ON THE HELTON CLASS OF p -HYPONORMAL OPERATORS

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ABSTRACT. In this paper we show that the Helton class of p -hyponormal operators has scalar extensions. As a corollary we get that each operator in the Helton class of p -hyponormal operators has a nontrivial invariant subspace if its spectrum has its interior in the plane.

1. INTRODUCTION

Let H be a complex (separable) Hilbert space and let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on H . If $T \in \mathcal{L}(H)$, we write $\sigma(T)$ for the spectrum of T .

An operator $T \in \mathcal{L}(H)$ is said to be p -hyponormal, $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$, where T^* is the adjoint of T . If $p = 1$, T is called hyponormal and if $p = \frac{1}{2}$, T is called semi-hyponormal. Semi-hyponormal operators were introduced by Xia (see [15]), and p -hyponormal operators for a general p , $0 < p < 1$, have been studied by Aluthge. Any p -hyponormal operators are q -hyponormal if $q \leq p$ by Löwner's theorem (see [13]). But there are examples to show that the converse of the above statement is not true (see [2]).

In [8] J. W. Helton initiated the study of operators T which satisfy an identity of the form

$$(1) \quad T^{*m} - \binom{m}{1} T^{*m-1}T + \cdots + (-1)^m T^m = 0.$$

We need further study for this class of operators based on (1). Let R and S be in $\mathcal{L}(H)$ and let $C(R, S) : \mathcal{L}(H) \rightarrow \mathcal{L}(H)$ be defined by $C(R, S)(A) = RA - AS$. Then

$$(2) \quad C(R, S)^k(I) = \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} R^j S^{k-j}.$$

Definition 1.1. Let $R \in \mathcal{L}(H)$. If there is an integer $k \geq 1$ such that an operator S satisfies $C(R, S)^k(I) = 0$, we say that S belongs to the Helton class of R with order k . We denote this by $S \in \text{Helton}_k(R)$.

We remark that $C(R, S)^k(I) = 0$ does not imply $C(S, R)^k(I) = 0$ in general.

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Example 1.2. Let S and R be operators in $\mathcal{L}(H \oplus H \oplus H)$ defined by the following:

$$S = \begin{pmatrix} 0 & A & B \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 0 & 0 & C \\ 0 & 0 & D \\ 0 & 0 & 0 \end{pmatrix},$$

where A , B , C , and D are bounded linear operators defined on H . Then it is easy to calculate that $C(R, S)^2(I) = 0$, but $C(S, R)^2(I) \neq 0$.

A bounded linear operator S on H is called scalar of order m if it possesses a spectral distribution of order m , i.e., if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(C) \longrightarrow \mathcal{L}(H)$$

such that $\Phi(z) = S$, where z stands for the identity function on C , and $C_0^m(C)$ stands for the space of compactly supported functions on C , continuously differentiable of order m , $0 \leq m \leq \infty$. An operator is subscalar if it is similar to the restriction of a scalar operator to an invariant subspace.

In this paper we show that the Helton class of p -hyponormal operators has scalar extensions. As a corollary we get that the Helton class of p -hyponormal operators has a nontrivial invariant subspace if its spectrum has its interior in the plane.

2. PRELIMINARIES

Let z be the coordinate in the complex plane C and let $d\mu(z)$ denote the planar Lebesgue measure. Fix a complex (separable) Hilbert space H and a bounded (connected) open subset U of C . We shall denote by $L^2(U, H)$ the Hilbert space of measurable functions $f : U \rightarrow H$, such that

$$\|f\|_{2,U} = \left(\int_U \|f(z)\|^2 d\mu(z) \right)^{\frac{1}{2}} < \infty.$$

The space of functions $f \in L^2(U, H)$ which are analytic functions in U (i.e., $\bar{\partial}f = 0$) is defined by

$$A^2(U, H) = L^2(U, H) \cap \mathcal{O}(U, H),$$

where $\mathcal{O}(U, H)$ denotes the Fréchet space of H -valued analytic functions on U with respect to uniform topology. $A^2(U, H)$ is called the Bergman space for U . Note that $A^2(U, H)$ is a Hilbert space. We denote by P the orthogonal projection of $L^2(U, H)$ onto $A^2(U, H)$.

