boundary map

$$\bigoplus_{i \in I}^\infty \delta_i^n : \bigoplus_{i \in I}^\infty \mathcal{L}^{n-1}(\ell^1(G), \ell^\infty(\mathcal{C}_i)) \rightarrow \bigoplus_{i \in I}^\infty \mathcal{L}^n(\ell^1(G), \ell^\infty(\mathcal{C}_i))$$

is defined by

$$\bigoplus_{i \in I}^\infty \delta_i^n (\{\phi_i\}_{i \in I}) = \{\delta_i^n \phi_i\}_{i \in I}.$$

Lemma 3.5. *There exists an isometric isomorphism from $\mathcal{L}^n(\ell^1(G), \ell^\infty(S))$ onto $\bigoplus_{i \in I}^\infty \mathcal{L}^n(\ell^1(G), \ell^\infty(\mathcal{C}_i))$.*

Proof. Let $\phi \in \mathcal{L}^n(\ell^1(G), \ell^\infty(S))$. Then for every $i \in I$ and $c \in \mathcal{C}_i$ we define ϕ_i by

$$\phi_i(g_1, \dots, g_n)[c] = \phi(g_1, \dots, g_n)(c).$$

Now define

$$T_n : \mathcal{L}^n(\ell^1(G), \ell^\infty(S)) \rightarrow \bigoplus_{i \in I}^{\infty} \mathcal{L}^n(\ell^1(G), \ell^\infty(\mathcal{C}_i))$$

by $T_n(\phi) = \{\phi_i\}_{i \in I}$.

(i) T_n is injective; if $T_n(\phi_1) = T_n(\phi_2)$, then by the definition of T_n we have $\phi_{1_i} = \phi_{2_i}$ for every $i \in I$, which implies that $\phi_1 = \phi_2$ on $G \times \dots \times G \times \mathcal{C}_i$ for every $i \in I$, but $S = \bigcup_{i \in I} \mathcal{C}_i$ and $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ if $i \neq j$, so $\phi_1 = \phi_2$ on $G \times \dots \times G \times S$.

(ii) T_n is surjective; let $\{\phi_i\}_{i \in I}$ be an element of $\bigoplus_{i \in I}^{\infty} \mathcal{L}^n(\ell^1(G), \ell^\infty(\mathcal{C}_i))$. For every $x \in S$ there exists a unique $i \in I$ such that $x \in \mathcal{C}_i$. Now we define $\phi : \ell^1(G) \times \dots \times \ell^1(G) \rightarrow \ell^\infty(S)$ by $\phi(g_1, \dots, g_n)(x) = \phi_i(g_1, \dots, g_n)[x]$, since $\sup_{i \in I} \|\phi_i\| < \infty$; then $\phi \in \mathcal{L}^n(\ell^1(G), \ell^\infty(S))$ and $T_n(\phi) = \{\phi_i\}_{i \in I}$.

Finally

$$\begin{aligned} \|T_n\phi\| &= \|\{\phi_i\}_{i \in I}\| \\ &= \sup_{i \in I} \{\|\phi_i\|\} = \|\phi\|. \end{aligned}$$

Hence T_n is isometric. □

Lemma 3.6. $\bigoplus_{i \in I} \delta_i^n \circ T_{n-1} = T_n \circ \delta^n$.

Proof. Let $\phi \in \mathcal{L}^n(\ell^1(G), \ell^\infty(S))$. Then

$$\begin{aligned} \bigoplus_{i \in I} \delta_i^n \circ T_{n-1}(\phi) &= \bigoplus_{i \in I} \delta_i^n(T_{n-1}(\phi)) \\ &= \bigoplus_{i \in I} \delta_i^n(\{\phi_i\}_{i \in I}) = \{\delta_i^n \phi_i\}_{i \in I}. \end{aligned}$$

On the other hand

$$T_n \circ \delta^n(\phi) = T_n(\delta^n \phi) = \{\delta_i^n \phi_i\}_{i \in I}.$$

□

Corollary 3.7. *An isomorphism of vector spaces*

$$\mathcal{H}^n(\ell^1(G), \ell^\infty(S)) \cong \bigoplus_{i \in I}^{\infty} \mathcal{H}^n(\ell^1(G), \ell^\infty(\mathcal{C}_i))$$

holds.

