A q -SAMPLING THEOREM AND PRODUCT FORMULA FOR CONTINUOUS q -JACOBI FUNCTIONS

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ABSTRACT. In this paper we derive a q -analogue of the sampling theorem for Jacobi functions. We also establish a product formula for the nonterminating version of the q -Jacobi polynomials. The proof uses recent results in the theory of q -orthogonal polynomials and basic hypergeometric functions.

1. INTRODUCTION

The classical Jacobi functions are defined by

$$R_t^{(\alpha, \beta)}(x) = {}_2F_1(-t, t + \alpha + \beta + 1; \alpha + 1; \frac{1-x}{2}), \quad -1 < x < 1,$$

where $\alpha, \beta > -1$ and $t \in \mathbb{C}$. A more usual normalization and notation are

$$P_t^{(\alpha, \beta)}(x) = \frac{\Gamma(t + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(t + 1)} R_t^{(\alpha, \beta)}(x).$$

The function $t \rightarrow R_t^{(\alpha, \beta)}(x)$ is even entire, and for $n = 0, 1, \dots$, $R_t^{(\alpha, \beta)}(x)$ becomes the normalized Jacobi polynomials $R_n^{(\alpha, \beta)}(x)$. The Jacobi polynomials satisfy the orthogonality relation

$$\int_0^\pi R_m^{(\alpha, \beta)}(\cos \theta) R_n^{(\alpha, \beta)}(\cos \theta) w_{\alpha, \beta}(\theta) d\theta = \delta_{m, n} \left(w_n^{(\alpha, \beta)} \right)^{-1},$$

where

$$w_n^{(\alpha, \beta)} = \begin{cases} \frac{\Gamma(2\lambda + 1)}{\Gamma(\alpha + 1)\Gamma(\beta + 1)}, & n = 0, \\ \frac{(2n + 2\lambda)\Gamma(n + \alpha + 1)\Gamma(n + 2\lambda)}{\Gamma^2(\alpha + 1)\Gamma(n + 1)\Gamma(n + \beta + 1)}, & n = 1, 2, \dots \end{cases}$$

and

$$\lambda = \frac{\alpha + \beta + 1}{2}.$$

The continuous q -Jacobi polynomials is defined by (see 15.5.1, [7])

(1)

$$P_n^{(\alpha, \beta)}(\cos \theta | q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{\frac{2\alpha+1}{4}} e^{i\theta}, q^{\frac{2\alpha+1}{4}} e^{-i\theta} \\ q^{\alpha+1}, -q^{\frac{\alpha+\beta+1}{2}}, -q^{\frac{\alpha+\beta+2}{4}} \end{matrix} \middle| q; q \right).$$

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$\{P_n^{(\alpha,\beta)}(\cos \theta | q)\}$ are q -analogues of the Jacobi polynomials in the sense that [17]

$$\lim_{q \rightarrow 1} P_n^{(\alpha,\beta)}(\cos \theta | q) = \frac{P_n^{(\alpha,\beta)}(\cos \theta)}{P_n^{(\alpha,\beta)}(1)}.$$

For $\alpha > -1/2$ and $\beta > -1/2$, we have

$$\int_{-1}^1 P_m^{(\alpha,\beta)}(x | q) P_n^{(\alpha,\beta)}(x | q) w(x | q^\alpha, q^\beta) dx = h_n \delta_{n,m},$$

where

$$h_n = \frac{2\pi \left(q^{\frac{\alpha+\beta+2}{2}}, q^{\frac{\alpha+\beta+3}{2}}; q \right)_\infty}{\left(q, q^{\alpha+1}, q^{\beta+1}, -q^{\frac{\alpha+\beta+1}{2}}, -q^{\frac{\alpha+\beta+2}{2}}; q \right)_\infty} \times \frac{(1 - q^{\alpha+\beta+1}) \left(q^{\alpha+1}, q^{\beta+1}, -q^{\frac{\alpha+\beta+3}{2}}; q \right)_n}{(1 - q^{2n+\alpha+\beta+1}) \left(q, q^{\alpha+\beta+1}, -q^{\frac{\alpha+\beta+1}{2}}; q \right)_n}$$

and

$$w(\cos \theta | q^\alpha, q^\beta) = \frac{1}{\sin \theta} \left| \frac{(e^{i\theta}, -e^{i\theta}; q^{1/2})_\infty}{\left(q^{\frac{2\alpha+1}{4}} e^{i\theta}, -q^{\frac{2\beta+1}{4}} e^{i\theta}; q^{1/2} \right)_\infty} \right|^2.$$

