DRIFTED LAPLACE OPERATORS ON HOMOGENEOUS TREES

ENRICO CASADIO TARABUSI AND ALESSANDRO FIGÀ-TALAMANCA

(Communicated by Michael T. Lacey)

ABSTRACT. We determine the spectrum and the resolvent operator of a drifted Laplace operator on a homogeneous tree, obtaining qualitatively different results according to the sign of the drift in the direction of a boundary point.

1. INTRODUCTION

Let $\mathfrak{T}$ be a homogeneous tree of degree $q + 1$, and $\mathfrak{X}$ the set of its vertices. The (isotropic) *Laplace operator* on $\mathfrak{T}$ maps a complex-valued function f on $\mathfrak{X}$ to the function Lf given by

$$Lf(x) = \frac{1}{q+1} \sum_{y \sim x} f(y) \quad \text{for every } x \in \mathfrak{X},$$

where $y \sim x$ means that $y \in \mathfrak{X}$ is adjacent to x . In apparently different contexts, including its realization as a convolution operator on free groups, the operator L (or $L - I$, as it is sometimes introduced) has been the object of thorough investigation, for instance in [K, C, FP1, MZ, FP2, FN]. A more complete list of references may be found in [FP2, FS, FN].

A basic property of L is its invariance with respect to the group $\text{Aut}(\mathfrak{T})$ of all automorphisms of the tree $\mathfrak{T}$. From this point of view it is natural to extend the investigation to average operators that are invariant for an appropriate subgroup G of $\text{Aut}(\mathfrak{T})$. This was done in [A, FS] for G discrete, acting transitively on $\mathfrak{X}$, and isomorphic to a free group or a free power of the two-element group.

In this paper we study a class of *drifted* Laplace operators invariant for the group G of automorphisms that fix a given element ω_0 of the boundary Ω of $\mathfrak{T}$. Besides the choice of ω_0 , every such operator L_θ depends on a real parameter $\theta \in (0, 1)$ that determines the G -invariant drift in the direction of ω_0 . Indeed, denoting by x^- the unique neighbor vertex of $x \in \mathfrak{X}$ that lies between x and ω_0 , define

$$L_\theta f(x) = \theta f(x^-) + \frac{1-\theta}{q} \sum_{y \in \mathfrak{X}: y^- = x} f(y) \quad \text{for every } x \in \mathfrak{X}.$$

To a drifted Laplace operator is associated a nearest-neighbor random walk, once we interpret θ as the probability of moving from x to x^- , and $(1-\theta)/q$ as the probability of moving from x to each one of the other neighbors of x . The isotropic,

Received by the editors March 21, 2006.

2000 *Mathematics Subject Classification.* Primary 43A85; Secondary 05C05.

Key words and phrases. Homogeneous trees, Laplace operator, drifts, spectrum, resolvent operator.

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driftless case is the operator $L = L_{1/(q+1)}$. In the case $\theta < 1/(q+1)$ the drift drags away from ω_0 , while in the case $\theta > 1/(q+1)$ the drift drags towards ω_0 . (According to some authors, the drift is absent for $\theta = 1/2$ rather than for $\theta = 1/(q+1)$.)

For several questions in harmonic analysis a homogeneous tree can be considered as a natural discrete analogue of the hyperbolic half-plane $\Pi = \{x+iy \in \mathbf{C} : y > 0\}$. The role of ω_0 is played by the point at infinity, and that of L_θ by the *drifted Laplace-Beltrami operator*

$$L_b = \frac{y^2}{2} \Delta + by \frac{\partial}{\partial y}$$

for real $b < 1/2$. This operator is invariant for the group of isometries of Π that fix ∞ , i.e., the group $\{z \mapsto \alpha z + \beta, z \mapsto -\alpha \bar{z} + \beta : \alpha > 0, \beta \in \mathbf{R}\}$. The analogy between the two settings was exploited in [BCF] (for $\theta < 1/(q+1)$ and $b < 0$), where these drifted Laplace operators were used for parallel characterizations of symmetric stable distributions.

The main result of this paper (Theorem 3.2) is the computation of the spectrum of L_θ for every $0 < \theta < 1$ and of the resolvent operator, as well as, more generally, of the one-sided left or right resolvents where they exist (Proposition 3.4). These computations are carried out in several steps.