4. QUASIDERIVATIONS AND PSEUDODERIVATIONS

Let G be a discrete group, and let S be a G -set with trivial action on the left-hand side. The inclusion homomorphism $C_b^n(G, \ell^\infty(S)) \rightarrow C^n(G, \ell^\infty(S))$ induces a homomorphism $\zeta : \mathcal{H}_b^n(G, \ell^\infty(S)) \rightarrow H^n(G, \ell^\infty(S))$, which in general is neither injective nor surjective. The image of this homomorphism is called the bounded part of $H^n(G, \ell^\infty(S))$ and will be denoted by $\mathcal{H}_{b,1}^n(G, \ell^\infty(S))$. Note that

$$\mathcal{H}_{b,1}^n(G, \ell^\infty(S)) = \frac{\ker \delta_b^{n+1}}{\text{Im } \delta^n \cap \ker \delta_b^{n+1}}.$$

The seminorm on $\mathcal{H}_b^n(G, X)$ induces a seminorm on the bounded part of $H^n(G, X)$ by

$$\|[f]\|_{\mathcal{H}_{b,1}^n} = \inf\{\|[f+g]\|_{\mathcal{H}_b^n} : g \in \text{Im } \delta^n \cap \ker \delta_b^{n+1}\}.$$

Proposition 4.1. *The space $\mathcal{H}_{b,1}^n(G, \ell^\infty(S))$ with the induced seminorm is a Banach space.*

Proof. Let $f \in C_b^n(G, \ell^\infty(S))$. We denote by $[f]$ the class of the element f in $\mathcal{H}_{b,1}^n(G, \ell^\infty(S))$. Then

$$(4.1) \quad \|[f]\|_{\mathcal{H}_{b,1}^n} = \inf\left\{\|[f+g]\|_{\mathcal{H}_b^n} : g \in \text{Im } \delta^n \cap \ker \delta_b^{n+1}\right\}.$$

But

$$(4.2) \quad \|[f+g]\|_{\mathcal{H}_b^n} = \inf\{\|f+g+h\|_\infty : h \in \text{Im } \delta_b^n\}.$$

Since $[f]$ is a nonzero element of $\mathcal{H}^n(G, \ell^\infty(S)) \cong \mathcal{H}_n(G, \ell^1(S))^*$, then there is $[x] \in \mathcal{H}_n(G, \ell^1(S))$ such that $f(x) \neq 0$. But since $g \in \text{Im } \delta^n \cap \ker \delta_b^{n+1}$ and $h \in \text{Im } \delta_b^n$, then there exist $g_1 \in C^{n-1}(G, \ell^\infty(S))$ and $h_1 \in C_b^{n-1}(G, \ell^\infty(S))$ such that $g = \delta^n g_1$ and $h = \delta_b^n h_1$. Since $x \in \ker d_n$ we have

$$(f+g+h)(x) = f(x) + \delta^n g_1(x) + \delta_b^n h_1(x) = f(x) + g_1(d_n(x)) + h_1(d_n(x)) = f(x).$$

Then by (4.1) and (4.2) $\|f\|_{\mathcal{H}_{b,1}^n} \geq |f(x)| > 0$. So we proved that if $[f] \neq 0$ in $\mathcal{H}_{b,1}^n(G, \ell^\infty(S))$, then $\|f\|_{\mathcal{H}_{b,1}^n} \neq 0$. $\square$

Now we define $\mathcal{H}_{b,2}^n(G, \ell^\infty(S)) \subseteq \mathcal{H}_b^n(G, \ell^\infty(S))$ to be the subspace

$$\mathcal{H}_{b,2}^n(G, \ell^\infty(S)) = \frac{\text{Im } \delta^n \cap \ker \delta_b^{n+1}}{\text{Im } \delta_b^n}.$$

Proposition 4.2. *A natural isomorphism of vector spaces with the seminorms*

$$\mathcal{H}_{b,1}^n(G, \ell^\infty(S)) \cong \frac{\mathcal{H}_b^n(G, \ell^\infty(S))}{\mathcal{H}_{b,2}^n(G, \ell^\infty(S))}$$

holds.

Proof. The following exact sequence exists:

$$0 \rightarrow \frac{\text{Im } \delta^n \cap \ker \delta_b^{n+1}}{\text{Im } \delta_b^n} \rightarrow \frac{\ker \delta_b^{n+1}}{\text{Im } \delta_b^n} \rightarrow \frac{\ker \delta_b^{n+1}}{\text{Im } \delta^n \cap \ker \delta_b^{n+1}} \rightarrow 0.$$

So by definition we have the result. The seminorm is defined as a seminorm on a quotient space. $\square$

Corollary 4.3. *An isomorphism of vector spaces*

$$\mathcal{H}_b^n(G, \ell^\infty(S)) \cong \mathcal{H}_{b,1}^n(G, \ell^\infty(S)) \oplus \mathcal{H}_{b,2}^n(G, \ell^\infty(S))$$

holds.