The q -shifted factorial is defined by [6]

$$(2) \quad (a; q)_0 = 1, \quad (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k), \quad n = 1, 2, \dots, \infty,$$

and more generally [6]

$$(a_1, \dots, a_s; q)_n = \prod_{k=1}^s (a_k; q)_n, \quad n = 0, 1, 2, \dots, \infty,$$

$$h_q(x, a) = \prod_{n=0}^{\infty} (1 - 2ax \cos \theta + a^2 q^{2n}) = (ae^{i\theta}, ae^{-i\theta}; q)_\infty, \quad x = \cos \theta.$$

A basic hypergeometric series is

$${}_r\varphi_s \left(\begin{matrix} a_1, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix} ; q, z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k} \left[(-1)^k q^{\binom{k}{2}} \right]^{1+s-r} z^k.$$

We also employ the compact notation

$${}_{r+1}W_r(a_1; a_4, \dots, a_{r+1}; q, z),$$

for the very-well poised ${}_{r+1}\varphi_r$ series by

$${}_{r+1}\varphi_r \left[\begin{matrix} a_1, q\sqrt{a_1}, -q\sqrt{a_1}, a_4, \dots, a_{r+1} \\ \sqrt{a_1}, -\sqrt{a_1}, q\frac{a_1}{a_4}, \dots, q\frac{a_1}{a_{r+1}} \end{matrix} ; q, z \right].$$

Given a function f defined on $(-1, 1)$ we set $\check{f}(e^{i\theta}) := f(x)$, $x = \cos \theta$. In other words we think of $f(\cos \theta)$ as a function of $e^{i\theta}$. In this notation the Askey-Wilson divided difference operator $\mathcal{D}_q$ is defined by [7]

$$(3) \quad (\mathcal{D}_q f)(x) = \frac{\check{f}(q^{\frac{1}{2}} e^{i\theta}) - \check{f}(q^{-\frac{1}{2}} e^{-i\theta})}{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})i \sin \theta}.$$

In particular $\mathcal{D}_q$ is well defined on $H_{\frac{1}{2}}$, where

$$H_{\nu} := \{f : f(\frac{1}{2}(z + \frac{1}{z})) \text{ is analytic for } q^{\nu} < |z| < q^{-\nu}\}.$$

2. THE CONTINUOUS q -JACOBI FUNCTIONS

The continuous q -Jacobi functions are special cases of the Askey-Wilson functions [10], [14], [18] and [19]

$$(4) \quad u_{\lambda}(\cos \theta, a, b, c, d) = \frac{(abcq^{\lambda}/z, q^{1-\lambda}z/d; q)_{\infty}}{(abcz, qz/d; q)_{\infty}} \\ \times {}_8W_7(abcz/q; q^{-\lambda}, abcdq^{\lambda-1}, az, bz, cz; q, q/dz).$$

We take $a = q^{\frac{2\alpha+1}{4}}$, $b = q^{\frac{2\alpha+3}{4}}$, $c = -q^{\frac{2\beta+1}{4}}$, $d = -q^{\frac{2\beta+3}{4}}$ in the definition of the Askey-Wilson functions, and let

$$(5) \quad P_{\mu}^{(\alpha, \beta)}(x; q) = u_{\mu}(x, q^{(2\alpha+1)/4}, q^{(2\alpha+3)/4}, -q^{(2\beta+1)/4}, -q^{(2\beta+3)/4}).$$

When μ is a nonnegative integer, $P_{\mu}^{(\alpha, \beta)}(x, q)$ becomes

$$(6) \quad P_n^{(\alpha, \beta)}(x; q) = q^{-n(n+\alpha+\beta+1)/2} \frac{(q; q)_n}{(q^{\alpha+1}; q)_n} P_n^{(\alpha, \beta)}(x | q).$$