Let $T \in \mathcal{L}(H)$. The study of the operator $T - z$ on the space $\mathcal{O}(U, H)$ led E. Bishop to fundamental results in spectral theory. Among other things he isolated in [4] the single-valued extension property, which means by definition that the operator $T - z$ acts one-to-one on $\mathcal{O}(U, H)$ for an arbitrary open subset U of C , and the property (β) , which requires that $T - z$ should be one-to-one and with closed range on $\mathcal{O}(U, H)$ for every open set U .

Let us define now a special Sobolev type space. Let U be again a bounded open subset of C and m be a fixed nonnegative integer. The vector-valued Sobolev space $W^m(U, H)$ with respect to $\bar{\partial}$ and of order m will be the space of those functions $f \in L^2(U, H)$ whose derivatives $\bar{\partial}f, \dots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U, H)$. Endowed with the norm

$$\|f\|_{W^m}^2 = \sum_{i=0}^m \|\bar{\partial}^i f\|_{2,U}^2,$$

$W^m(U, H)$ becomes a Hilbert space contained continuously in $L^2(U, H)$.

Let U be a (connected) bounded open subset of C and let m be a nonnegative integer. The linear operator M of multiplication by z on $W^m(U, H)$ is continuous and it has a spectral distribution of order m , defined by the functional calculus

$$\Phi_M : C_0^m(C) \longrightarrow \mathcal{L}(W^m(U, H)), \Phi_M(f) = M_f.$$

Therefore, M is a scalar operator of order m .

3. SOME PROPERTIES

Let H be a complex (separable) Hilbert space with an orthonormal basis $\{e_0, e_1, \dots\}$ and let S be the unilateral shift defined by $Se_n = e_{n+1}$ for all $n \geq 0$. If $\{\alpha_n\}_{n=0}^\infty$ is any bounded sequence of nonnegative numbers, a unilateral weighted shift W with a weight sequence $\{\alpha_n\}$ is defined by $We_n = \alpha_n e_{n+1}$ for $n \geq 0$.

Proposition 3.1. *Let S be the unilateral shift and let W be a unilateral weighted shift with a weight sequence $\{\alpha_n\}_{n=0}^\infty$. Then $W \in \text{Helton}_k(S)$ if and only if*

$$\sum_{j=0}^{k-1} (-1)^{k-j} \binom{k}{j} \alpha_n \cdots \alpha_{n+k-j-1} + 1 = 0$$

for $n \geq 0$.

Proof. From the definition, we know that $W \in \text{Helton}_k(S)$ if and only if

$$\sum_{j=0}^k (-1)^{k-j} \binom{k}{j} S^j W^{k-j} = 0.$$

Since $S^j W^{k-j} e_n = \alpha_n \alpha_{n+1} \cdots \alpha_{n+k-j-1} e_{n+k}$ and $S^k e_n = e_{n+k}$ for $n \geq 0$, we get that $W \in \text{Helton}_k(S)$ if and only if

$$\sum_{j=0}^{k-1} (-1)^{k-j} \binom{k}{j} \alpha_n \cdots \alpha_{n+k-j-1} + 1 = 0$$

for $n \geq 0$. □

The next corollary shows that $\text{Helton}_2(S)$ contains non- p -hyponormal operators.

Corollary 3.2. *Let S be the unilateral shift and let W be a unilateral weighted shift with a weight sequence $\{\alpha_n\}_{n=0}^\infty = \{2, \frac{3}{2}, \frac{4}{3}, \dots, \frac{n+2}{n+1}, \dots\}$ for $n \geq 1$. Then $W \in \text{Helton}_2(S)$, but is not a p -hyponormal operator.*

Proof. Since a weight sequence is decreasing, W is not a p -hyponormal operator. Since $\alpha_n \alpha_{n+1} - 2\alpha_n + 1 = 0$ for $n \geq 0$, from Proposition 3.1, $W \in \text{Helton}_2(S)$. □

Remark. $C(W, S)^k(I) \neq 0$ for all positive integers k , where S is the unilateral shift and W is the unilateral weighted shift in Corollary 3.2. Indeed, since

$$W^j S^{k-j} e_n = \alpha_{n+k-j} \alpha_{n+k-j+1} \cdots \alpha_{n+k-1} e_{n+k}$$

and $\alpha_n = \frac{n+2}{n+1}$ for all $n \geq 0$, we obtain

$$\begin{aligned} & 1 + \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} \alpha_{n+k-j} \cdots \alpha_{n+k-1} \\ &= 1 + \sum_{j=1}^k (-1)^{k-j} \binom{k}{j} \frac{n+k+1}{n+k-j+1} \neq 0. \end{aligned}$$

Hence, $C(W, S)^k(I) \neq 0$ for all positive integers k .