First of all we introduce, for any two vertices x, y of the tree, a *bi-distance* $\mathbf{d}(x, y)$ as a pair of non-negative integers that express the relative position of x, y with respect to ω_0 . The group G is characterized as the group of all automorphisms of the tree that preserve $\mathbf{d}$. It becomes natural at this point to consider, for every pair (j, k) of non-negative integers, the elementary summation operator $\Lambda_{j,k}$ whose kernel is the indicator function of the set $\{(x, y) \in \mathfrak{X} \times \mathfrak{X} : \mathbf{d}(x, y) = (j, k)\}$. The operators $\Lambda_{j,k}$ span the space of all G -invariant operators, and their norm is easily computed by using a result of L. Saloff-Coste and W. Woess [SW] concerning G -invariant operators with non-negative kernels. From this we finally prove, for every applicable $\gamma \in \mathbf{C}$, the boundedness of the natural candidates for the (two- or one-sided) inverse of the operator $\gamma - L_\theta$, which is of course G -invariant.

2. PRELIMINARIES

It will be assumed that $q > 1$, so that $\mathfrak{T}$ is infinite and not isomorphic to $\mathbf{Z}$. Unless otherwise stated, the notation and terminology of [FN] will be used. The choice of ω_0 determines a partial ordering on $\mathfrak{X}$, by which $x \in \mathfrak{X}$ is an *ancestor* of $y \in \mathfrak{X}$ if x lies on the *half-geodesic* $[y, \omega_0)$, the one-sided-infinite *chain* of vertices from x to ω_0 . (The boundary point ω_0 is sometimes called the *mythical ancestor* of all of $\mathfrak{X}$; cf. [C].) Each $x \in \mathfrak{X}$ can be identified with a subset of $\Omega' = \Omega \setminus \{\omega_0\}$: in this identification, a boundary point $\omega \neq \omega_0$ belongs to x if the (doubly-infinite) *geodesic* (ω_0, ω) from ω_0 to ω contains x . Then $y \subseteq x$ if and only if x is an ancestor of y , so the order relation is inclusion. The *parent* x^- of $x \in \mathfrak{X}$ is consequently the smallest vertex containing x properly; of course x is a *child* of x^- , and every vertex has exactly q children. The union of two vertices is not always a vertex. In general, the *join* of $x, y \in \mathfrak{X}$ with respect to ω_0 is the smallest vertex $x \vee y$ that contains x, y ; similarly $x \vee \omega$ (respectively $\omega \vee \omega'$) is the smallest vertex that contains $x \in \mathfrak{X}$ and $\omega \in \Omega'$ (respectively $\omega, \omega' \in \Omega'$).

Let $\mathbf{N}$ be the set of non-negative integers. Let $d(x, y)$ be the usual tree distance between $x, y \in \mathfrak{X}$. We introduce their *bi-distance* (with respect to ω_0) $\mathbf{d}(x, y) =$

$(d^-(x, y), d^+(x, y)) \in \mathbf{N} \times \mathbf{N}$ by

$$\begin{aligned} d^-(x, y) &= d(x, x \vee y), \\ d^+(x, y) &= d(x \vee y, y) = d^-(y, x). \end{aligned}$$

Consequently if $\mathbf{d}(x, y) = (j, k)$, then $\mathbf{d}(y, x) = (k, j)$. Furthermore

$$d(x, y) = d^-(x, y) + d^+(x, y),$$

because $x \vee y$ lies on the finite chain between x, y . An automorphism g of $\mathfrak{T}$ belongs to G if and only if it preserves bi-distances, i.e., $\mathbf{d}(gx, gy) = \mathbf{d}(x, y)$ for every $x, y \in \mathfrak{X}$. On the other hand, for every two pairs $(x, y), (x', y')$ of vertices such that $\mathbf{d}(x, y) = \mathbf{d}(x', y')$ there is $g \in G$ such that $gx = x'$ and $gy = y'$; that is, $\mathfrak{X}$ is two-point homogenous under the action of G with respect to the bi-distance.