The lowering operator is

$$(7) \quad \mathcal{D}_q P_{\mu}^{(\alpha, \beta)}(x; q) \\ = \frac{2q^{-(2\beta+1)/4}}{(1-q)} \frac{(1-q^{-\mu})(1-q^{\mu+\alpha+\beta+1})}{(1-q^{\alpha+1})(1+q^{(\alpha+\beta+1)/2})(1+q^{(\alpha+\beta+2)/2})} P_{\mu-1}^{(\alpha+1, \beta+1)}(x; q),$$

while the raising operator is

$$(8) \quad \mathcal{D}_q[w(x | q^{\alpha+1}, q^{\beta+1}) P_{\mu-1}^{(\alpha+1, \beta+1)}(x | q)] \\ = 2q^{(2\beta+3)/4} \frac{(1-q^{\alpha+1})(1+q^{(\alpha+\beta+1)/2})(1+q^{(\alpha+\beta+2)/2})}{(1-q)} \\ \times w(x | q^{\alpha}, q^{\beta}) P_{\mu}^{(\alpha, \beta)}(x | q).$$

From the formulas (7) and (8), we get the q -difference equation for the continuous q -Jacobi functions $P_{\mu}^{(\alpha, \beta)}(x | q)$ written in a self-adjoint form

$$(9) \quad L_q^{(\alpha, \beta)} y(x) = \frac{1}{w(x; \beta | q)} \mathcal{D}_q(w(x; \beta q | q) \mathcal{D}_q y(x)) \\ = \frac{4q}{(1-q)^2} (1-q^{-\mu})(1-q^{\mu+\alpha+\beta+1}) y(x).$$

3. A q -SAMPLING THEOREM

We denote by H_σ the set of functions f that are even and entire on $\mathbb{C}$ and satisfy

$$\sup_{z \in \mathbb{C}} \left(1 + |z|^2\right)^m |f(z)| \exp(-\sigma |\operatorname{Im} z|) < \infty, \quad m \in \mathbb{N}_0, \sigma > 0.$$

A function is said to be bandlimited to $[-\sigma, \sigma]$ if and only if $f \in H_\sigma$. A nice characterization of a function bandlimited to $[-\sigma, \sigma]$ is given by the Paley-Wiener type theorem for Jacobi Fourier series (Theorem 5.1, [15]). It may be stated as follows: Let $\alpha > \beta > -1/2$. A function f belongs to H_σ if and only if it is representable in the form

$$f(t) = \frac{1}{2} \int_{-\sigma}^{\sigma} g(\cos \theta) R_{t-\lambda}^{(\alpha, \beta)}(\cos \theta) w_{\alpha, \beta}(\theta) d\theta \quad \text{for some function } g \in \mathcal{D}_\sigma(\mathbb{R}),$$

where

$$\mathcal{D}_\sigma(\mathbb{R}) = \{g \in C^\infty(-\sigma, \sigma) : g \text{ is even with compact support}\}.$$

To derive the sampling theorem for Jacobi transform, recall that if $f(t)$ is bandlimited to $[-\pi, \pi]$, then by the Paley-Wiener theorem

$$f(t) = \int_0^\pi g(\cos \theta) R_{t-\lambda}^{(\alpha, \beta)}(\cos \theta) w_{\alpha, \beta}(\theta) d\theta, \quad \text{for some function } g \in \mathcal{D}_\pi(\mathbb{R}).$$

By [16, Lemma 3.1], we have

$$g(x) = \sum_{n=0}^{\infty} g_n R_n^{(\alpha, \beta)}(x),$$

where

$$g_n = w_n^{(\alpha, \beta)} \int_0^\pi g(\cos \theta) R_n^{(\alpha, \beta)}(\cos \theta) w_{\alpha, \beta}(\theta) d\theta.$$

Thus,

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} g_n w_n^{(\alpha, \beta)} \int_0^\pi R_n^{(\alpha, \beta)}(\cos \theta) R_{t-\lambda}^{(\alpha, \beta)}(\cos \theta) w_{\alpha, \beta}(\theta) d\theta \\ (10) \quad &= \sum_{n=0}^{\infty} f(\lambda + n) s_n^{(\lambda)}(t - \lambda), \end{aligned}$$

where

$$s_n^{(\lambda)}(t - \lambda) = w_n^{(\alpha, \beta)} \int_0^\pi R_n^{(\alpha, \beta)}(\cos \theta) R_{t-\lambda}^{(\alpha, \beta)}(\cos \theta) w_{\alpha, \beta}(\theta) d\theta$$

and obtain from [21] that

$$(11) \quad s_n^{(\lambda)}(t - \lambda) = \frac{(2n + 2\lambda) \Gamma(n + 2\lambda)}{\Gamma(n + 1)} \frac{\sin \pi(t - \lambda - n) \Gamma(t - \lambda + 1)}{\pi(t^2 - (\lambda + n)^2) \Gamma(t + \lambda)}.$$

To derive a q -analogue of the sampling expansion (10), we start by defining a q -analogue of bandlimitedness.