In general, when R is a p -hyponormal operator, there exists a non- p -hyponormal operator T which satisfies $C(R, T)^k(I) = 0$, but $C(T, R)^k(I) \neq 0$ for some k .

Example 3.3. Let S be a unilateral weighted shift with a weight sequence $\{\alpha_n\}_{n=0}^\infty = \{\frac{1}{2}, \frac{2}{3}, \dots\}$ and let W be a unilateral weighted shift with a weight sequence $\{\beta_n\}_{n=0}^\infty = \{2, \frac{7}{6}, \frac{15}{14}, \frac{26}{25}, \dots\}$ which satisfies $\beta_{n+1}\beta_n - \frac{2(n+2)}{n+3}\beta_n + \frac{n+1}{n+3} = 0$. Then it is easy to show that $C(S, W)^2(I) = 0$, but $C(W, S)^2(I) \neq 0$.

4. SUBSCALARITY

In this section we consider the scalar extensions of the Helton class of p -hyponormal operators. We start this section with the definition of the Aluthge transform of an operator. An arbitrary operator $T \in \mathcal{L}(H)$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the appropriate partial isometry satisfying $\ker U = \ker |T|$ and $\ker U^* = \ker T^*$. Associated with T is a related operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, called the Aluthge transform of T , and denoted throughout this paper by $\tilde{T}$. Lemma 4.1 is essential for the proof of our main theorem.

Lemma 4.1. *Let $T \in \mathcal{L}(H)$ be a p -hyponormal operator where $0 < p \leq 1$. If $\{f_n\}$ is a sequence in $W^{4k}(D, H)$ such that $\lim_{n \rightarrow \infty} \|(T-z)^k \bar{\partial}^i f_n\|_{2,D} = 0$ for $i = 1, \dots, 4k$, then $\lim_{n \rightarrow \infty} \|(T-z) \bar{\partial}^i f_n\|_{2,D_0} = 0$ for $i = 1, \dots, 4$ and $\lim_{n \rightarrow \infty} \|(I-P)f_n\|_{2,D_0} = 0$ for every disk D_0 strictly contained in D .*

Proof. Assume that $\{f_n\}$ is a sequence in $W^{4k}(D, H)$ such that

$$(3) \quad \lim_{n \rightarrow \infty} \|(T-z)^k \bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, \dots, 4k$. Using the induction, we will show that

$$\lim_{n \rightarrow \infty} \|(T-z)^{k-1} \bar{\partial}^i f_n\|_{2,D_0} = 0$$

for $i = 1, \dots, 4(k-1)$, where a disk D_0 is strictly contained in D . Since $|T|^{1/2}T = \tilde{T}|T|^{1/2}$, we have

$$(4) \quad \lim_{n \rightarrow \infty} \|(\tilde{T}-z)^k |T|^{\frac{1}{2}} \bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, \dots, 4k$. Since $\tilde{T}$ is semihyponormal, by [10],

$$(5) \quad \lim_{n \rightarrow \infty} \|(\tilde{T}-z)^*(\tilde{T}-z)^{k-1} |T|^{\frac{1}{2}} \bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, \dots, 4k$. By an application of [14, Proposition 2.1], we get

$$(6) \quad \lim_{n \rightarrow \infty} \|(I-P)(\tilde{T}-z)^{k-1} |T|^{\frac{1}{2}} \bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, \dots, 4k - 2$. From (4) and (5), we have

$$\lim_{n \rightarrow \infty} \|(\tilde{T} - z)P(\tilde{T} - z)^{k-1}|T|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, \dots, 4k - 2$, where P denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$. Since $\tilde{T}$ is semihyponormal, it satisfies the property (β) from [11]. Hence

$$(7) \quad \lim_{n \rightarrow \infty} \|P(\tilde{T} - z)^{k-1}|T|^{\frac{1}{2}}\bar{\partial}^i f_n\|_{2,D_0} = 0$$

for $i = 1, \dots, 4k - 2$, where a disk D_0 is strictly contained in D . Since $|T|^{1/2}T = \tilde{T}|T|^{1/2}$, (6) and (7) imply that $\lim_{n \rightarrow \infty} \||T|^{\frac{1}{2}}(T - z)^{k-1}\bar{\partial}^i f_n\|_{2,D} = 0$ for $i = 1, \dots, 4k - 2$. Since $T = U|T|$, from (3) we have

