COMPOSITION OPERATORS ON SPACES OF ENTIRE FUNCTIONS

GERARDO A. CHACÓN, GERARDO R. CHACÓN, AND JOSÉ GIMÉNEZ

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ABSTRACT. In this paper we study composition operators on spaces of entire functions. We determine which entire functions induce bounded composition operators on the Paley-Wiener space, L^2_π , and on the $E^2(\gamma)$ spaces. In addition, we characterize compact composition operators on these spaces. We also study the cyclic properties of composition operators acting on L^2_π .

1. INTRODUCTION AND PRELIMINARIES

Let $\text{Hol}(D)$ denote the space of all analytic functions in a domain $D \subset \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets of D , and let $\mathcal{H}$ be a linear subspace of $\text{Hol}(D)$. If φ is an analytic self-map of D such that $f \circ \varphi$ belongs to $\mathcal{H}$ for all $f \in \mathcal{H}$, then φ induces a linear operator $C_\varphi : \mathcal{H} \rightarrow \mathcal{H}$ defined as $C_\varphi(f) := f \circ \varphi$.

C_φ is called the *composition operator* with symbol φ .

A Hilbert subspace $\mathfrak{H}$ of $\text{Hol}(D)$ is said to be a *functional Hilbert space* if for all $w \in D$, the evaluation functional:

$$\delta_w : \mathfrak{H} \rightarrow \mathbb{C}; \quad f \mapsto f(w),$$

is continuous. In this case, as a consequence of the Riesz representation theorem, for each $w \in D$, there exists a function $k_w \in \mathfrak{H}$ such that

$$f(w) = \langle f, k_w \rangle, \quad f \in \mathfrak{H}.$$

Each function k_w ($w \in D$) is known as a *reproducing kernel* of $\mathfrak{H}$.

A straightforward application of the closed graph theorem shows that a holomorphic function $\varphi : D \rightarrow D$ induces a continuous operator $C_\varphi : \mathfrak{H} \rightarrow \mathfrak{H}$ if and only if $f \circ \varphi \in \mathfrak{H}$ for each $f \in \mathfrak{H}$.

In this paper we will study bounded composition operators acting on certain functional Hilbert spaces of entire functions. In a recent paper ([6]), composition operators acting on the Fock space of entire functions were studied.

We refer the reader to [3] or [13] for the background on entire functions that we will use here. As usual, given an entire function f , we define $M_f(r) := \max_{|z|=r} |f(z)|$. We recall that an entire function f is said to be of *finite order* if the inequality $M_f(r) < \exp(r^k)$ holds for sufficiently large values of r , for some

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$k > 0$. The *order* of an entire function f of finite order is the greatest lower bound of those values of k for which this asymptotic inequality is satisfied.

An entire function f , of order ρ , is said to be of *finite type* if for some $A > 0$ the inequality $M_f(r) < e^{Ar^\rho}$ holds for sufficiently large values of r . The greatest lower bound of those values of A for which this asymptotic inequality is satisfied is called the *type* of the function f . Following [3], we will say that an entire function f is of *exponential type* σ if it is of order $\rho \leq 1$ and type $\sigma \in (0, \infty)$.

Another interesting issue in this context is the study of the cyclic properties of composition operators. See, for example [5] and [10]. A bounded operator T acting on the Hilbert space $\mathfrak{H}$ is called *cyclic* if there is a vector $x \in \mathfrak{H}$ whose orbit under T , $\text{Orb}(T, x) := \{T^n x : n \in \mathbb{N}\}$, has dense linear span. If the set of all scalar multiples of $\text{Orb}(T, x)$ is dense in $\mathfrak{H}$, then T is called *supercyclic*, and if the $\text{Orb}(T, x)$ itself is dense in $\mathfrak{H}$, then T is called *hypercyclic*. As usual, x is called a cyclic (resp. supercyclic, hypercyclic) vector for T .

2. COMPOSITION OPERATORS ON THE PALEY-WIENER SPACE

2.1. The Paley-Wiener space. The *Paley-Wiener space* L_π^2 is the space of those entire functions of exponential type less than or equal to π whose restriction to the real line belongs to the space $L^2(-\infty, \infty)$ (cf. [3, 13]). L_π^2 with the norm given by

$$\|f\|_{L^2(-\infty, \infty)}^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx$$

is a closed subspace of $L^2(-\infty, \infty)$ and thus, it is a Hilbert space. In fact, L_π^2 is a functional Hilbert space. Its reproducing kernels are given by (cf. [13])

$$k_w(z) = \text{sinc}(z - \overline{w}) := \frac{\sin \pi(z - \overline{w})}{\pi(z - \overline{w})} \quad (w \in \mathbb{C})$$

and the set $\{k_n\}_{n \in \mathbb{Z}}$ is an orthonormal basis in L_π^2 . Therefore by Parseval's identity,

$$\|f\|_{L^2(-\infty, \infty)}^2 = \sum_{n \in \mathbb{Z}} |f(n)|^2.$$

As a consequence of the well-known Paley-Wiener theorem, the Unitary Fourier transform (the usual Fourier transform normalized) $\hat{f} := \mathfrak{F}(f)$ of a function $f \in L_\pi^2$ is supported in $[-\pi, \pi]$. Each function $f \in L_\pi^2$ admits the representation

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(t) e^{itz} dt, \quad \hat{f} \in L^2(-\pi, \pi),$$

where $L^2(-\pi, \pi)$ denotes the closed subspace of $L^2(-\infty, \infty)$ consisting of those functions which vanish a.e. outside of $(-\pi, \pi)$. Then,

$$L_\pi^2 = \mathfrak{F}^{-1}(L^2(-\pi, \pi)).$$

In fact, Parseval's identity

$$\|f\|_{L^2(-\infty, \infty)} = \|\hat{f}\|_{L^2(-\pi, \pi)}$$

shows that the Unitary Fourier transform $\mathfrak{F} : L_\pi^2 \rightarrow L^2(-\pi, \pi)$ is an isometric isomorphism.