Definition 1. We say that a function $f(x)$ is q -bandlimited to $[-1, 1]$ if it can be written in the form

$$(12) \quad f(\mu) = \int_{-1}^1 g(x) P_\mu^{(\alpha, \beta)}(x; q) w(x | q^\alpha, q^\beta) dx$$

where

$$g \in L^2(-1, 1) \text{ .}$$

Now we state the q -analogue of the sampling expansion (10) .

Theorem 1. *Let f be a q -bandlimited function. Then f has the sampling expansion*

$$f(\mu) = \sum_{n=0}^{\infty} s_n^{(\alpha, \beta)}(\mu, q) f(n),$$

where

$$s_n^{(\alpha, \beta)}(\mu, q) = \frac{(-1)^n q^{n(n+1)/2} (1 - q^{\alpha+\beta+2n+1})}{(q; q)_n (q, q^{\alpha+\beta+n+1}, q)_{\infty}} \frac{(q^{-\mu}, q^{\alpha+\beta+\mu+2}; q)_{\infty}}{(1 - q^{-\mu+n})(1 - q^{\alpha+\beta+\mu+n+1})}.$$

Proof. The function f is q -bandlimited. Then

$$(13) \quad f(\mu) = \int_{-1}^1 g(x) P_{\mu}^{(\alpha, \beta)}(x; q) w(x | q^{\alpha}, q^{\beta}) dx,$$

where

$$g \in L^2(-1, 1) \text{ .}$$

Since $\{P_n^{(\alpha, \beta)}(x | q)\}_{n=0}^{\infty}$ is a complete orthogonal set in a weighted $L^2(-1, 1)$, we have

$$(14) \quad g(x) = \sum_{n=0}^{\infty} \hat{g}_n P_n^{(\alpha, \beta)}(x | q),$$

where

$$(15) \quad \hat{g}_n = \frac{1}{h_n} \int_{-1}^1 g(x) P_n^{(\alpha, \beta)}(x | q) w(x | q^{\alpha}, q^{\beta}) dx.$$

Substituting (14) into (13),

$$(16) \quad f(\mu) = \sum_{n=0}^{\infty} \hat{g}_n \int_{-1}^1 P_n^{(\alpha, \beta)}(x | q) P_{\mu}^{(\alpha, \beta)}(x; q) w(x | q^{\alpha}, q^{\beta}) dx.$$

But in view of (6) and (15), equation (16) becomes

$$f(\mu) = \sum_{n=0}^{\infty} f(n) s_n^{(\alpha, \beta)}(\mu, q),$$

where

$$(17) \quad s_n^{(\alpha, \beta)}(\mu, q) = \frac{q^{n(n+\alpha+\beta+1)/2} (q^{\alpha+1}; q)_n}{h_n (q; q)_n} \times \int_{-1}^1 P_n^{(\alpha, \beta)}(x | q) P_{\mu}^{(\alpha, \beta)}(x; q) w(x | q^{\alpha}, q^{\beta}) dx.$$

The function $s_n^{(\alpha, \beta)}(\mu, q)$ is a q -analogue of $s_n^{(\alpha, \beta)}(t)$ in (11). To evaluate the last integral, we use ([4, Proposition 6]) to obtain

$$(18) \quad s_n^{(\alpha, \beta)}(\mu, q) = \frac{(-1)^n q^{n(n+1)/2} (1 - q^{\alpha+\beta+2n+1})}{(q; q)_n (q, q^{\alpha+\beta+n+1}, q)_{\infty}} \frac{(q^{-\mu}, q^{\alpha+\beta+\mu+2}; q)_{\infty}}{(1 - q^{-\mu+n})(1 - q^{\alpha+\beta+\mu+n+1})}.$$