The *relative horocycle index* of $x, y \in \mathfrak{X}$ with respect to ω_0 is

$$h(x, y) = d^-(x, y) - d^+(x, y) = -h(y, x),$$

and is readily proved to satisfy the cocycle relation $h(x, z) = h(x, y) + h(y, z)$ for every $x, y, z \in \mathfrak{X}$, so that $h(x, y) = 0$ is an equivalence relation on $\mathfrak{X}$. The equivalence classes are the *horocycles* through ω_0 , and, once a base horocycle $\mathfrak{H}_0$ is arbitrarily chosen, each one may be indexed as $\mathfrak{H}_n$ with $n \in \mathbf{Z}$, adopting the convention that n increases as the horocycle is farther from ω_0 ; thus $h(x, y) = m - n$ for $x \in \mathfrak{H}_m$ and $y \in \mathfrak{H}_n$; cf. [FN, p. 35]. With respect to this choice of $\mathfrak{H}_0$ one can also define the *absolute horocycle index* of $x \in \mathfrak{X}$ with respect to ω_0 by setting $h(x) = h(x, y)$ for any $y \in \mathfrak{H}_0$; i.e., $h(x) = n$ if $x \in \mathfrak{H}_n$.

The number of vertices that lie at bi-distance (j, k) from a given vertex x , that is the cardinality $n_{j,k}$ of the set $\{y \in \mathfrak{X} : \mathbf{d}(x, y) = (j, k)\}$, is easily seen to be

$$(2.1) \quad n_{j,k} = \begin{cases} q^k & \text{if } j = 0 \text{ or } k = 0, \\ (q-1)q^{k-1} & \text{if } j, k > 0. \end{cases}$$

A bounded linear operator A on the space $\ell^2(\mathfrak{X})$ is associated to a summation kernel $A(x, y) = A\delta_y(x)$, where δ_y is the Kronecker delta at $y \in \mathfrak{X}$. If $f \in \ell^2(\mathfrak{X})$, then

$$Af(x) = \sum_{y \in \mathfrak{X}} A(x, y)f(y) \quad \text{for every } x \in \mathfrak{X}.$$

Therefore we can refer to A as an operator or as a kernel. The adjoint A^* of A is associated to the kernel $A^*(x, y) = \overline{A(y, x)}$. The operator A is invariant for $g \in \text{Aut}(\mathfrak{X})$ if and only if its kernel is invariant under the diagonal action of g , that is, $A(gx, gy) = A(x, y)$ for every $x, y \in \mathfrak{X}$. A G -invariant operator A has *finite support* if its kernel $A(x, y)$ vanishes for large enough $d(x, y)$.

We shall make use of the elementary G -invariant kernels that take the value 1 on pairs of vertices at a given bi-distance, and vanish otherwise. For $j, k \in \mathbf{N}$ let $\Lambda_{j,k}$ be the G -invariant operator given for $x, y \in \mathfrak{X}$ by

$$\Lambda_{j,k}(x, y) = \begin{cases} 1 & \text{if } \mathbf{d}(x, y) = (j, k), \\ 0 & \text{if } \mathbf{d}(x, y) \neq (j, k). \end{cases}$$

Lemma 2.1. *The operator norm of $\Lambda_{j,k}$ on $\ell^2(\mathfrak{X})$ is*

$$(2.2) \quad \|\Lambda_{j,k}\| = \sqrt{n_{j,k}n_{k,j}} = q^{(j+k)/2} \cdot \begin{cases} 1 & \text{if } j = 0 \text{ or } k = 0, \\ \frac{q-1}{q} & \text{if } j, k > 0. \end{cases}$$

Moreover

$$(2.3) \quad \Lambda_{j,k}^* = \Lambda_{k,j} \quad \text{for every } j, k \in \mathbf{N},$$

and

$$(2.4) \quad \Lambda_{j,k} = \begin{cases} \text{identity} & \text{if } j = k = 0, \\ \Lambda_{1,0}^j & \text{if } k = 0, \\ \Lambda_{0,1}^k & \text{if } j = 0, \\ \Lambda_{1,0}^j \Lambda_{0,1}^k - \Lambda_{1,0}^{j-1} \Lambda_{0,1}^{k-1} & \text{if } j, k > 0. \end{cases}$$