2.2. Bounded operators and compact operators on L^2_π . As it was pointed out in [1], composition operators on the Paley-Wiener space appear also in the field of *signal processing*. A function f in $L^2(-\infty, \infty)$ is considered as a *signal* (with “finite energy”) and belongs to the Paley-Wiener space if its frequency domain (the domain of its Unitary Fourier transform) is limited to the band $[-\pi, \pi]$.

In this setting the symbol φ is called a *warping function* and the operator $f \mapsto f \circ \varphi$ is a *warping operator*. In [1] (cf. also [2], a shorter version published) the authors considered the case $\varphi : \mathbb{R} \rightarrow \mathbb{R}$. We will study the case in which φ is an entire function. In the first place, we will consider the problem of characterizing the symbols φ such that C_φ acts as a bounded operator on L^2_π .

In order to characterize the bounded composition operators on the Paley-Wiener space we will use the following results proved in [15] (see also [1, 11]).

Theorem 2.1. *Let f and φ be entire functions with $\varphi(0) = 0$. Let $F = f \circ \varphi$. Then there is a constant $c \in (0, 1)$ such that*

$$M_F(r) \geq M_f(cM_\varphi(r/2)).$$

Theorem 2.2 (Pólya’s Theorem). *Let f and φ be entire functions such that $F = f \circ \varphi$ is of finite order. Then either*

- (1) φ is a polynomial and f is of finite order, or
- (2) φ is not a polynomial (but a function of finite order) and f is of order 0.

Pólya’s theorem implies that if a warping function $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ maps every bandlimited functions to bandlimited functions, then φ is affine (cf. [1, 2]). In the proof of the following lemma we use the ideas given in [1] and [2].

Lemma 2.3. *Let $\varphi : \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic map (no null). If the operator C_φ maps L^2_π into itself, then φ is an affine map.*

Proof. Suppose that C_φ maps L^2_π into itself. Then the function $\text{sinc} \circ \varphi$ is in L^2_π and since sinc has order exactly one, Theorem 2.2 implies that φ is a polynomial. Let n denotes the degree of φ . We are going to show that $n = 1$.

Without loss of generality we may assume that $\varphi(0) = 0$. Then there is a positive constant a such that $M_\varphi(r) \geq ar^n$ for r large, and by Theorem 2.1 there exists a constant c , $0 < c < 1$, such that

$$M_{f \circ \varphi}(r) \geq M_f(acr^n/2^n),$$

for each function of order one in L^2_π . Let $0 < b < 1$. If the order of f is one, then there are arbitrarily large values of R for which the inequality $M_f(R) \geq \exp R^b$ holds. Thus, there are arbitrarily large values of r such that

$$M_f(acr^n/2) \geq \exp(a^b cr^{nb}/2^{nb}).$$

If $f \circ \varphi$ is of order $\rho \leq 1$, then there exist constants A, B such that

$$M_{f \circ \varphi}(r) \leq A \exp(Br),$$

for all r . Thus, there are arbitrarily large values of r such that $\exp(a^b cr^{nb}/2^{nb}) \leq A \exp(Br)$. It follows that $nb \leq 1$. Since b is any positive number less than one, we must have $n = 1$ ($n > 0$ because a constant function cannot be a symbol). $\square$

Theorem 2.4. *Let φ be a nonconstant entire function. The operator C_φ is bounded on L^2_π if and only if $\varphi(z) = az + b$ ($z \in \mathbb{C}$) with $0 < |a| \leq 1$, and $a \in \mathbb{R}$.*

Proof. It is easy to see that the order and type of entire functions are preserved by translations. The Plancherel-Pólya theorem (cf. [13, Section 7.4]) shows that

$$\int |f(x + s + it)|^2 dx = \int |f(x + it)|^2 dx \leq \|f\|_{L^2(-\infty, \infty)}^2 e^{2\pi|t|}.$$

Therefore, the space of entire functions of exponential type $\leq \pi$ that belong to $L^2(-\infty, \infty)$ is invariant under translations.

Now,

$$\int |f(ax)|^2 dx = (1/|a|) \int |f(x)|^2 dx \quad (a \in \mathbb{R}).$$

On the other hand, if the order of $f(z)$ is ρ , then the order of $f(az)$ is also ρ , while if the type of $f(z)$ is σ , then the type of $f(az)$ is $|a|^\rho \sigma$.

This shows, in addition, that if $C_\varphi(L_\pi^2) \subset L_\pi^2$, then $\varphi(z) = az + b$ with $0 < |a| \leq 1$. In order to see that $a \in \mathbb{R}$ it suffices to observe that if $\psi(z) = iz$, then the function $f(z) = \operatorname{sinc} z$ belongs to L_π^2 but $f \circ \psi \notin L_\pi^2$. $\square$

Corollary 2.5. *No bounded composition operator on L_π^2 is compact.*

Proof. Clearly, the operator C_φ is compact in L_π^2 if and only if the operator $C_{\varphi-\varphi(0)}$ is compact.

Since $f(n) \rightarrow 0$ when $n \rightarrow \pm\infty$ for all $f \in L_\pi^2$, the sequence of orthonormal vectors $\{k_n\}_{n=0}^\infty$ converges weakly to zero. However, an easy computation shows that $\|C_{\varphi-\varphi(0)}(k_n)\| = 1/\sqrt{|a|}$ for all n . Consequently, C_φ cannot be compact. $\square$

2.3. The adjoint of a composition operator on L_π^2 . Recall that by the Paley-Wiener Theorem, L_π^2 and $L^2(-\pi, \pi)$ are isometrically isomorphic via $\mathfrak{F}$. If $f \in L_\pi^2$ we have:

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(t) e^{itz} dt,$$

with $\hat{f} \in L^2(-\pi, \pi)$. Thus, if $\varphi(z) = az + b$, $b = \lambda + i\eta$, $a \in \mathbb{R}$, $0 < a \leq 1$ (for the sake of simplicity we may assume that $a > 0$), then we have

$$\begin{aligned} C_\varphi(f)(z) &= f(az + \lambda + i\eta) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(t) e^{itaz} e^{i\lambda t} e^{-\eta t} dt \\ &= \frac{1}{a\sqrt{2\pi}} \int_{-a\pi}^{a\pi} \hat{f}\left(\frac{x}{a}\right) e^{i\lambda(\frac{x}{a})} e^{-\eta(\frac{x}{a})} e^{ixz} dx. \end{aligned}$$

