A NOTE ON k -INTERSECTION BODIES

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ABSTRACT. The concept of an intersection body is central for the dual Brunn-Minkowski theory and has also played an important role in the solution of the Busemann-Petty problem. A more general concept of k -intersection bodies is related to the generalization of the Busemann-Petty problem. In this note, we compare classes of k -intersection bodies for different k and examine the conjecture that these classes increase with k . In particular, we construct a 4-intersection body that is not a 2-intersection body.

1. INTRODUCTION

Let D, L be origin symmetric star bodies in $\mathbb{R}^n$. Following Lutwak [LU], we say that D is the intersection body of L if the radius of D in every direction is equal to the $(n-1)$ -dimensional volume of the section of L by the central hyperplane orthogonal to this direction, i.e. for every $\xi \in \mathbf{S}^{n-1}$,

$$(1) \quad \rho_D(\xi) = \|\xi\|_D^{-1} = \text{vol}_{n-1}(L \cap \xi^\perp),$$

where ρ_d is the radial function of D , $\|x\|_D = \min\{a \geq 0 : x \in aD\}$ is the Minkowski functional of D , and $\xi^\perp$ is the hyperplane orthogonal to ξ and passing through the origin. A more general class of intersection bodies can be defined as the closure in the radial metric of the class of intersection bodies of star bodies. The more general concept of a k -intersection body comes from [K4]. For $1 \leq k < n$, we say that D is a k -intersection body of L if for every $(n-k)$ -dimensional subspace H of $\mathbb{R}^n$,

$$\text{vol}_k(D \cap H^\perp) = \text{vol}_{n-k}(L \cap H).$$

When $k = 1$, this corresponds to (1) up to a constant of 2. k -intersection bodies are also the closure in the radial metric of k -intersection bodies of star bodies. The concepts of intersection bodies are important for the Busemann-Petty problem and its generalizations (see [K3, Chap. 5]). We are interested in comparing classes of k -intersection bodies for different k . Koldobsky's conjecture is that these classes increase with k . This conjecture comes from the connection of k -intersection bodies and embedding in L_p with $p < 0$. The concept of embedding in L_p with $p < 0$ was introduced in [K2] as an extension of the same properties for $p > 0$.

Definition 1. Let $0 < p < n$. Let D be an origin symmetric star body in $\mathbb{R}^n$. We say that the space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-p} if there exists a finite Borel measure

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μ on $\mathbf{S}^{n-1}$ so that for every even test function $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$(2) \quad \int_{\mathbb{R}^n} \|x\|_D^{-k} \phi(x) dx = \int_{\mathbf{S}^{n-1}} d\mu(\xi) \int_0^\infty t^{k-1} \hat{\phi}(t\xi) dt.$$

For $-1 < p < 0$, (2) is equivalent to the existence of a measure μ on $\mathbf{S}^{n-1}$ so that for every x

$$\|x\|^p = \int_{\mathbf{S}^{n-1}} |(x, \xi)|^p d\mu(\xi).$$

This says the norm $\|x\|$ admits the Levy representation with exponent p . A special construction of a normed space with its Levy representation was used in [K1] to show there is a Banach space embedding isometrically into $L_{1/2}$ but not into L_1 . The construction also yielded an example of a Banach space embedding into $L_{1/4}$ but not into $L_{1/2}$. J. Borwein and the Center for Computational Mathematics at Simon Fraser University (unpublished) showed by computer methods that this algorithm yields examples of Banach spaces embedding into $L_{a/64}$ but not into $L_{(a+1)/64}$ for $a = 1, 2, \dots, 63$. Recently, N. Kalton and A. Koldobsky ([KK]) showed that for every $0 < p < 1$ one can find a Banach space embedding isometrically into L_p but not into any L_q for $p < q \leq 1$. In section 3, we use the construction ideas of [K1] to construct a space that embeds isometrically in $L_{-1/3}$, while at the same time it does not embed in $L_{-1/6}$.