We also have

$$(2.5) \quad \begin{aligned} \Lambda_{j,k} \Lambda_{1,0} &= \begin{cases} \Lambda_{j+1,0} & \text{if } k = 0, \\ (q-1)\Lambda_{j,0} & \text{if } k = 1 \text{ and } j > 0, \\ q\Lambda_{j,k-1} & \text{if } k > j = 0 \text{ or } k > 1, \end{cases} \\ \Lambda_{j,k} \Lambda_{0,1} &= \begin{cases} \Lambda_{j-1,0} + \Lambda_{j,1} & \text{if } j > 0 = k, \\ \Lambda_{j,k+1} & \text{if } j = 0 \text{ or } k > 0; \end{cases} \end{aligned}$$

similar expressions for $\Lambda_{1,0}\Lambda_{j,k}$ and $\Lambda_{0,1}\Lambda_{j,k}$ may be obtained from (2.5) by using (2.3). In particular

$$\begin{aligned} \Lambda_{1,0}\Lambda_{0,1} &= \Lambda_{0,0} + \Lambda_{1,1}, \\ \Lambda_{0,1}\Lambda_{1,0} &= q\Lambda_{0,0}, \end{aligned}$$

whence the commutator of $\Lambda_{1,0}, \Lambda_{0,1}$ is

$$[\Lambda_{1,0}, \Lambda_{0,1}] = \Lambda_{1,1} - (q-1)\Lambda_{0,0}.$$

Proof. We shall prove (2.2) only; the remaining identities follow from direct computation. Fix a Haar measure $|\cdot|$ on G , and let G_x be the isotropy subgroup of G at $x \in \mathfrak{X}$. Then $|G_y|/|G_x| = q^{h(x,y)}$ for every $y \in \mathfrak{X}$. By [SW, (3.3) of Theorem 3.1], for any $x \in \mathfrak{X}$ we have

$$\|\Lambda_{j,k}\| = \sum_{y \in \mathfrak{X}} \sqrt{\frac{|G_y|}{|G_x|}} \Lambda_{j,k}(x, y) = \sum_{\substack{y \in \mathfrak{X}: \\ \mathbf{d}(x,y)=(j,k)}} \sqrt{q^{j-k}} = n_{j,k} q^{(j-k)/2},$$

which by (2.1) yields (2.2). $\square$

The value $A(x, y)$ of the kernel of a G -invariant bounded operator A on $\ell^2(\mathfrak{X})$ only depends on $\mathbf{d}(x, y)$, so that for each $j, k \in \mathbf{N}$ we can set

$$a_{j,k} = A(x, y) \quad \text{for any } x, y \in \mathfrak{X} \text{ such that } \mathbf{d}(x, y) = (j, k).$$

Thus we can write

$$A(x, y) = \sum_{j,k=0}^{\infty} a_{j,k} \Lambda_{j,k}(x, y) \quad \text{for every } x, y \in \mathfrak{X},$$

and the operator A is identified with the infinite matrix $(a_{j,k})_{j,k \in \mathbf{N}}$. If $a_{j,k} \geq 0$ for every $j, k \in \mathbf{N}$, again [SW, (3.3) of Theorem 3.1] implies that

$$\|A\| = \sum_{j,k=0}^{\infty} n_{j,k} q^{(j-k)/2} a_{j,k} \leq \sum_{j,k=0}^{\infty} a_{j,k} q^{(j+k)/2}.$$

More generally, in view of (2.2) we derive the following result.

Corollary 2.2. *Assume given $a_{j,k} \in \mathbf{C}$ for every $j, k \in \mathbf{N}$, and suppose that*

$$\sum_{j,k=0}^{\infty} |a_{j,k}| q^{(j+k)/2} < \infty.$$

Then the series

$$A = \sum_{j,k=0}^{\infty} a_{j,k} \Lambda_{j,k}$$

is convergent in the $\ell^2(\mathfrak{X})$ operator norm to a bounded G -invariant operator A with

$$\|A\| \leq \sum_{j,k=0}^{\infty} |a_{j,k}| q^{(j+k)/2}.$$

The kernel of the drifted Laplace operator L_θ under investigation in this paper is given by

$$L_\theta(x, y) = \begin{cases} \theta & \text{if } y = x^-, \\ (1 - \theta)/q & \text{if } x = y^-, \\ 0 & \text{otherwise.} \end{cases}$$

We first show that L_θ itself and its adjoint L_θ^* generate (as do the elementary operators $\Lambda_{1,0}, \Lambda_{0,1}$) all G -invariant operators.