Clearly

$$s_n^{(\alpha, \beta)}(k, q) = \delta_{n, k}.$$

The continuous q -Jacobi function has the expansion formula in terms of q -Jacobi polynomials. $\square$

Proposition 1. *We have*

$$P_{\mu}^{(\alpha, \beta)}(x; q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{-n(\alpha+\beta+1)/2} (q^{\alpha+1}; q)_n (1 - q^{\alpha+\beta+2n+1})}{(q, q^{\alpha+\beta+n+1}; q)_{\infty}} \\ \times \frac{(q^{-\mu}, q^{\alpha+\beta+\mu+2}; q)_{\infty}}{(1 - q^{-\mu+n})(1 - q^{\alpha+\beta+\mu+n+1})} P_n^{(\alpha, \beta)}(x | q).$$

4. PRODUCT FORMULA FOR THE CONTINUOUS q -JACOBI FUNCTIONS

Product formulas are very useful identities associated with a special functions or orthogonal polynomials. It is a very important tool in harmonic analysis. Note that the right-hand side of a product formula usually defines a generalized translation operator and that this in turn can be used to establish an appropriate convolution structure on the space and often a hypergroup structure on the corresponding interval.

In recent work M. E. H. Ismail and D. Stanton (see [12]) solve the interpolation problem for the sequence $\{x_{2n}\}$, defined by

$$x_n = [aq^{n/2} + q^{-n/2}/a]/2, \quad 0 < a < 1.$$

For an entire function f satisfying (see [12, Theorem 4.1])

$$\lim_{r \rightarrow +\infty} \sup \frac{\ln M(r; f)}{\ln^2 r} < \frac{1}{2 \ln q^{-1}},$$

we have

$$(19) \quad f(x) = \sum_{k=0}^{\infty} S_n(x, a) f(x_{2k}),$$

where

$$S_n(x, a) = \frac{(-1)^k q^{k(k+1)/2} (1 - a^2 q^{2k})}{(q; q)_k (q, a^2 q^k; q)_{\infty}} \frac{(ae^{i\theta}, ae^{-i\theta}; q)_{\infty}}{(1 - aq^k e^{i\theta})(1 - aq^k e^{-i\theta})}$$

and

$$M(r; f) = \sup \{|f(z)| : |z| \leq r\}.$$

We use these result to extend the product formula for the q -Jacobi polynomials established by Rahman [17], to the nonterminating cases.

Lemma 1. *If f is an entire function satisfying the condition*

$$(20) \quad M(\cosh u, f) \leq K \prod_{j=1}^m (-b_j e^u, -b_j e^{-u}; q^m)_{\infty}, \quad b_j > 0, \quad K > 0,$$

with

$$(b_1 \dots b_m)^{\frac{1}{m}} q^{(1-m)/2} < a,$$

then the expansion (19) holds.

Proof. The result follow from [12, Theorem 5.1]. $\square$

Lemma 2. *Suppose f is an entire function satisfying (20) and suppose*

$$f(x_{2k}) = 0, \quad k = 0, 1, 2, \dots$$

Then f is identically zero.

The continuous q -Jacobi polynomials satisfy the product formula (see [17])

$$(21) \quad \begin{aligned} & P_n^{(\alpha, \beta)}(x | q) P_n^{(\alpha, \beta)}(y | q) \\ &= \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \int_{-1}^1 K(x, y, z; q^\alpha, q^\beta) P_n^{(\alpha, \beta)}(z | q) dz, \end{aligned}$$