It was proved in [K2] that if D is an origin symmetric star body in $\mathbb{R}^n$, then the space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_p , $p < -1$, if and only if $\|\cdot\|_D^p$ is a positive definite distribution. A distribution is positive definite if its Fourier transform is a positive distribution, that is, $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function ϕ . A direct connection between k -intersection bodies and embeddings in L_{-k} was established in [K4] stated as the following theorem.

Theorem 1. *Let $1 \leq k < n$. An origin symmetric star body D in $\mathbb{R}^n$ is a k -intersection body if and only if the space $(\mathbb{R}^n, \|\cdot\|_D)$ embeds in L_{-k} . Both conditions are equivalent to $\|\cdot\|_D^{-k}$ being a positive definite distribution.*

The advantage of this connection is that we can now try to extend known results on L_p spaces to negative values of p . Every such extension gives new information about intersection bodies. For example, a well-known fact is that for $0 < p < q \leq 2$, the space L_q embeds isometrically in L_p , therefore L_p -spaces become larger when p decreases from 2. This result was also extended to negative values of p , in [K2]. Namely, every finite-dimensional subspace of L_q , with $0 < q \leq 2$, embeds in L_{-p} for every $p \in (0, n)$. By Theorem 1, the unit ball of every finite-dimensional subspace of L_q , with $0 < q \leq 2$, is a k -intersection body for every $k \in (0, n)$. One question still to answer is: For $-n < p < q < 0$, does every normed subspace that embeds in L_q also embed in L_p ? Theorem 1 allows us to restate the question in terms of intersection bodies. For integers $0 < q < p < n$, is a q -intersection body also a p -intersection body? If q divides p , then every q -intersection body is a p -intersection body. Therefore, every 2-intersection body is a 4-intersection body (see [M]). In section 4, we construct a normed space that shows that there exist 4-intersection bodies that are not 2-intersection bodies.

2. MAIN IDEA

The idea of the construction originates from [SCH], [K1]. Let f be an infinitely differentiable function on the unit sphere $\mathbf{S}^{n-1}$ in $\mathbb{R}^n$. We perturb the Euclidean norm $\|x\|_2$ by means of the function f . Given $\lambda > 0$, consider the function

$$(3) \quad \mathcal{N}_\lambda(x) = \|x\|_2 \left(1 + \lambda f \left(\frac{x}{\|x\|_2} \right) \right), x \in \mathbb{R}^n.$$

One can choose λ small enough so that $\mathcal{N}_\lambda$ is a norm in $\mathbb{R}^n$. This follows from considering the one-dimensional case: if $a, b \in \mathbb{R}$, g is a convex function on $[a, b]$ with $g'' > \delta > 0$ on $[a, b]$ and $h \in C^2[a, b]$, then the function $g + \lambda h$ has positive second derivatives on $[a, b]$ for sufficiently small λ 's and, therefore, is convex on $[a, b]$.

Let $\lambda_f = \sup\{\lambda > 0 : \mathcal{N}_\lambda \text{ is a norm in } \mathbb{R}^n \text{ for every } t \leq \lambda\}$. For each $\lambda \leq \lambda_f$, we denote by X_λ the n -dimensional normed space with the norm $\mathcal{N}_\lambda$. For $q < 0$, let $\lambda_q = \sup\{\lambda > 0 : X_t \text{ embeds in } L_q \text{ for every } t \leq \lambda\}$. Given $-n < p < q < 0$, we want to find a Banach subspace of L_p that is not isometric to a subspace of L_q . If we find a function, f , so that λ_q is strictly less than λ_p and λ_p is strictly less than λ_f , then we get an example of a Banach space which embeds in L_p but does not embed in L_q .

We will use the following characterization of finite-dimensional spaces that embed in L_p , $p > -1$; see [K1] for the proof.

Lemma 1. *Let $p > -1$, p not an even integer, let $(X, \|\cdot\|)$ be an n -dimensional Banach space, and suppose there exists a continuous function b on the sphere $\mathbf{S}^{n-1}$ in $\mathbb{R}^n$ such that, for every $x \in \mathbb{R}^n$,*

$$\|x\|^p = \int_{\mathbf{S}^{n-1}} |(x, \xi)|^p b(\xi) d\xi,$$

where (x, ξ) stands for the scalar product in $\mathbb{R}^n$. Then X embeds in L_p if and only if b is a non-negative (not identically zero) function.