Therefore, the composition operator C_φ corresponds, via $\mathfrak{F}$, to the operator $\hat{C}_\varphi : L^2(-\pi, \pi) \rightarrow L^2(-\pi, \pi)$ defined as

$$(2.1) \quad (\hat{C}_\varphi g)(x) := \frac{1}{a} g\left(\frac{x}{a}\right) \chi_{(-a\pi, a\pi)}(x) e^{i\lambda(\frac{x}{a})} e^{-\eta(\frac{x}{a})}.$$

Now let $f, g \in L^2(-\pi, \pi)$. Then

$$\begin{aligned} \langle \hat{C}_\varphi g, f \rangle &= \frac{1}{2a\pi} \int_{-a\pi}^{a\pi} g\left(\frac{x}{a}\right) e^{i\lambda\frac{x}{a}} e^{-\eta\frac{x}{a}} \overline{f(x)} dx \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) e^{i\lambda t} e^{-\eta t} \overline{f(at)} dt \\ &= \langle g, \hat{C}_\varphi^* f \rangle. \end{aligned}$$

Consequently, the operator C_φ^* corresponds, via $\mathfrak{F}$, to the operator

$$(2.2) \quad (\hat{C}_\varphi^* f)(x) = e^{-i\lambda x} e^{-\eta x} f(ax), \quad f \in L^2(-\pi, \pi).$$

Since $\mathfrak{F}$ is an isometry, we can compute C_φ^* acting on the Paley-Wiener space as follows: Let $f \in L_\pi^2$ and $\hat{f} \in L^2(-\pi, \pi)$ be its respective Unitary Fourier transform. Then since $0 < a \leq 1$, we have $\hat{C}_\varphi^* \hat{f} \in L^2(-\pi, \pi)$ and $\mathfrak{F}^{-1}(\hat{C}_\varphi^* \hat{f}) \in L_\pi^2$. Now,

$$\begin{aligned} (\mathfrak{F}^{-1}(\hat{C}_\varphi^* \hat{f}))(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-i\lambda x} e^{-\eta x} \hat{f}(ax) e^{izx} dx \\ &= \frac{1}{a\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(t) e^{-i\lambda \frac{t}{a}} e^{-\eta \frac{t}{a}} e^{iz \frac{t}{a}} dt \\ &= \frac{1}{a\sqrt{2\pi}} \int_{-\pi}^{\pi} \hat{f}(t) e^{it(\frac{z-\lambda+i\eta}{a})} dt = \frac{1}{a} f\left(\frac{z-\lambda+i\eta}{a}\right). \end{aligned}$$

Therefore,

$$(2.3) \quad (C_\varphi^* f)(z) = \frac{1}{a} f\left(\frac{z-\lambda+i\eta}{a}\right), \quad f \in L_\pi^2.$$

As a consequence of equation (2.3) we have the following.

Proposition 2.6. *A bounded composition operator C_φ on L_π^2 is normal if and only if $\varphi'(0) = 1$.*

Proof. Let $\varphi(z) = az + b$. Using (2.3) it is straightforward to check that $C_\varphi C_\varphi^* - C_\varphi^* C_\varphi = 0$ if and only if $b = 0$. $\square$

2.4. Cyclic behavior of composition operators on L_π^2 . We shall now use equation (2.1) in order to prove the following result.

Theorem 2.7. *No bounded composition operator on L_π^2 is supercyclic.*

Proof. Let $\varphi(z) = az + (\lambda + i\eta)$ and suppose that C_φ is supercyclic. For the sake of simplicity we will assume $a > 0$ and $\eta > 0$ (in the general case, the modifications needed are straightforward). Then there exists a function $g \in L^2(-\pi, \pi)$, a sequence $\{\alpha_k\}$ in $\mathbb{C}$, and a sequence $\{n_k\}$ in $\mathbb{N}$ such that $\{\alpha_k \hat{C}_\varphi^{n_k}(g)\}$ converges in $L^2(-\pi, \pi)$ to the function $f \equiv 1$. By taking a subsequence, if necessary, we may assume that $\alpha_k \hat{C}_\varphi^{n_k}(g)(x) \rightarrow 1$ a.e., but it is easy to see, using (2.1) that $\lim_{k \rightarrow \infty} \alpha_k \hat{C}_\varphi^{n_k}(g)(x) = 0$ if $a < 1$.

If $a = 1$, then $C_\varphi(g)(x) = g(x)e^{i\lambda x}e^{-\eta x}$. That is, C_φ is the multiplication operator $M_h : g \mapsto hg$ on $L^2(-\pi, \pi)$ with $h(x) = e^{i\lambda x}e^{-\eta x}$. Now we are going to use the angle criterion for supercyclic vectors [10]. Let $g \in L^2(-\pi, \pi) \setminus \{0\}$. Then Bessel's inequality implies that for every $k \in \mathbb{Z}^+$

$$|\langle \frac{\chi_{(-\pi, 0)}}{\sqrt{\pi}}, e^{-k\eta(\cdot)} |g| \rangle|^2 + |\langle \frac{\chi_{(0, \pi)}}{\sqrt{\pi}}, e^{-k\eta(\cdot)} |g| \rangle|^2 \leq \|e^{-k\eta(\cdot)} g\|^2,$$

and therefore there exists a constant $\varepsilon > 0$, depending on g , such that for every $k \in \mathbb{Z}^+$,

$$\sqrt{\pi} \left(\int_{-\pi}^{\pi} e^{-2k\eta x} |g(x)|^2 dx \right)^{\frac{1}{2}} > \int_0^{\pi} e^{-k\eta x} |g(x)| dx + \varepsilon.$$

Thus

$$\sup_k \frac{\int_0^\pi e^{-k\eta x} |g(x)| dx}{\sqrt{\pi} \left(\int_{-\pi}^\pi e^{-2k\eta x} |g(x)|^2 dx \right)^{\frac{1}{2}}} < \sup_k \frac{\int_0^\pi e^{-k\eta x} |g(x)| dx + \varepsilon}{\sqrt{\pi} \left(\int_{-\pi}^\pi e^{-2k\eta x} |g(x)|^2 dx \right)^{\frac{1}{2}}} \leq 1.$$

Therefore,

$$\sup_k \frac{|\langle M_h^k g, \chi_{(0,\pi)} \rangle|}{\|M_h^k g\| \|\chi_{(0,\pi)}\|} < 1.$$

Consequently, g cannot be a supercyclic vector for M_h . $\square$

2.5. Composition operators on L_π^p . Let us now consider the general Paley-Wiener spaces L_π^p , $1 \leq p < \infty$, i.e., L_π^p is the space of those entire functions of exponential type $\leq \pi$ whose restrictions to the real line belongs to the space $L^p(-\infty, \infty)$.