where

$$\begin{aligned} & K(x, y, z; q^\alpha, q^\beta) \\ &= \frac{(q, q^{(2\alpha+3)/2}, q^{(2\beta+1)/2}; q)_\infty (q^{1/2}, q^{(\alpha+\beta+2)/2}, q^{(\alpha-\beta+1)/2}; q^{1/2})_\infty}{4\pi^2 (q^{\alpha+\beta+2}; q)_\infty (q^{(2\alpha+3)/2}; q^{1/2})_\infty} \\ & \times \left| \left(q^{(2\alpha+3)/2} e^{i\theta}, q^{(2\alpha+3)/2} e^{i\varphi}; q^{1/2} \right)_\infty \right| \\ & \times \frac{(1-z^2)^{-1/2} h_q(z, 1) h_q(z, -1) h_q\left(z, q^{1/2}\right) h_q\left(z, -q^{1/2}\right)}{h_q\left(z, q^{(2\alpha+3)/4} e^{i(\theta+\varphi)}\right) h_q\left(z, q^{(2\beta+3)/4} e^{-i(\theta+\varphi)}\right)} \\ & \times \int_{-1}^1 \frac{h_{q^{1/2}}(z, 1) h_{q^{1/2}}(z, -1) h_{q^{1/2}}\left(z, q^{1/4}\right) h_{q^{1/2}}\left(z, -q^{1/4}\right)}{h_{q^{1/2}}\left(\tau, q^{(2\beta+1)/8} e^{i\psi/2}\right) h_{q^{1/2}}\left(\tau, q^{(2\beta+1)/8} e^{-i\psi/2}\right) h_{q^{1/2}}\left(\tau, -q^{(2\beta+1)/8} e^{i\psi/2}\right)} \\ & \times \frac{h_{q^{1/2}}\left(\tau, -q^{(\alpha+\beta+2)/4} e^{i(\theta+\varphi)}\right) h_{q^{1/2}}\left(\tau, -q^{(\alpha+\beta+2)/4} e^{-i(\theta+\varphi)}\right) (1-\tau^2)^{-1/2} d\tau}{h_{q^{1/2}}\left(\tau, -q^{(2\beta+1)/8} e^{-i\psi/2}\right) h_{q^{1/2}}\left(\tau, q^{(\alpha-\beta+1)/4} e^{i(\theta-\varphi)}\right) h_{q^{1/2}}\left(\tau, q^{(\alpha-\beta+1)/4} e^{i(\varphi-\theta)}\right)}, \end{aligned}$$

with $x = \cos \theta$ and $y = \cos \varphi$. For the integral on the right side to exist it is sufficient that $\beta > \alpha > -1/2$.

Proposition 2. For $\alpha + \beta > 0$, the continuous q -Jacobi functions satisfy the product formula

$$P_\mu^{(\alpha, \beta)}(x; q) P_\mu^{(\alpha, \beta)}(y; q) = \int_{-1}^1 \Delta(x, y, z; q^\alpha, q^\beta) P_\mu^{(\alpha, \beta)}(z; q) dz,$$

where

$$\Delta(x, y, z; q^\alpha, q^\beta) = \frac{(q^{\frac{1}{2} + \mu + \alpha + \beta + 1}, q^{\frac{1}{2} - (\mu + \alpha + \beta + 1)}; q)_\infty}{(-q^{(\alpha+\beta+2)/2}, -q^{(\alpha-\beta+1)/2}; q)_\infty} K(x, y, z; q^\alpha, q^\beta).$$

Proof. The function

$$\begin{aligned} & \varphi(v, x, a, b, c, d; q) \\ &= \frac{\left(qa/d, \sqrt{qbc/ade}e^{i\vartheta}, \sqrt{qabc/de}e^{i(\theta-\vartheta)}, \sqrt{qabc/de}e^{-i(\theta+\vartheta)} \right)_{\infty}}{(a\sqrt{qabcde}e^{-i\vartheta}/d, bc, q/d, q/de^{i\vartheta}; q)_{\infty}} \\ & \times {}_8W_7(a\sqrt{abcd/q}e^{i\vartheta}/d; \sqrt{abcd/q}e^{-i\vartheta}, \sqrt{qac/bde}e^{-i\vartheta}, \\ & \sqrt{qab/cde}e^{-i\vartheta}, ae^{i\vartheta}, ae^{-i\vartheta}, \sqrt{qbc/ade}e^{i\vartheta}) \\ &= \frac{(\sqrt{qbc/ade}e^{i\vartheta}, \sqrt{qbc/ade}e^{-i\vartheta}; q)_{\infty}}{(bc, qa/d; q)_{\infty}} \\ & \times {}_4\phi_3 \left(\begin{matrix} ae^{i\vartheta}, ae^{-i\vartheta}, \sqrt{abcd/q}e^{i\vartheta}, \sqrt{abcd/q}e^{-i\vartheta} \\ ab, ac, ad \end{matrix} \middle| q; q \right) \\ & + \frac{(ae^{i\vartheta}, ae^{-i\vartheta}, \sqrt{abcd/q}e^{i\vartheta}, \sqrt{abcd/q}e^{-i\vartheta}, qb/d, qc/d; q)_{\infty}}{(qz/d, q/dz, ab, ac, bc, ad/q; q)_{\infty}} \\ & \times {}_4\phi_3 \left(\begin{matrix} qe^{i\vartheta}/d, q/de^{i\vartheta}, \sqrt{qbc/ade}e^{i\vartheta}, \sqrt{qbc/ade}e^{-i\vartheta} \\ qb/d, qc/d, q^2/ad \end{matrix} \middle| q; q \right) \end{aligned}$$