We are going to choose special norms for which it is possible to calculate the function b and then check if b is non-negative. We will need the following lemma from [K1] that has been expanded to meet our needs here.

Lemma 2. *For every $x = (x_1, x_2, \dots, x_n)$ from the unit sphere $\mathbf{S}^{n-1}$ in $\mathbb{R}^n$ and every $q > -1$, $q \neq 0$, $q \neq 2$, we have*

$$(4) \quad x_n^2 = \frac{\Gamma(\frac{n+q}{2})}{2\pi^{\frac{n-1}{2}}\Gamma(\frac{q+1}{2})} \int_{\mathbf{S}^{n-1}} |(x, \xi)|^q \left(\frac{n+q}{q} \xi_n^2 - \frac{1}{q} \right) d\xi.$$

Furthermore,

$$(5) \quad x_n^4 = \frac{\Gamma(\frac{n+q}{2})}{2\pi^{\frac{n-1}{2}}\Gamma(\frac{q+1}{2})} \int_{\mathbf{S}^{n-1}} |(x, \xi)|^q \left(\frac{(n+q+2)(n+q)(q+3)}{2q(q-2)} \xi_n^4 - \frac{6(n+q)}{q(q-2)} \xi_n^2 + \frac{3}{q(q-2)} \right) d\xi.$$

Proof. It is a well-known fact, [K3, p. 56], that for every $x \in \mathbb{R}^n$ and every $k > 0$,

$$(6) \quad (x_1^2 + \dots + x_n^2)^k = \frac{\Gamma((n+2k)/2)}{2\pi^{(n-1)/2}\Gamma((2k+1)/2)} \int_{\mathbf{S}^{n-1}} |(x, \xi)|^{2k} d\xi.$$

Differentiate both sides of (6) by x_n twice, and then use that $x \in \mathbf{S}^{n-1}$ to get

$$1 + (2k - 2)x_n^2 = (2k - 1) \frac{\Gamma((n + 2k)/2)}{2\pi^{(n-1)/2}\Gamma((2k + 1)/2)} \int_{\mathbf{S}^{n-1}} |(x, \xi)|^{2k-2} \xi_n^2 d\xi,$$

for $2k - 2 > -1$. Set $q = 2k - 2$ and use $\Gamma(x + 1) = x\Gamma(x)$ to get (4).

Now to get (5), differentiate both sides of (6) by x_n four times, remembering that $x \in \mathbf{S}^{n-1}$:

$$\begin{aligned} & 3 + (2k - 6)(2k - 4)x_n^4 + 6(2k - 4)x_n^2 \\ &= (2k - 1)(2k - 3) \frac{\Gamma((n + 2k)/2)}{2\pi^{(n-1)/2}\Gamma((2k + 1)/2)} \int_{\mathbf{S}^{n-1}} |(x, \xi)|^{2k-4} \xi_n^4 d\xi, \end{aligned}$$

for $2k - 4 > -1$. Now using the properties of the Γ function, setting $q = 2k - 4$ and using (4) we get (5). $\square$

It is a well-known fact due to L. Schwartz (see [GV, p. 152]), that a distribution is positive definite if and only if its Fourier transform is a positive distribution. Using the above construction of $\mathcal{N}_\lambda(x)$, we will take the Fourier transform in the sense of distributions of $(\mathcal{N}_\lambda(x))^p$ with $p < -1$ and check for which values of λ is $(\widehat{(\mathcal{N}_\lambda(x))^p})(\xi)$ positive everywhere on $\mathbf{S}^{n-1}$. Therefore by Theorem 1, for these values of λ the space X_λ embeds isometrically in L_p , $p < -1$. We will need the following lemma, which is based on the formula for the Fourier transform of the Euclidean norm [GS, p. 192]:

$$\widehat{(\|x\|_2^p)}(\xi) = \frac{2^{n+p}\pi^{n/2}\Gamma(\frac{n+p}{2})}{\Gamma(\frac{-p}{2})} \|\xi\|_2^{-n-p}.$$