In a standard way (cf. [6]) the boundedness and compactness of C_φ is closely related to the properties of a certain Carleson-type measure associated with φ . However, a careful revision of the proof of Theorem 2.4 will show to the reader that the result can be easily generalized to the setting of L_π^p spaces. The next two results involving Carleson measures have interest by themselves.

In what follows, we will assume that $1 \leq p \leq q < \infty$.

We shall say that a measure μ on $\mathbb{R}$ is a (p, q) -Carleson measure (for the Paley-Wiener space) if the space L_π^p is boundedly contained in $L^q(\mu)$, that is, if the inclusion map $i : L_\pi^p \rightarrow L^q(\mu)$ is bounded. If the map i is compact, we will say that μ is a (p, q) -vanishing Carleson measure (for the Paley-Wiener space).

For a given entire function φ , the weighted pullback measure μ_φ on $\mathbb{R}$ is given by

$$\mu_\varphi(E) := l(\varphi^{-1}(E)),$$

for every Borel subset E of $\mathbb{R}$, where l denotes the arc-length measure. Since one can easily check that $\|C_\varphi(f)\|_{L^q(-\infty, \infty)} = \|f\|_{L^q(\mu_\varphi)}$ for every $f \in L_\pi^p$, we have that if the operator $C_\varphi : L_\pi^p \rightarrow L_\pi^q$ is bounded, then μ_φ is a (p, q) -Carleson measure. Furthermore, if $1 < p$, then C_φ is compact if and only if μ_φ is a (p, q) -vanishing Carleson measure.

In this section, we shall characterize both Carleson and vanishing Carleson measures on Paley-Wiener spaces. In [14], a similar result is mentioned.

Now, we list some well-known facts about Paley-Wiener spaces L_π^p ($1 \leq p < \infty$) that we will require (cf. [13, 20.1]).

Remark 2.8. (i) As a consequence of the Plancherel-Pólya theorem and since $|f|^p$ is a subharmonic function, then for any $x, y \in \mathbb{R}$ we obtain

$$|f(x + iy)|^p \leq \frac{2}{\pi} e^{\pi(|y|+1)} \|f\|_{L^p(-\infty, \infty)}^p.$$

In particular, there is a constant K (depending on p , but neither on x nor on f) such that

$$|f(x)|^p \leq K \|f\|_{L^p(-\infty, \infty)}^p.$$

(ii) Let $\{\lambda_n\} \subset \mathbb{C}$ be a sequence and let H, δ be positive numbers such that

$$|\operatorname{Im} \lambda_n| \leq H < \infty, \quad |\lambda_n - \lambda_m| \geq \delta \quad \text{for } n \neq m.$$

Then, for each $f \in L_\pi^p$,

$$\sum_n |f(\lambda_n)|^p \leq C \|f\|_{L^p(-\infty, \infty)}^p,$$

where the constant C depends on δ and H , but not on f .

- (iii) If $1 < p < \infty$, then for any sequence $\{c_k\}$ in $l^p(\mathbb{Z})$ there exists a unique solution in L_π^p for the interpolation problem $f(k) = c_k$, $k \in \mathbb{Z}$. In addition, the norms of f and the corresponding $\{c_k\}$ are comparable, i.e. there exist constants m, M such that

$$m \left(\sum_{k=-\infty}^{\infty} |f(k)|^p \right)^{1/p} \leq \|f\|_{L^p(-\infty, \infty)} \leq M \left(\sum_{k=-\infty}^{\infty} |f(k)|^p \right)^{1/p},$$

for all functions $f \in L_\pi^p$. In particular, this last inequality implies that if $1 \leq p < q < \infty$, then $L_\pi^p \subset L_\pi^q$.

Actually, we are only going to need the result for a simpler problem: given $x_0 \in \mathbb{R}$ we want to find a function $f \in L_\pi^p$ such that $f(x_0) = 1$. This is obtained by considering the sequence $c_k = 0$ if $k \neq 0$ and $c_0 = 1$ and translating the function with $f(k) = c_k$ by x_0 .

- (iv) *Bernstein Inequality* (cf. [3, 11.1.2]). If f is an entire function of exponential type τ and $|f(x)| \leq M$ for all $x \in \mathbb{R}$, then $|f'(x)| \leq M\tau$.

$D(x, r)$ will denote the usual open ball in $\mathbb{R}$ with center x and radius r .

Theorem 2.9. *Let $1 \leq p \leq q < \infty$. The following statements are equivalent:*

- (a) *The measure μ is a (p, q) -Carleson measure.*
- (b) *For each $r > 0$ there exists a constant C such that $\mu(D(x, r)) \leq C$ for all $x \in \mathbb{R}$.*
- (c) *There exist $r > 0$ and a constant C such that $\mu(D(x, r)) \leq C$ for all $x \in \mathbb{R}$.*

Proof. Assertions (b) and (c) are clearly equivalent.