is entire in $v = \cos \vartheta$.

Indeed

$$(\xi e^{i\vartheta}, \xi e^{-i\vartheta}; q)_n = \prod_{k=0}^{n-1} (1 - 2\xi v q^k + \xi^2 q^{2k})$$

is an entire function of v with $\xi = \sqrt{qbc/ad}, \sqrt{abcd/q}$ and $n = 1, 2, \dots$ or ∞ .

From the trigonometric formula

$$\cos 4\vartheta = 8 \cos^4 \vartheta - 8 \cos^2 \vartheta + 1$$

we deduce that the function Ψ defined by

$$\Psi(\nu, x) = \varphi(8\nu^4 - 8\nu^2 + 1, x, q^{\frac{2\alpha+1}{4}}, q^{\frac{2\alpha+3}{4}}, -q^{\frac{2\beta+1}{4}}, -q^{2\beta+3}, q^4)$$

is an entire function and for

$$\lambda_n = (q^{n+(\alpha+\beta+1)/2} + q^{-n-(\alpha+\beta+1)/2})/2, \quad n = 0, 1, \dots$$

The function $\Psi(\nu, x)$ is a multiple of the q -Jacobi polynomials (1) of the n -th degree,

$$\Psi(\lambda_n, x) = \frac{(q^{2+4n+\alpha+\beta+2}, q^{2-4n-(\alpha+\beta+2)}; q^4)_{\infty} (q^4; q^4)_n}{(-q^{2(\alpha+\beta+2)}, -q^{2(\alpha-\beta+1)}; q^4)_{\infty} (q^{4(\alpha+1)}; q^4)_n} P_n^{(\alpha, \beta)}(x | q^4).$$

If we put

$$\begin{aligned} \Lambda(\lambda) &= \Psi(\lambda, x)\Psi(\lambda, y) \\ (22) \quad & - \frac{(q^2 e^{4i\vartheta}, q^2 e^{-4i\vartheta}; q^4)_{\infty}}{(-q^{2(\alpha+\beta+2)}, -q^{2(\alpha-\beta+1)}; q^4)_{\infty}} \int_{-1}^1 \Psi(\lambda, z) K(x, y, z; \beta | q^4) dz, \end{aligned}$$

we obtain

$$\Lambda(\lambda_n) = 0, \quad n = 0, 1, 2, \dots$$

On the other hand

$$M(\cosh u/4, \Lambda) \leq K(q^2 e^u, q^2 e^{-u}, q^{2(\alpha+\beta+1)} e^u, q^{2(\alpha+\beta+1)} e^{-u}; q^4)_{\infty}^2.$$

Then according to Lemma 5, for $\alpha + \beta > 0$, we have

$$\Lambda = 0,$$

change q^4 by q , and put $\lambda = (q^{\mu+\alpha+\beta+1} + q^{-(\mu+\alpha+\beta+1)})/2$ in (23) to obtain

$$P_{\mu}^{(\alpha,\beta)}(x; q) P_{\mu}^{(\alpha,\beta)}(y; q) = \frac{(q^{\frac{1}{2}+\mu+\alpha+\beta+2}, q^{\frac{1}{2}-(\mu+\alpha+\beta+2)}; q)_{\infty}}{(-q^{(\alpha+\beta+2)/2}, -q^{(\alpha-\beta+1)/2}; q)_{\infty}} \\ \times \int_{-1}^1 K(x, y, z; q^{\alpha}, q^{\beta}) P_{\mu}^{(\alpha,\beta)}(z; q) dz.$$

This completes the proof. $\square$

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