$$(8) \quad \lim_{n \rightarrow \infty} \|z(T - z)^{k-1}\bar{\partial}^i f_n\|_{2,D_0} = 0$$

for $i = 1, \dots, 4k - 2$. By an application of [14, Proposition 2.1] with $T = 0$,

$$(9) \quad \lim_{n \rightarrow \infty} \|(I - P)(T - z)^{k-1}\bar{\partial}^i f_n\|_{2,D_0} = 0$$

for $i = 1, \dots, 4k - 4$. From (8) and (9), we obtain

$$(10) \quad \lim_{n \rightarrow \infty} \|zP(T - z)^{k-1}\bar{\partial}^i f_n\|_{2,D_0} = 0$$

for $i = 1, \dots, 4k - 4$. By [6], there exists a constant $c > 0$ such that

$$c\|P(T - z)^{k-1}\bar{\partial}^i f_n\|_{2,D_0} \leq \|zP(T - z)^{k-1}\bar{\partial}^i f_n\|_{2,D_0}$$

for $i = 1, \dots, 4k - 4$. Hence from (10) we have $\lim_{n \rightarrow \infty} \|P(T - z)^{k-1}\bar{\partial}^i f_n\|_{2,D_0} = 0$ for $i = 1, \dots, 4k - 4$. Then (9) implies that $\lim_{n \rightarrow \infty} \|(T - z)^{k-1}\bar{\partial}^i f_n\|_{2,D_0} = 0$ for $i = 1, \dots, 4k - 4$. Thus $\lim_{n \rightarrow \infty} \|(T - z)\bar{\partial}^i f_n\|_{2,D_0} = 0$ for $i = 1, \dots, 4$. In particular, if $k = 1$ in (9), we have $\lim_{n \rightarrow \infty} \|(I - P)f_n\|_{2,D_0} = 0$. So we complete the proof. $\square$

Now we are ready to prove our main theorem.

Theorem 4.2. *Let $T \in \mathcal{L}(H)$ be a p -hyponormal operator where $0 < p \leq 1$. If $S \in \text{Helton}_k(T)$, then S is a subscalar operator of order $4k$.*

Proof. Consider an arbitrary bounded open disk D of the complex plane C containing $\sigma(S)$ and 0 and the quotient space

$$H(D) = W^{4k}(D, H) / \overline{(z - S)W^{4k}(D, H)}$$

endowed with the Hilbert space norm. The class of a vector f or an operator A on $H(D)$ will be denoted by $\tilde{f}$, respectively $\tilde{A}$. Note that M , the multiplication operator with z on $W^{4k}(D, H)$, leaves invariant $\text{ran}(z - S)$; hence $\tilde{M}$ is well defined. Moreover, the spectral distribution Φ of M commutes with $z - S$; therefore, $\tilde{M}$ is still a scalar operator of order $4k$, with $\tilde{\Phi}$ as spectral distribution.

Let V be the operator $V(h) = \widetilde{1 \otimes h}$, from H into $H(D)$, denoting by $1 \otimes h$ the constant function h . Then $VS = \tilde{M}V$. Indeed, $VSh = \widetilde{1 \otimes Sh} = \widetilde{z \otimes h} = \tilde{M}(\widetilde{1 \otimes h}) = \tilde{M}Vh$ for any $h \in H$. Since $\text{ran}V$ is an invariant subspace for $\tilde{M}$, it suffices to show that V is one-to-one and has closed range.

If $h_n \in H$ and $f_n \in W^{4k}(D, H)$ are sequences such that

$$(11) \quad \lim_{n \rightarrow \infty} \|(z - S)f_n + 1 \otimes h_n\|_{W^{4k}} = 0,$$

then we have

$$(12) \quad \lim_{n \rightarrow \infty} \|(z - S)\bar{\partial}^i f_n\|_{2,D} = 0$$