Throughout we drop the subscript i on δ_i and the elements of $\mathcal{L}^1(\ell^1(G), \ell^\infty(\mathcal{C}_i))$.

Definition 4.4. A function $\psi \in C^n(G, \ell^\infty(S))$ is called a quasiderivation if there is a constant $C \geq 0$ such that for every $g, h \in G$ and $c \in \mathcal{C}_i$

$$|\psi(h)(cg) - \psi(gh)(c) + \psi(g)(c)| \leq C,$$

we denote the space of quasiderivations by $QD(G, \ell^\infty(S))$. Note that a function $\psi \in QD(G, \ell^\infty(S))$ if and only if $\delta^1(\psi)$ is bounded.

Definition 4.5. A function $\psi \in QD(G, \ell^\infty(S))$ is called a pseudoderivation if $\psi(g^n)(c) = n\psi(g)(c)$ for every $g \in \text{Stab}_G(c)$, $c \in S$ and $n \in \mathbb{Z}$. We denote by $PD(G, \ell^\infty(S))$ the space of pseudoderivations, and by $D(G, \ell^\infty(S))$ the space of all derivations from G to $\ell^\infty(S)$.

Proposition 4.6. If $c \in \mathcal{C}_i$ is some fixed element, then a vector spaces isomorphism

$$\mathcal{H}_{b,2}^2(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i)) \cong \frac{QD(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i))}{D(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i)) \oplus C_b^1(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i))}$$

holds.

Proof. We abbreviate $\text{Stab}_G(c)$ by H . To begin note that if $\psi \in D(H, \ell^\infty(\mathcal{C}_i))$ for every $g, h \in H$ we have $\psi(gh)[c] = \psi(h)[c] + \psi(g)[c]$. So by induction for every $n \in \mathbb{N}$ we have $\psi(g^n)[c] = n\psi(g)[c]$; now if $\psi \in C_b^1(H, \ell^\infty(\mathcal{C}_i))$, then $\psi(g)[c] = 0$, that is, $D(H, \ell^\infty(\mathcal{C}_i)) \cap C_b^1(H, \ell^\infty(\mathcal{C}_i)) = \{0\}$. We have a sequence

$$QD(H, \mathcal{C}_i) \xrightarrow{\delta^1} \text{Im } \delta^1 \cap \ker \delta_b^2 \xrightarrow{\pi} \frac{\text{Im } \delta^1 \cap \ker \delta_b^2}{\text{Im } \delta_b^1},$$

where δ^1 is onto and π is the canonical mapping. For every $\psi \in QD(H, \mathcal{C}_i)$

$$\begin{aligned} \pi(\delta^1(\psi)) = 0 &\iff \delta^1\psi \in \text{Im } \delta_b^1 \\ &\iff \delta^1(\psi) = \delta_b^1(g) \quad \text{for some } g \in C_b^1(H, \ell^\infty(\mathcal{C}_i)) \\ &\iff \delta^1(\psi - g) = 0 \iff \psi - g \in D(H, \mathcal{C}_i). \end{aligned}$$

So $\psi = (\psi - g) + g \in D(H, \ell^\infty(\mathcal{C}_i)) \oplus C_b^1(H, \ell^\infty(\mathcal{C}_i))$, that is, we have proved that $\ker \pi \circ \delta^1 = D(H, \ell^\infty(\mathcal{C}_i)) \oplus C_b^1(H, \ell^\infty(\mathcal{C}_i))$, and this finishes the proof. $\square$

Proposition 4.7. Let ψ be a real-valued submultiplicative map on a Banach algebra X . Then $\lim_{n \rightarrow \infty} \psi(x^n)^{1/n}$ exists.

Proof. Set $r = \inf_{n \geq 1} \psi(x^n)^{1/n}$. We will show that $\overline{\lim}_n \psi(x^n)^{1/n} \leq r$. For any $\epsilon > 0$, choose m such that $\psi(x^m)^{1/m} \leq r + \epsilon$. For arbitrary n , write $n = pm + q$ where $0 \leq q \leq m - 1$. Then

$$\psi(x^n)^{1/n} \leq \psi(x^m)^{p/n} + \psi(x)^{q/n} \leq (r + \epsilon)^{pm/n} + \psi(x)^{q/n}.$$

Since $pm/n \rightarrow 1$ and $q/n \rightarrow 0$ as $n \rightarrow \infty$, we have $\psi(x^n)^{1/n} \leq r + \epsilon$. Since ϵ was arbitrary, we have $\overline{\lim}_n \psi(x^n)^{1/n} \leq r$. $\square$

Corollary 4.8. If $\psi \in QD(G, \ell^\infty(S))$, then $\lim_{n \rightarrow \infty} \psi(g^n)(c)/n$ exists for every $g \in \text{Stab}_G(c)$.