(c) $\Rightarrow$ (a): Let $C > 0$ be such that $\mu(D(x, 1)) < C$ for each $x \in \mathbb{R}$ and let $x_n \in [n, n+1]$ be such that $|f(x_n)| = \max_{[n, n+1]} |f(x)|$. Then, we decompose the sequence $\{x_n\}$ in two subsequences: $\{\lambda_n\}$ and $\{\lambda'_n\}$, with $|\lambda_n - \lambda_m| \geq 1/2$ and $|\lambda'_n - \lambda'_m| \geq 1/2$ if $n \neq m$. Now,

$$\begin{aligned} \left(\int |f|^q d\mu \right)^{\frac{1}{q}} &= \left(\sum_{n=-\infty}^{\infty} \int_n^{n+1} |f|^q d\mu \right)^{\frac{1}{q}} \leq \left(\sum_{n=-\infty}^{\infty} |f(x_n)|^q \mu([n, n+1]) \right)^{\frac{1}{q}} \\ &\leq C^{\frac{1}{q}} \left(\sum_{n=-\infty}^{\infty} |f(x_n)|^q \right)^{\frac{1}{q}} \leq C^{\frac{1}{q}} \left(\sum_{n=-\infty}^{\infty} |f(x_n)|^p \right)^{\frac{1}{p}} \\ &= C^{\frac{1}{q}} \left(\sum_{n=-\infty}^{\infty} |f(\lambda_k)|^p + \sum_{n=-\infty}^{\infty} |f(\lambda'_k)|^p \right)^{\frac{1}{p}} \\ &\leq C^{\frac{1}{q}} (2K)^{\frac{1}{p}} \|f\|_{L^p(-\infty, \infty)}, \end{aligned}$$

where the constant K is as in Remark 2.8(ii).

(a) $\Rightarrow$ (c): Let K and M be constants such that $|f(x)| \leq K \|f\|_{L^p(-\infty, \infty)} \leq KM \left(\sum_{k=-\infty}^{\infty} |f(k)|^p \right)^{1/p}$, for all $x \in \mathbb{R}$, $f \in L_\pi^p$ (cf. Remark 2.8(i,iii)). Let $r > 0$

be such that $\pi KMr < 1$. If (c) is false, then for each $n \in \mathbb{N}$ there exists x_n in $\mathbb{R}$ such that $\mu(D(x_n, r)) > n$. But, by (ii) above, for each $n \in \mathbb{N}$, there exists $f_n \in L^p_\pi$ such that $f_n(x_n) = 1$ and $\|f_n\|_{L^p(-\infty, \infty)} \leq M$ (cf. Remark 2.8(iii)). Bernstein's Inequality (Remark 2.8 (iv)) then implies that $|f'_n(x)| \leq \pi KM$ for all $x \in \mathbb{R}, n \in \mathbb{N}$, and an application of the Mean Value Theorem gives:

$$1 - |f_n(x)| \leq |1 - f_n(x)| = |f_n(x_n) - f_n(x)| = |f'_n(\xi)| |x - x_n| \leq \pi KMr,$$

for all $x \in D(x_n, r)$ and some real number ξ in this ball.

Finally, we have

$$\int |f_n|^q d\mu \geq \int_{D(x_n, r)} |f_n|^q d\mu \geq (1 - \pi KMr)^q n,$$

which contradicts the fact that μ is a Carleson measure and hence there must exist a constant $C > 0$ such that $\|f_n\|_{L^q(\mu)} \leq C \|f_n\|_{L^p(-\infty, \infty)} \leq CM$. $\square$

For vanishing Carleson measures, we have the following.

Theorem 2.10. *Let $1 < p \leq q < \infty$. The following statements are equivalent:*

- (a) *The measure μ is a (p, q) -vanishing Carleson measure.*
- (b) *For each $r > 0$, $\mu(D(x, r)) \rightarrow 0$ as $x \rightarrow \infty$.*
- (c) *There exists $r > 0$ such that $\mu(D(x, r)) \rightarrow 0$ as $x \rightarrow \infty$.*

Proof. Again, assertions (b) and (c) are clearly equivalent.

(c) $\Rightarrow$ (a): Let $\{f_n\}$ be a sequence in L^p_π weakly convergent to zero; this is, the sequence $\{\|f_n\|_{L^p(-\infty, \infty)}\}$ is bounded (without loss of generality we may assume that it is bounded by 1), and f_n converges uniformly on compact sets to zero. We shall see that $\int |f_n|^q d\mu \rightarrow 0$.

As in the proof of Theorem 2.9, for each n we choose the sequence $\{x_{n,k}\}_k$ as follows: $x_{n,k}$ will be a number in $[k, k+1]$ where $|f_n|$ reaches its greatest value. Then, we decompose this sequence in two subsequences $\{\lambda_{n,k}\}$ and $\{\lambda'_{n,k}\}$ so that $|\lambda_{n,k} - \lambda_{n,j}| \geq 1/2$ and $|\lambda'_{n,k} - \lambda'_{n,j}| \geq 1/2$ when $k \neq j$. By Remark 2.8(ii), there exists a constant C such that $\sum_k |f_n(\lambda_{n,k})|^p \leq C \|f_n\|_{L^p(-\infty, \infty)}^p$ (and $\sum_k |f_n(\lambda'_{n,k})|^p \leq C \|f_n\|_{L^p(-\infty, \infty)}^p$) for all n . Then for all $n \in \mathbb{N}$:

$$\left(\sum_{k \in \mathbb{Z}} |f_n(x_{n,k})|^q \right)^{\frac{1}{q}} \leq \left(\sum_{k \in \mathbb{Z}} |f_n(x_{n,k})|^p \right)^{\frac{1}{p}} \leq (2C)^{\frac{1}{p}} \|f_n\|_{L^p(-\infty, \infty)} \leq (2C)^{\frac{1}{p}}.$$

Given $\varepsilon > 0$, we take $N \in \mathbb{N}$ such that $\mu(D(x, 1)) < \frac{\varepsilon^q}{(4C)^{\frac{q}{p}}}$ if $|x| \geq N$, and if n is large enough such that $|f_n(x)|^q \leq \frac{\varepsilon}{2\mu([-N, N])}$ for all x in the compact set $[-N, N]$, then

$$\int_{-N}^N |f_n|^q d\mu \leq \frac{\varepsilon}{2}.$$

Hence

$$\begin{aligned} & \left(\int_{|x|>N} |f_n|^q d\mu \right)^{\frac{1}{q}} \\ & \leq \left(\sum_{k=N}^{\infty} |f_n(x_{n,k})|^q \mu([k, k+1]) + \sum_{k=N}^{\infty} |f_n(x_{n,-k-1})|^q \mu([-k-1, -k]) \right)^{\frac{1}{q}} \\ & \leq \left(\frac{\varepsilon}{8C} \sum_{k=N}^{\infty} |f(x_{n,k})|^q + \frac{\varepsilon}{8C} \sum_{k=N}^{\infty} |f(x_{n,-k-1})|^q \right)^{\frac{1}{q}} \leq \varepsilon/2, \end{aligned}$$

thus, $\int |f_n|^q d\mu < \varepsilon$ for such n .