for $i = 1, 2, \dots, 4k$. Now for $i = 1, 2, \dots, 4k$,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left\| \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} T^j S^{k-j} \bar{\partial}^i f_n - (T - z)^k \bar{\partial}^i f_n \right\|_{2,D} \\ &= \lim_{n \rightarrow \infty} \left\| \sum_{j=0}^k \binom{k}{j} (T - z)^j (z - S)^{k-j} \bar{\partial}^i f_n - (T - z)^k \bar{\partial}^i f_n \right\|_{2,D} \\ &= \lim_{n \rightarrow \infty} \left\| \sum_{j=0}^{k-1} \binom{k}{j} (T - z)^j (z - S)^{k-j} \bar{\partial}^i f_n \right\|_{2,D} \\ &\leq \lim_{n \rightarrow \infty} \left\| \sum_{j=0}^{k-1} \binom{k}{j} (T - z)^j (z - S)^{k-j-1} \right\| \|(z - S)\bar{\partial}^i f_n\|_{2,D} \\ &= 0. \end{aligned}$$

Hence we have $\lim_{n \rightarrow \infty} \|(T - z)^k \bar{\partial}^i f_n\|_{2,D} = 0$ for $i = 1, 2, \dots, 4k$. Then Lemma 4.1 can be applied to get

$$(13) \quad \lim_{n \rightarrow \infty} \|(I - P)f_n\|_{2,D_0} = 0,$$

where a disk D_0 is strictly contained in D . Now we apply (13) to (11). Then we get

$$\lim_{n \rightarrow \infty} \|(z - S)Pf_n + 1 \otimes h_n\|_{2,D_0} = 0.$$

Let Γ be a curve in D_0 surrounding $\sigma(S)$. Then for $z \in \Gamma$,

$$\lim_{n \rightarrow \infty} \|Pf_n(z) + (z - S)^{-1}(1 \otimes h_n)\| = 0$$

uniformly. Hence by the Riesz-Dunford functional calculus we have

$$\lim_{n \rightarrow \infty} \left\| \frac{1}{2\pi i} \int_{\Gamma} Pf_n(z) dz + h_n \right\| = 0.$$

But since $\int_{\Gamma} Pf_n dz = 0$ by Cauchy's theorem, $\lim_{n \rightarrow \infty} h_n = 0$. Hence V is one-to-one and has closed range. $\square$

Corollary 4.3. *Every p -hyponormal operator is subscalar of order 4.*

Proof. If $k = 1$ in Theorem 4.2, then $S = T$. Hence the proof follows from Theorem 4.2. $\square$

If U is a bounded open set in the complex plane $\mathbf{C}$, recall that a subset $\Lambda \subset U$ is said to be *dominating* for U if every function $h(z)$ analytic and bounded on U satisfies

$$\sup_{z \in U} |h(z)| = \sup_{z \in U \cap \Lambda} |h(z)|.$$

Corollary 4.4. *Let $S \in \text{Helton}_k(T)$ where $T \in \mathcal{L}(H)$ is a p -hyponormal operator with $0 < p \leq 1$. If there exists a nonempty open set U in the complex plane $\mathbf{C}$ such that $\sigma(S) \cap U$ is dominating for U , then S has a nontrivial invariant subspace.*

Proof. This follows from Theorem 4.2 and [7]. $\square$

From Theorem 4.2 we get the following corollary.

Corollary 4.5. *If $S \in \text{Helton}_k(T)$ where $T \in \mathcal{L}(H)$ is a p -hyponormal operator with $0 < p \leq 1$, then*

- (a) *S has the property (β) ;*
- (b) *S has the single-valued extension property.*

Proof. Since every scalar operator satisfies the property (β) and the property (β) is transmitted from an operator to its restrictions to closed invariant subspaces, it follows from Theorem 4.2 that S satisfies the property (β) . In particular, it has the single-valued extension property. $\square$

We close this paper with the following remark pointed out by the referee.

Remark. T^* being in the Helton class of order k of T translates to (in Helton's terminology) that T^* is coadjoint of order $k - 1$, from which it follows (as in [9]) that $\|e^{isT^*}\| = O(|s|^{k-1})$. Hence T^* (and also T) possesses a continuous $C^k(\mathbf{R})$ -functional calculus. But then it follows that T is decomposable (see [5]). By definition any decomposable operator having spectrum with more than one point has nontrivial invariant subspaces. Helton conjectured and proved in some special case while Agler provided a proof for the general case (at least for $k = 3$) (see also the paper of Ball and Helton, [3]) an alternative characterization of coadjoint operators as *sub-Jordan operators*. Sub-Jordan operators appear to be related to but a priori different from subscalar operators.

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