Lemma 3. *Let $x \in \mathbb{R}^n$ and let $\|x\|_2$ be the Euclidean norm. Then*

$$(7) \quad \widehat{(x_n^j \|x\|_2^p)}(\xi) = \frac{2^{n+p}\pi^{n/2}\Gamma(\frac{n+p}{2})}{\Gamma(\frac{-p}{2})} (-i)^j \frac{\partial^j}{\partial \xi_n^j} \|\xi\|_2^{-n-p}.$$

Furthermore,

$$(8) \quad \frac{\partial^2}{\partial \xi_n^2} \|\xi\|_2^{-n-p} = (n+p)(n+p+2)\xi_n^2 \|\xi\|_2^{-n-p-2} - (n+p)\|\xi\|_2^{-n-p-1}$$

and

$$\begin{aligned} (9) \quad \frac{\partial^4}{\partial \xi_n^4} \|\xi\|_2^{-n-p} &= 3(n+p)(n+p+2)\|\xi\|_2^{-n-p-2} \\ &\quad - 6(n+p)(n+p+2)(n+p+4)\xi_n^2 \|\xi\|_2^{-n-p-3} \\ &\quad + (n+p)(n+p+2)(n+p+4)(n+p+6)\xi_n^4 \|\xi\|_2^{-n-p-4}. \end{aligned}$$

Proof. We use the connection between the Fourier transform and differentiation: $\widehat{(x_n^j f(x))}(\xi) = (-i)^j \frac{\partial^j}{\partial \xi_n^j} \widehat{f}(\xi)$. Using this to compute $\widehat{(x_n^j \|x\|_2^p)}(\xi)$ we get (7). In order to obtain the rest of the results, we differentiate $\|\xi\|_2^{-n-p}$ two and four times, respectively. After collecting terms, we get the result. $\square$

3. FIRST EXAMPLE

For every $\lambda > 0$ define a function $\mathcal{N}_\lambda$ on $\mathbb{R}^n$ by

$$(10) \quad \mathcal{N}_\lambda(x) = (x_1^2 + \cdots + x_n^2)^{1/2} \left(1 + \lambda \frac{x_1^2 + \cdots + x_{n-1}^2 + 2x_n^2}{x_1^2 + \cdots + x_n^2} \right)^{-6}, x \in \mathbb{R}^n.$$

Lemma 4. $\mathcal{N}_\lambda$ is a convex function if and only if $\lambda \leq \frac{1}{11}$.

Proof. $\mathcal{N}_\lambda$ is convex if and only if the following function in two dimensions is convex:

$$f(x, y) = (x^2 + y^2)^{1/2} \left(1 + \lambda \frac{x^2 + 2y^2}{x^2 + y^2} \right)^{-6}.$$

The function f is convex if and only if $a^2 \frac{d^2 f}{dx^2} + 2ab \frac{d^2 f}{dxdy} + b^2 \frac{d^2 f}{dy^2}$ is a non-negative function for any choice of a, b . Taking the second derivatives of the function f and considering all choices of a, b, x , and y , we have the following expression:

$$\begin{aligned} & [a^2 \frac{d^2 f}{dx^2} + 2ab \frac{d^2 f}{dxdy} + b^2 \frac{d^2 f}{dy^2}] \\ &= \frac{(bx-ay)^2(x^2+y^2)^{9/2}(x^4(1-10\lambda-11\lambda^2)+y^4(1+16\lambda+28\lambda^2)+2x^2y^2(1+3\lambda+80\lambda^2))}{(x^2(1+\lambda)+y^2(1+2\lambda))^8}. \end{aligned}$$

Since we only consider positive values of λ , we choose λ so that each of the following expressions is positive: $1 - 10\lambda - 11\lambda^2$, $1 + 16\lambda + 28\lambda^2$, and $1 + 3\lambda + 80\lambda^2$. The latter two are always positive when $\lambda > 0$, and the first is positive if $\lambda \leq \frac{1}{11}$. Therefore, $f(x, y)$ is convex if and only if $\lambda \leq \frac{1}{11}$. $\square$

Theorem 2. Let $n \geq 4$. If

$$(11) \quad \frac{1}{6n-8} < \lambda \leq \frac{63n-105-\sqrt{21}\sqrt{153n^2-390n+428}}{36n^2-240n+97} \leq \frac{1}{11},$$

then the Banach space $\mathbf{X}_\lambda$ embeds in $L_{-1/3}$ and, at the same time $\mathbf{X}_\lambda$ does not embed in $L_{-1/6}$.