Proposition 2.3. *For $\theta \neq 1/(q+1)$ the algebra of polynomials in L_θ, L_θ^* is weakly dense in the von Neumann algebra of G -invariant operators on $\ell^2(\mathfrak{X})$.*

Proof. Clearly L_θ, L_θ^* may be expressed as linear combinations of the elementary operators $\Lambda_{1,0}, \Lambda_{1,0}$ as

$$(2.6) \quad \begin{aligned} L_\theta &= \theta \Lambda_{1,0} + \frac{1-\theta}{q} \Lambda_{0,1}, \\ L_\theta^* &= \frac{1-\theta}{q} \Lambda_{1,0} + \theta \Lambda_{0,1}. \end{aligned}$$

It follows that, if $\theta \neq 1/(q+1)$, then L_θ, L_θ^* are linearly independent, so that $\Lambda_{1,0}, \Lambda_{0,1}$ may conversely be expressed as linear combinations of L_θ, L_θ^* .

Every finitely supported G -invariant operator is a linear combination of the operators $\Lambda_{j,k}$ for $j, k \in \mathbf{N}$, which are expressed as polynomials in $\Lambda_{1,0}, \Lambda_{0,1}$ by (2.4). Therefore every G -invariant operator of finite support is a polynomial in L_θ, L_θ^* .

It remains to prove that every bounded G -invariant operator on $\ell^2(\mathfrak{X})$ may be approximated in the weak topology by finitely supported G -invariant operators. Indeed G -invariant operators may be put into a one-to-one correspondence with right convolution operators on $L^2(G)$ with respect to the left Haar measure [SW]. This correspondence is isometric and maps finitely supported operators onto the algebra of right convolution operators by continuous functions on G with compact

support, which is dense in the von Neumann algebra of operators on $L^2(G)$ that commute with left translations. $\square$

Remark 2.4. As shown in the proof above, the algebra of polynomials in L_θ, L_θ^* coincides with the algebra of polynomials in $\Lambda_{1,0}, \Lambda_{0,1}$. From (2.6) the relation between commutators is

$$[L_\theta, L_\theta^*] = \left(\theta^2 - \frac{(1-\theta)^2}{q^2} \right) [\Lambda_{1,0}, \Lambda_{0,1}].$$

The following remark shows that the adjoint L_θ^* of L_θ is a multiple of $L_{\theta'}$ for a suitable drift θ' , with $\theta' \neq \theta$ unless $\theta = 1/(q+1)$.

Remark 2.5. For $0 < \theta < 1$, if

$$b_\theta = q\theta + \frac{1-\theta}{q}, \quad \theta' = \frac{1}{1 + q^2\theta/(1-\theta)},$$

then

$$L_\theta^* = b_\theta L_{\theta'}.$$

The function $\theta \mapsto \theta'$ is strictly decreasing on the interval $(0, 1)$ onto itself, so that θ, θ' always lie on opposite sides with respect to the fixed point $1/(q+1)$.

For $a > 0$ denote by M_a the operator of multiplication by a^h ; observe that $M_a^{-1} = M_{1/a}$. The following result may be regarded as a direct consequence of the reversibility of the random walk associated to L_θ ; cf. [W, p. 126].

Proposition 2.6. *For $0 < \theta < 1$, if*

$$(2.7) \quad a = \sqrt{\frac{1-\theta}{\theta q}}, \quad \mu = (q+1) \sqrt{\frac{\theta(1-\theta)}{q}},$$

then

$$L_\theta = \mu M_a^{-1} L M_a.$$

Therefore a (not necessarily bounded) function f is an eigenfunction of L of eigenvalue λ if and only if $M_{1/a}f$ is an eigenfunction of L_θ of eigenvalue $\mu\lambda$.

Proof. For every $x \in \mathfrak{X}$ we have

$$\begin{aligned} \mu M_a^{-1} L M_a f(x) &= \mu a^{-h(x)} L(a^h f)(x) \\ &= \mu a^{-h(x)} \frac{1}{q+1} \left(a^{h(x^-)} f(x^-) + \sum_{y^-=x} a h(y) f(y) \right) \\ &= \frac{\mu}{q+1} \left(\frac{f(x^-)}{a} + \sum_{y^-=x} a f(y) \right) \\ &= \theta f(x^-) + \frac{1-\theta}{q} \sum_{y^-=x} f(y) = L_\theta f(x). \end{aligned} \quad \square$$