(a) $\Rightarrow$ (c): As in the proof of (a) $\Rightarrow$ (c) in Theorem 2.9 (with the notation of that proof) we take $r > 0$ such that $\pi KMr < 1$. If (c) is not true, then there exists $\varepsilon > 0$ and a sequence $\{x_n\} \subset \mathbb{R}$ such that $x_n \rightarrow \infty$ and $\mu(D(x_n, r)) \geq \varepsilon$. Of course, we can assume that these discs are disjoint. We now take $f_n \in L^p_\pi$ with $f_n(x_n) = 1$ and $\|f_n\|_{L^p(-\infty, \infty)}^p \leq M$.

Since the inclusion operator $i : L^p_\pi \rightarrow L^q(\mu)$ is compact, we can assume (by taking a subsequence if necessary) that $f_n \rightarrow f \in L^q(\mu)$.

Again, as in the proof of Theorem 2.9, by using Bernstein Inequality and the mean value theorem, we see that $|f_n(x)| \geq 1 - \pi KMr$ in the disc $D(x_n, r)$ for each n , and then

$$\int |f_n|^q d\mu \geq \int_{D(x_n, r)} |f_n|^q d\mu \geq (1 - \pi KMr)^q \varepsilon.$$

Now, by observing that

$$\begin{aligned} \|f_n - f\|_{L^q(\mu)} & \geq \left(\int_{D(x_n, r)} |f_n - f|^q d\mu \right)^{1/q} \\ & \geq \left(\int_{D(x_n, r)} |f_n|^q d\mu \right)^{1/q} - \left(\int_{D(x_n, r)} |f|^q d\mu \right)^{1/q} \\ & \geq (1 - \pi KMr) \varepsilon^{1/p} - \left(\int_{D(x_n, r)} |f|^q d\mu \right)^{1/q}, \end{aligned}$$

we obtain $\left(\int_{D(x_n, r)} |f|^q d\mu \right)^{1/q} \geq (1 - \pi KMr) \varepsilon^{1/p} - \|f_n - f\|_{L^q(\mu)}$, and since $\|f_n - f\|_{L^q(\mu)} \rightarrow 0$, then there exists $n \geq N$ such that

$$\int_{D(x_n, r)} |f|^q d\mu \geq \left(\frac{(1 - \pi KMr) \varepsilon^{1/p}}{2} \right)^q.$$

But

$$\int |f|^q \geq \sum_{n \geq N} \int_{D(x_n, r)} |f|^q d\mu,$$

which is a contradiction because $f \in L^q(\mu)$. $\square$

An important consequence of these characterizations is that the Carleson (vanishing Carleson) condition is actually independent of p and q , $1 \leq p \leq q < \infty$. From this last result and from section 2.2 we may conclude the following.

Theorem 2.11. *Let $1 < p \leq q < \infty$ and suppose that C_φ acts from L_π^p to L_π^q . Then*

- (1) C_φ is bounded if and only if $\varphi(z) = az + b$, with $0 < |a| \leq 1, a \in \mathbb{R}$.
- (2) C_φ is not compact.

3. COMPOSITION OPERATORS ON THE SPACES $E^2(\gamma)$

3.1. Spaces $E^2(\gamma)$. According to Chan and Shapiro [7], an entire function $\gamma(z) = \sum \gamma_n z^n$ is called a *comparison function* if $\gamma_n > 0$ for all n and if the sequence of ratios γ_{n+1}/γ_n decreases to zero as $n \rightarrow \infty$. In case that the sequence $(n+1)\gamma_{n+1}/\gamma_n$ decreases monotonically to $\tau \geq 0$, then γ is said to be an *admissible comparison function*.

For each comparison function γ , we define $E^2(\gamma)$ as the Hilbert space of entire functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$, such that

$$\|f\|_{2,\gamma}^2 := \sum_{n=0}^{\infty} \gamma_n^{-2} |a_n|^2 < \infty.$$

The functions $e_n(z) := \gamma_n z^n$, $n = 0, 1, 2, \dots$, form an orthonormal basis for $E^2(\gamma)$. Moreover, if $f(z) = \sum a_n z^n \in E^2(\gamma)$, then the following inequality [7, Prop. 1.4] shows that the spaces $E^2(\gamma)$ are functional Hilbert spaces:

$$\begin{aligned} |f(z)| &\leq \sum_{n=0}^{\infty} |a_n| |z|^n = \sum_{n=0}^{\infty} \frac{|a_n|}{\gamma_n} \gamma_n |z|^n \\ &\leq \|f\|_{2,\gamma} \left(\sum_{n=0}^{\infty} \gamma_n^2 |z|^{2n} \right)^{1/2} \leq \|f\|_{2,\gamma} \gamma(|z|). \end{aligned}$$

In fact, the reproducing kernels of $E^2(\gamma)$ are given by

$$K_w(z) = \hat{\gamma}(\overline{w}z),$$

where $\hat{\gamma}(z) := \sum \gamma_n^2 z^n$.

From [7, Prop. 1.3], it follows that the order and type of comparison functions γ affects the behavior of the functions in the corresponding Hilbert space $E^2(\gamma)$. Actually, every element of $E^2(\gamma)$ has order and type no more than that of γ .