Proof. First, we will prove show that $\mathbf{X}_\lambda$ does not embed in $L_{-1/6}$ if $\lambda > \frac{1}{6n-8}$. For $x \in \mathbf{S}^{n-1}$, we use (4) with $q = -1/6$ to obtain the following representation of the norm $\mathcal{N}_\lambda$:

$$\begin{aligned} \mathcal{N}_\lambda^{-1/6} &= (1 + \lambda) + \lambda x_n^2 \\ &= \frac{\Gamma(\frac{n-1/6}{2})}{2\pi^{\frac{n-1}{2}} \Gamma(5/12)} \int_{\mathbf{S}^{n-1}} |(x, \xi)|^{-1/6} b(\xi) d\xi, \end{aligned}$$

where

$$b(\xi) = (1 + 7\lambda) - \xi_n^2 \lambda (6n - 1).$$

By Lemma 1, the space $\mathbf{X}_\lambda$ embeds in $L_{-1/6}$ if and only if $b(\xi)$ is non-negative. The function $b(\xi)$ is linear in $\xi_n^2 \in [0, 1]$ with $b(0) > 0$ and is non-negative if $\lambda \leq \frac{1}{6n-8}$. Therefore, if $\lambda > \frac{1}{6n-8}$, then $\mathbf{X}_\lambda$ does not embed in $L_{-1/6}$. This proves the left-hand inequality.

For the right-hand side, we use Lemma 2 with $q = -1/3$ to get the following representation for the norm $\mathcal{N}_\lambda$:

$$\begin{aligned}\mathcal{N}_\lambda^{-1/3} &= (1 + 2\lambda + \lambda^2) + (2\lambda + 2\lambda^2)x_n^2 + \lambda^2 x_n^4 \\ &= \frac{\Gamma(\frac{n-1/3}{2})}{2\pi^{\frac{n-1}{2}}\Gamma(1/3)} \int_{\mathbf{S}^{n-1}} |(x, \xi)|^{-1/3} b(\xi) d\xi,\end{aligned}$$

where

$$b(\xi) = (1 + 8\lambda + \frac{76}{7}\lambda^2) - \xi_n^2(3n-1)(2\lambda + \frac{32}{7}\lambda^2) + \xi_n^4(3n-1)(3n+5)(\frac{4}{21}\lambda^2).$$

Lemma 1 says the space $\mathbf{X}_\lambda$ embeds in $L_{-1/3}$ if and only if $b(\xi)$ is non-negative. The function $b(\xi)$ is quadratic in ξ_n^2 with positive leading coefficient. Since $b(0) > 0$, it is enough to make sure that the roots of $b(\xi)$ do not lie in the interval $[0, 1]$. If we choose λ so that the roots occur for $\xi_n^2 \geq 1$, then $b(\xi)$ is non-negative for all $\xi \in \mathbf{S}^{n-1}$. Remembering that $b(\xi)$ is quadratic in $\xi_n^2 \in [0, 1]$, we get the following inequality:

$$\begin{aligned}(12) \quad & \frac{8}{21}(3n-1)(3n+5)\lambda^2 - (3n-1)(2\lambda + \frac{32}{7}\lambda^2) \\ & \leq \sqrt{(3n-1)^2(2\lambda + \frac{32}{7}\lambda^2)^2 - \frac{16}{21}(3n-1)(3n+5)\lambda^2(1 + 8\lambda + \frac{76}{7}\lambda^2)}.\end{aligned}$$

Solving for λ we get

$$\lambda \leq \frac{63n - 105 - \sqrt{21}\sqrt{153n^2 - 390n + 428}}{36n^2 - 240n + 97}.$$