Proof. Decomposing ψ into its real and imaginary parts, we can reduce the problem to the case in which ψ is real-valued. Since $\psi \in QD(G, \ell^\infty(S))$, then for every $g \in \text{Stab}_G(c)$ and $h \in G$ we have

$$-C \leq \psi(h)(c) - \psi(gh)(c) + \psi(g)(c) \leq C$$

and so

$$e^{(\psi(gh)(c)+C)} \leq e^{(\psi(g)(c)+C)} e^{(\psi(h)(c)+C)},$$

then by Proposition 4.7 $\lim_{n \rightarrow \infty} (e^{\psi(g^n)(c)+C})^{1/n}$ exists. Therefore

$$\lim_{n \rightarrow \infty} \frac{(\psi(g^n)(c) + C)}{n} = \lim_{n \rightarrow \infty} \frac{\psi(g^n)(c)}{n}$$

exists. $\square$

Remark 4.9. Let $\psi \in QD(G, \ell^\infty(\mathcal{C}_i))$. For every $g \in \text{Stab}_G(c)$ we define $\tilde{\psi} : \ell^1(\text{Stab}_G(c)) \rightarrow \ell^\infty(\mathcal{C}_i)$ by

$$(4.3) \quad \tilde{\psi}(g)[c] = \lim_{n \rightarrow \infty} \frac{\psi(g^n)[c]}{n}.$$

Note that for every $g \in \text{Stab}_G(c)$

$$(4.4) \quad \tilde{\psi}(g^k)[c] = k \lim_{n \rightarrow \infty} \frac{\psi(g^{kn})[c]}{kn} = k\tilde{\psi}(g)[c].$$

Since for every $g \in \text{Stab}_G(c)$

$$(4.5) \quad \sum_{n=0}^{\infty} \frac{\delta\psi(g^{2^n}, g^{2^n})[c]}{2^{n+1}} = \psi(g)[c] - \frac{\psi(g^{2^N})[c]}{2^N} + \sum_{n=N}^{\infty} \frac{\delta\psi(g^{2^n}, g^{2^n})[c]}{2^{n+1}},$$

so $\psi(g)[c] - \tilde{\psi}(g)[c] = \sum_{n=0}^{\infty} \frac{\delta\psi(g^{2^n}, g^{2^n})[c]}{2^{n+1}}$. But since $\psi \in QD(G, \ell^\infty(\mathcal{C}_i))$ we have $|\delta\psi(g^{2^n}, g^{2^n})[c]| < C$. Then

$$(4.6) \quad \psi(g)[c] \approx \tilde{\psi}(g)[c],$$

where the notation $\approx$ means $|\psi(g)[c] - \tilde{\psi}(g)[c]|$ is bounded for every $g \in G$ and $c \in \mathcal{C}_i$.

Next by Remark 3.3 for every $h \in G$ and $t_1 \in T$ there exist unique elements $t_2 \in T, g \in \text{Stab}_G(c)$ such that $h = t_1^{-1}gt_2$ and since $\psi \in QD(G, \ell^\infty(S))$ we have

$$(4.7) \quad \begin{aligned} \psi(t_1^{-1}gt_2)[ct_1] &\approx \psi(gt_2)[c] + \psi(t_1^{-1})[ct_1] \\ &\approx \psi(t_2)[c] + \psi(g)[c] + \psi(t_1^{-1})[ct_1]. \end{aligned}$$

But from

$$\begin{aligned} \psi(e)[ct_1] &= \psi(t_1^{-1}t_1)[ct_1] \approx \psi(t_1)[c] + \psi(t_1^{-1})[ct_1] \\ &\approx \psi(e)[ct_1] + \psi(e)[ct_1] \end{aligned}$$

we have $\psi(e)[ct_1] \approx 0$ and so $\psi(t_1^{-1})[ct_1] \approx -\psi(t_1)[c]$. Hence by (4.7)

$$(4.8) \quad \psi(t_1^{-1}gt_2)[ct_1] \approx \psi(g)[c] + \psi(t_2)[c] - \psi(t_1)[c].$$

Using (4.6) and (4.8) we obtain

$$(4.9) \quad \psi(h)[ct_1] = \psi(t_1^{-1}gt_2)[ct_1] \approx \tilde{\psi}(g)[c] + \psi(t_2)[c] - \psi(t_1)[c].$$