On the other hand, in [7, Prop. 1.3], it is proved that if $(n+1)\gamma_{n+1}/\gamma_n \downarrow \tau$, $\tau > 0$, then γ is of order one and type τ . The case in which $\tau = 0$ is more delicate. For example, if we take $\gamma_1(z) := 1 + \sum_{n=1}^{\infty} \left(\frac{e}{2n}\right)^{2n} z^n$, then γ_1 is clearly an admissible comparison function with $\tau = 0$, order $1/2$ and type 1 . However, the function $\gamma_2(z) := \sum_{n=0}^{\infty} e^{-n^2} z^n$ is also an admissible comparison function with $\tau = 0$ but of order zero; see [13, Chapter 1]. This shows that the order and type of an admissible comparison function γ is not determined by τ in this case.

3.2. Bounded composition operators on $E^2(\gamma)$. In [7], Chan and Shapiro showed that if the sequence $\{(n+1)\gamma_{n+1}/\gamma_n\}$ is bounded, then each translation operator is bounded on $E^2(\gamma)$. In this section, we will consider the problem of characterizing the symbols that induce bounded composition operators and compact composition operators on $E^2(\gamma)$ when $\tau > 0$.

First, we need to observe that for each $\sigma < \tau$, the function $f_\sigma(z) := e^{\sigma z}$ belongs to $E^2(\gamma)$. This follows from the fact that, for each n , the inequality $(n+1)\gamma_{n+1}/\gamma_n \geq \tau$

implies that $\gamma_n \geq \tau^n \gamma_0 / n!$ and thus

$$\|f_\sigma\|_{2,\gamma}^2 = \sum_{n=0}^{\infty} \frac{1}{\gamma_n^2} \frac{\sigma^{2n}}{(n!)^2} \leq \frac{1}{\gamma_0^2} \sum_{n=0}^{\infty} \frac{(n!)^2}{\tau^{2n}} \frac{\sigma^{2n}}{(n!)^2} = \frac{1}{\gamma_0^2} \sum_{n=0}^{\infty} \left(\frac{\sigma}{\tau}\right)^{2n} < \infty.$$

Theorem 3.1. *Let φ be an entire function. The operator C_φ is bounded on $E^2(\gamma)$ if and only if $\varphi(z) = az + b$ with $|a| \leq 1$.*

Proof. Suppose that $f \in E^2(\gamma)$ is of nonzero finite order. If $f \circ \varphi$ belongs to $E^2(\gamma)$, then it is of finite order, and by Pólya's Theorem (2.1), φ must be a polynomial. Let $\varphi(z) = a_n z^n + \dots + a_0$, $a_n \neq 0$, $n \geq 1$. If $\sigma < \tau$, then

$$f_\sigma \circ \varphi(z) = \exp(\sigma(a_n z^n + \dots + a_0))$$

belongs to $E^2(\gamma)$, which is a function of order n and type $\sigma|a_n|$. Thus $n = 1$ and $\sigma|a_n| \leq \tau$. Since σ is arbitrary, it has to be $|a_n| \leq 1$.

In order to see that the condition is sufficient, it suffices to show that the symbol $\varphi(z) = az$, $|a| \leq 1$, induces a bounded composition operator on $E^2(\gamma)$, since every translation operator is bounded in $E^2(\gamma)$ (cf. [7]). Indeed, if $f(z) = \sum c_n z^n$ belongs to $E^2(\gamma)$, then

$$\|C_\varphi f\|_{2,\gamma}^2 = \sum_{n=0}^{\infty} \gamma_n^{-2} |a|^{2n} |c_n|^2 \leq \sum_{n=0}^{\infty} \gamma_n^{-2} |c_n|^2 = \|f\|_{2,\gamma}^2.$$

□

3.3. Compact composition operators on $E^2(\gamma)$. In order to characterize the symbols that induce compact composition operators on $E^2(\gamma)$, we shall first show that this space is closely related to the space $\text{Hol}(\mathbb{C}) \cap L^2(\mathbb{C}, e^{-2\tau|z|} dA)$ where dA denotes the area measure on $\mathbb{C}$. The ideas and computations below follow those in [16, 6.8] where it was considered the space $E(e^z)$.

In general, $\mathfrak{H}_W$ is defined as the Hilbert space of entire functions such that

$$\|f\|_W^2 := \int_{\mathbb{C}} |f(z)|^2 W(|z|) dA(z) < \infty,$$

where W is a certain weight function.

Lemma 3.2. *Let $\gamma(z) = \sum \gamma_n z^n$ be an admissible comparison function such that $0 < \tau = \lim_{n \rightarrow \infty} (n+1)\gamma_{n+1}/\gamma_n$. The space $\mathfrak{H}_{W_\tau}$, where $W_\tau(|z|) := e^{-2\tau|z|}$, is contained in $E^2(\gamma)$, and there exists a constant K such that $\|f\|_{2,\gamma} \leq K\|f\|_{W_\tau}$ for all $f \in \mathfrak{H}_{W_\tau}$.*

On the other hand, the space $E^2(\gamma)$ is contained in the space $\mathfrak{H}_{W_t}$, where $W_t(|z|) := e^{-2t|z|}$, $t > \tau$. In this case there exists a constant C such that $\|f\|_{W_t} \leq C\|f\|_{2,\gamma}$ for all $f \in E^2(\gamma)$.

Proof. As pointed out before, $\lim_{n \rightarrow \infty} (n+1)\gamma_{n+1}/\gamma_n = \tau > 0$ implies that

$$(3.1) \quad \gamma_n \geq \tau^n \gamma_0 / n! \quad n = 0, 1, 2, \dots$$

Moreover, if $t > \tau$, then we can find a constant C (depending only on t) such that

$$(3.2) \quad \gamma_n < C \frac{t^n}{n!} \quad n = 0, 1, 2, \dots$$

Now, if $W_t(|z|) = e^{-2t|z|}$, $t \geq \tau$, and $f(z) = \sum a_n z^n$, then

$$\|f\|_{W_t}^2 = 2\pi \sum_{n=0}^{\infty} p_n |a_n|^2$$

where $p_n := \int_0^{\infty} r^{2n+1} W_t(r) dr = (2n+1)!(2t)^{-2(n+1)}$ and thus

$$\frac{p_n}{(n!)^2} = (2n+1) \left[\frac{(2n)!}{(n!)^2} \right] (2t)^{-2(n+1)}, \quad n = 0, 1, 2, \dots$$

The bracketed term is (by Stirling's formula) asymptotically a constant times $4^n/\sqrt{n}$, and therefore p_n is asymptotically a constant times $\sqrt{n}(n!)^2 t^{-2n}$.