Thus, if we choose λ this way, $b(\xi)$ will be non-negative and the space X_λ embeds isometrically in $L_{-1/3}$. To prove the theorem, it suffices to note that, for every $n \geq 4$, $\frac{63n-105-\sqrt{21}\sqrt{153n^2-390n+428}}{36n^2-240n+97} \leq \frac{1}{11}$, and the condition

$$\frac{1}{6n-8} < \frac{63n-105-\sqrt{21}\sqrt{153n^2-390n+428}}{36n^2-240n+97},$$

holds if $n > \frac{-1+4\sqrt{15}}{6}$. Since $n \geq 4 > \frac{-1+4\sqrt{15}}{6}$, if we choose λ so that (11) is satisfied, then $\mathbf{X}_\lambda$ embeds isometrically in $L_{-1/3}$ but at the same time does not embed isometrically in $L_{-1/6}$. $\square$

Note that for $n \leq 3$, every normed space embeds in any L_p , $-1 < p < 0$, and this is why we only considered spaces with $n \geq 4$; see [K3, p. 78].

4. SECOND EXAMPLE

For every $\lambda > 0$ define a function $\mathcal{N}_\lambda$ on $\mathbb{R}^n$ by

$$(13) \quad \mathcal{N}_\lambda(x) = (x_1^2 + \cdots + x_n^2)^{1/2} \left(1 + \lambda \frac{x_1^2 + \cdots + x_{n-1}^2 + 5x_n^2}{x_1^2 + \cdots + x_n^2} \right)^{-1/2}, \quad x \in \mathbb{R}^n.$$

Lemma 5. $\mathcal{N}_\lambda$ is a convex function if and only if $\lambda \leq \frac{1}{3}$.

Proof. $\mathcal{N}_\lambda$ is convex if and only if the following function in two dimensions is convex:

$$f(x, y) = (x^2 + y^2)^{1/2} \left(1 + \lambda \frac{x^2 + 5y^2}{x^2 + y^2} \right)^{-1/2}.$$

The function f is convex if and only if $a^2 \frac{d^2 f}{dx^2} + 2ab \frac{d^2 f}{dx dy} + b^2 \frac{d^2 f}{dy^2}$ is a non-negative function for all choices of a, b . Taking the second derivatives of the function f and considering all choices of a, b, x , and y , we have the following expression:

$$a^2 \frac{d^2 f}{dx^2} + 2ab \frac{d^2 f}{dx dy} + b^2 \frac{d^2 f}{dy^2} = \frac{(bx - ay)^2 (x^2(1 - 2\lambda - 3\lambda^2) + y^2(1 + 14\lambda + 45\lambda^2))}{(x^2(1 + \lambda) + y^2(1 + 5\lambda))^{5/2}}.$$

Since we only consider positive values of λ , then choose λ so that each of the following expressions are always positive: $1 - 2\lambda - 3\lambda^2$ and $1 + 14\lambda + 45\lambda^2$. The latter expression is always positive when $\lambda > 0$, and the first is positive if $\lambda \leq \frac{1}{3}$. Therefore, $f(x, y)$ is convex if and only if $\lambda \leq \frac{1}{3}$. $\square$

Theorem 3. *Let $n \geq 7$. If*

$$(14) \quad \frac{1}{2n-7} < \lambda \leq \frac{3n-18-\sqrt{3}\sqrt{n^2-6n+5}}{2n^2-30n+103} \leq \frac{1}{3},$$

then the Banach space $\mathbf{X}_\lambda$ embeds in L_{-4} and, at the same time, $\mathbf{X}_\lambda$ does not embed in L_{-2} .