Now let us define $\bar{\psi} : \ell^1(G) \rightarrow \ell^\infty(\mathcal{C}_i)$

$$\bar{\psi}(h)[ct_1] = \tilde{\psi}(g)[c] + \psi(t_2)[c] - \psi(t_1)[c].$$

Then by (4.9) $\psi(h)[ct_1] \approx \bar{\psi}(h)[ct_1]$ for every $h \in G, t_1 \in T$, that is, $\psi - \bar{\psi} \in C_b^1(G, \ell^\infty(\mathcal{C}_i))$. We claim that $\bar{\psi} \in PD(G, \ell^\infty(S))$. To see this we have to show for every $h \in \text{Stab}_G(ct_0)$ and $t_0 \in T$

$$\bar{\psi}(h^k)[ct_0] = k\bar{\psi}(h)[ct_0].$$

Since $h \in \text{Stab}_G(ct_0)$ we have $t_0ht_0^{-1} \in \text{Stab}_G(c)$ and so $h^k = t_0^{-1}(t_0h^kt_0^{-1})t_0$. Therefore by (4.4) and the definition of $\bar{\psi}$ we have

$$\begin{aligned} \bar{\psi}(h^k)[ct_0] &= \bar{\psi}(t_0h^kt_0^{-1})[c] + \psi(t_0)[c] - \psi(t_0)[c] \\ &= k\bar{\psi}(t_0ht_0^{-1})[c] = k\bar{\psi}(h)[ct_0]. \end{aligned}$$

Theorem 4.10. *If $c \in \mathcal{C}_i$ is some fixed element, then a vector spaces isomorphism*

$$\frac{QD(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i))}{D(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i)) \oplus C_b^1(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i))} \cong \frac{PD(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i))}{D(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i))}$$

holds.

Proof. Again we abbreviate $\text{Stab}_G(c)$ by H . For every $\psi \in QD(H, \ell^\infty(\mathcal{C}_i))$ Remark 4.9 gives $\bar{\psi}$ in $PD(H, \ell^\infty(\mathcal{C}_i))$ such that $\psi - \bar{\psi} \in C_b^1(H, \ell^\infty(\mathcal{C}_i))$. Now we define

$$\xi : QD(H, \ell^\infty(\mathcal{C}_i)) \longrightarrow \frac{PD(H, \ell^\infty(\mathcal{C}_i))}{D(H, \ell^\infty(\mathcal{C}_i))}$$

by $\xi(\psi) = \bar{\psi} + D(H, \ell^\infty(\mathcal{C}_i))$. Note that $\psi \in \ker \xi$ if and only if $\bar{\psi} + D(H, \ell^\infty(\mathcal{C}_i)) = D(H, \ell^\infty(\mathcal{C}_i))$ if and only if $\bar{\psi} \in D(H, \ell^\infty(\mathcal{C}_i))$. Therefore $\psi = (\psi - \bar{\psi}) + \bar{\psi} \in D(H, \ell^\infty(\mathcal{C}_i)) \oplus C_b^1(H, \ell^\infty(\mathcal{C}_i))$, that is, $\ker \xi = D(H, \ell^\infty(\mathcal{C}_i)) \oplus C_b^1(H, \ell^\infty(\mathcal{C}_i))$. $\square$

The following theorem is a consequence of Proposition 4.6 and Theorem 4.10.

Theorem 4.11. *If $c \in \mathcal{C}_i$ is some fixed element, then an isomorphism of vector spaces*

$$\mathcal{H}_{b,2}^2 \left(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i) \right) \cong \frac{PD(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i))}{D(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i))}$$

holds.

Remark 4.12. One should note that the result of Grønbaek [3, Section 6] has shown that

$$\mathcal{H}^2(\ell^1(G), \ell^\infty(\mathcal{C}_i)) \cong \mathcal{H}^2(\ell^1(\text{Stab}_G(c)), \ell^\infty(\mathcal{C}_i)),$$

where $c \in \mathcal{C}_i$ is some fixed element. The Grønbaek argument is a generalization of the method of Johnson [6, 2.7].

Finally we apply Remark 4.12 and Theorem 4.10 for the following result.

Theorem 4.13. *If $c \in \mathcal{C}_i$ is some fixed element, then an isomorphism of vector spaces*

$$\mathcal{H}_{b,2}^2(\ell^1(G), \ell^\infty(\mathcal{C}_i)) \cong \frac{PD(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i))}{D(\text{Stab}_G(c), \ell^\infty(\mathcal{C}_i))}$$

holds.

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