If $t = \tau$, by using (3.1) we see that there is a constant C such that

$$\begin{aligned} \|f\|_{W_\tau}^2 &= C \sum_{n=0}^{\infty} \sqrt{n} \frac{(n!)^2}{\tau^{2n}} |a_n|^2 \geq C \sum_{n=0}^{\infty} \sqrt{n} \frac{1}{\gamma_n^2} |a_n|^2 \\ &\geq C \sum_{n=0}^{\infty} \frac{1}{\gamma_n^2} |a_n|^2 = C \|f\|_{2,\gamma}^2. \end{aligned}$$

On the other hand if $t > \tau$, we take $\tau < s < t$, and (3.2) gives:

$$\begin{aligned} \|f\|_{W_t} &= C \sum_{n=0}^{\infty} \sqrt{n} \frac{(n!)^2}{t^{2n}} |a_n|^2 = C \sum_{n=0}^{\infty} \frac{(n!)^2}{s^{2n}} \sqrt{n} \left(\frac{s}{t}\right)^{2n} |a_n|^2 \\ &\leq C \sum_{n=0}^{\infty} \frac{1}{\gamma_n^2} \sqrt{n} \left(\frac{s}{t}\right)^{2n} |a_n|^2 \leq C \sum_{n=0}^{\infty} \frac{1}{\gamma_n^2} |a_n|^2 = C \|f\|_{2,\gamma}^2. \quad \square \end{aligned}$$

The following lemma holds in any analytic functional Hilbert space (cf. [9]).

Lemma 3.3. *Let $\{f_n\}$ be a sequence in $E^2(\gamma)$. Then $\{f_n\}$ converges weakly to zero if and only if $\{f_n\}$ converges to zero uniformly on compact subsets of $\mathbb{C}$.*

Theorem 3.4. *Let φ be an entire function and $\gamma = \sum \gamma_n z^n$ an admissible comparison function such that $\lim(n+1)\gamma_{n+1}/\gamma_n = \tau > 0$. Then, C_φ is compact in $E^2(\gamma)$ if and only if $\varphi(z) = az + b$ where $|a| < 1$.*

Proof. Recall that the sequence $\{e_n\}$, $e_n(z) = \gamma(n)z^n$ is an orthonormal basis of $E^2(\gamma)$, in particular, $\{e_n\}$ is weakly convergent to zero. If $\varphi(z) = az$ with $|a| = 1$, then

$$\|C_\varphi e_n\|_{2,\gamma}^2 = \|a^n e_n\|_{2,\gamma}^2 = 1,$$

and C_φ cannot be compact. It then follows that if $\varphi(z) = az + b$ ($|a| = 1$), then C_φ cannot be compact either.

Now, we shall prove that if $\varphi(z) = az$, $|a| < 1$, then C_φ has to be compact. Let $\{f_n\}$ be a sequence in $E^2(\gamma)$ weakly convergent to zero. Then by Lemma 3.3, $\{f_n\}$ is bounded by $M > 0$ and converges to zero uniformly on compact subsets of $\mathbb{C}$. We must prove that $\|f_n \circ \varphi\|_{2,\gamma}$ converges to zero.

By lemma 3.2, there exists a constant $c > 0$ such that

$$\|f\|_{2,\gamma}^2 \leq c \int |f(z)|^2 e^{-\tau|z|} dA(z).$$

Thus, it suffices to show that $\int |f_n(az)|^2 e^{-2\tau|z|} dA(z)$ converges to zero, or by changing variables that

$$\int |f_n(w)|^2 e^{-2\tau|w|/|a|} dA(w) \rightarrow 0.$$

Given $\varepsilon > 0$, we take $\tau < t < \tau/|a|$. Again by Lemma 3.2, there exists a constant $C > 0$ such that

$$\int |f(w)|^2 e^{-t|w|} dA(w) \leq C \|f\|_{2,\gamma}^2.$$

Now we may take a compact set $K \subset \mathbb{C}$ such that $e^{-2\tau|w|/|a|} e^{t|w|} < \varepsilon/2CM$ for each w out of K ; therefore,

$$\begin{aligned} \int_{\mathbb{C} \setminus K} |f_n(w)|^2 e^{-2\tau|w|/|a|} dA(z) &= \int_{\mathbb{C} \setminus K} |f_n(w)|^2 e^{-2\tau|w|/|a|} e^{2t|w|} e^{-2t|w|} dA(w) \\ &\leq \frac{\varepsilon}{2CM} \int_{\mathbb{C} \setminus K} |f_n(w)|^2 e^{-2t|w|/|a|} dA(z) \\ &\leq \frac{\varepsilon}{2CM} C \|f\|_{2,\gamma}^2 \leq \frac{\varepsilon}{2CM} CM = \frac{\varepsilon}{2}. \end{aligned}$$

Finally, we may take n large enough such that, for each $w \in K$,

$$|f_n(w)| \leq \varepsilon/2 \int e^{-2\tau|z|} dA(z).$$

Then

$$\int_K |f_n(w)|^2 e^{-2\tau|w|/|a|} dA(w) \leq \varepsilon/2,$$

and this completes the proof. $\square$

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NÚCLEO TÁCHIRA, UNIVERSIDAD DE LOS ANDES, MERIDA, VENEZUELA

E-mail address: `gchacon@cantv.net`, `gchacon@ula.ve`

FACULTAD DE HUMANIDADES Y EDUCACIÓN, UNIVERSIDAD DE LOS ANDES, MERIDA, VENEZUELA

E-mail address: `grchacon@ula.ve`

FACULTAD DE CIENCIAS, UNIVERSIDAD DE LOS ANDES, MERIDA, VENEZUELA

E-mail address: `jgimenez@ula.ve`