Proof. First, we will show that $\mathbf{X}_\lambda$ is isometric to a subspace of L_{-2} if and only if $\lambda \leq \frac{1}{2n-7}$. We need to find the values of λ that will make $(\mathcal{N}_\lambda)^{-2}$ positive definite for all $\xi \in \mathbf{S}^{n-1}$. We will use Lemma 3 to compute the Fourier Transform of $(\mathcal{N}_\lambda)^{-2}$:

$$\begin{aligned} \mathcal{N}_\lambda^{-2} &= (1 + \lambda)\|x\|_2^{-2} + (4\lambda)x_n^2\|x\|_2^{-4}, \\ b(\xi) &= \widehat{(\mathcal{N}_\lambda^{-2})}(\xi) \\ &= 2^{n-2}\pi^{n/2}\Gamma\left(\frac{n-2}{2}\right) ((1 + 3\lambda) - \xi_n^2(n-2)(2\lambda)). \end{aligned}$$

The function $b(\xi)$ is linear in $\xi_n^2 \in [0, 1]$ with $b(0) > 0$. If we choose $\lambda \leq \frac{1}{2n-7}$, then $b(\xi)$ is non-negative for all $\xi \in \mathbf{S}^{n-1}$. Therefore, by Lemma 1, if $\lambda > \frac{1}{2n-7}$, then $\mathbf{X}_\lambda$ does not embed in L_{-2} . This proves the left-hand inequality.

For the right-hand side, we again use Lemma 3 to compute the Fourier Transform of the norm $\mathcal{N}_\lambda$ with exponent $q = -4$:

$$\begin{aligned} \mathcal{N}_\lambda^{-4} &= (1 + 2\lambda + \lambda^2)\|x\|_2^{-4} + (8\lambda + 8\lambda^2)x_n^2\|x\|_2^{-6} + 16\lambda^2x_n^4\|x\|_2^{-8}, \\ b(\xi) &= \widehat{(\mathcal{N}_\lambda^{-4})}(\xi) = 2^{n-4}\pi^{n/2}\Gamma\left(\frac{n-4}{2}\right) \\ &\quad \times [(1 + 4\lambda + 5\lambda^2) - \xi_n^2(n-4)(2\lambda + 6\lambda^2) + \xi_n^4(n-4)(n-2)\left(\frac{2}{3}\lambda^2\right)]. \end{aligned}$$

Lemma 1 says the space $\mathbf{X}_\lambda$ embeds in L_{-4} if and only if $b(\xi)$ is non-negative. The function $b(\xi)$ is quadratic in ξ_n^2 with positive leading coefficient. Since $b(0) > 0$, it is enough to make sure that the roots of $b(\xi)$ do not lie in the interval $[0, 1]$. If we choose λ so that the roots occur for $\xi_n^2 \geq 1$, then $b(\xi)$ is non-negative for all $\xi \in \mathbf{S}^{n-1}$. Remembering that $b(\xi)$ is quadratic in $\xi_n^2 \in [0, 1]$, we get the following inequality:

$$(15) \quad \begin{aligned} &2\lambda(n-4)(n-2) - 3(n-4)(1+3\lambda) \\ &\geq \sqrt{9(n-4)^2(1+3\lambda)^2 - 6(n-4)(n-2)(1+4\lambda+5\lambda^2)}. \end{aligned}$$

Solving for λ we get

$$\lambda \leq \frac{3n-18-\sqrt{3}\sqrt{n^2-6n+5}}{2n^2-30n+103}.$$

Thus, if we choose λ this way, $b(\xi)$ will be non-negative, and the space X_λ embeds isometrically in L_{-4} . To prove the theorem, it suffices to note that, for every $n \geq 7$, $\frac{3n-18-\sqrt{3}\sqrt{n^2-6n+5}}{2n^2-30n+103} \leq \frac{1}{3}$ and the condition $\frac{1}{2n-7} < \frac{3n-18-\sqrt{3}\sqrt{n^2-6n+5}}{2n^2-30n+103}$ holds if $n > \frac{15-\sqrt{19}}{2}$. Since $n \geq 7 > \frac{15-\sqrt{19}}{2}$ if we choose λ so that (11) is satisfied, then $\mathbf{X}_\lambda$ embeds isometrically in L_{-4} but at the same time does not embed isometrically in L_{-2} . $\square$

By Theorem 1, we have an example of a normed space whose unit ball is a 4-intersection body but is not a 2-intersection body. For $5 \leq n < 7$, every normed space embeds in L_{-4} (see [K3, p. 78]), so the example is for any space that does not embed in L_{-2} , for example ℓ_q^n , $q > 2$; see [K3, p. 83].

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