3. THE SPECTRUM AND THE RESOLVENT OPERATOR

For $0 < \theta < 1$, consider the function ϕ given by

$$\phi(\zeta) = (1 - \theta)\zeta + \theta/\zeta \quad \text{for every } \zeta \in \mathbf{C}^*.$$

This is a two-to-one holomorphic mapping of $\mathbf{C}^*$ onto $\mathbf{C}$ with branching points at $\pm\sqrt{\theta/(1-\theta)}$. For every $\rho > 0$ it maps the circle C_ρ of radius ρ and center at the origin—as well as the circle $C_{\theta/(1-\theta)\rho}$ —onto the ellipse $\phi(C_\rho) = \phi(C_{\theta/(1-\theta)\rho})$ with foci at

$$\begin{aligned} F_+ &= +2\sqrt{\theta(1-\theta)}, \\ F_- &= -2\sqrt{\theta(1-\theta)}, \end{aligned}$$

both independent of ρ , and with shorter (i.e., vertical) semi-axis of length

$$\sigma = |\phi(i\rho)| = |(1-\theta)\rho - \theta/\rho| \quad \text{for every } \rho > 0;$$

denote by $\mathcal{E}$ the family of these confocal ellipses. Conversely, the dependence of the two values ρ_1, ρ_2 of ρ on σ , with $\rho_1 \leq \rho_2$, is given by

$$\begin{aligned} \rho_1(\sigma) &= \frac{-\sigma + \sqrt{\sigma^2 + 4\theta(1-\theta)}}{2(1-\theta)} & \text{for every } \sigma \geq 0, \\ \rho_2(\sigma) &= \frac{+\sigma + \sqrt{\sigma^2 + 4\theta(1-\theta)}}{2(1-\theta)} & \text{for every } \sigma \geq 0. \end{aligned}$$

The function ρ_1 is strictly decreasing and the function ρ_2 strictly increasing on $[0, +\infty)$. They agree at $\sigma = 0$, because the circle of radius $\rho_\pm(0) = \sqrt{\theta/(1-\theta)}$ is mapped onto the degenerate ellipse equal to the segment joining F_+, F_- .

Let us introduce the constant

$$(3.1) \quad c_\theta = \frac{\theta q}{1-\theta} \begin{cases} < 1 & \text{if } \theta < 1/(q+1), \\ = 1 & \text{if } \theta = 1/(q+1), \\ > 1 & \text{if } \theta > 1/(q+1), \end{cases}$$

and single out the ellipse

$$E = \phi(C_{1/\sqrt{q}}) = \phi(C_{c_\theta/\sqrt{q}}).$$

Let $E^{\text{int}}, E^{\text{ext}}$ be the bounded, respectively unbounded component of the complement of E , and let

$$\hat{\sigma} = |\theta\sqrt{q} - (1-\theta)/\sqrt{q}|$$

be the value of σ on E . If $\theta = 1/(q+1)$, then $\hat{\sigma} = 0$ and the ellipse E degenerates to the segment, thus E^{int} is empty.

Lemma 3.1. *For $\gamma \in \mathbf{C}$ let ζ_1, ζ_2 be the solutions of the quadratic equation $\phi(\zeta) = \gamma$, with $|\zeta_1| \leq |\zeta_2|$, and set $\eta_j = \zeta_j/c_\theta$ for $j = 1, 2$. We have*

$$\begin{aligned} |\zeta_1| &< |\eta_1| < 1/\sqrt{q} < |\zeta_2| < |\eta_2| && \text{if } \gamma \in E^{\text{ext}} \text{ and } \theta < 1/(q+1), \\ |\zeta_1| &= |\eta_1| < 1/\sqrt{q} < |\zeta_2| = |\eta_2| && \text{if } \gamma \in E^{\text{ext}} \text{ and } \theta = 1/(q+1), \\ |\eta_1| &< |\zeta_1| < 1/\sqrt{q} < |\eta_2| < |\zeta_2| && \text{if } \gamma \in E^{\text{ext}} \text{ and } \theta > 1/(q+1), \\ |\zeta_1| &< |\zeta_2| = 1/\sqrt{q} = |\eta_1| < |\eta_2| && \text{if } \gamma \in E \text{ and } \theta < 1/(q+1), \\ |\zeta_1| &= |\zeta_2| = 1/\sqrt{q} = |\eta_1| = |\eta_2| && \text{if } \gamma \in E \text{ and } \theta = 1/(q+1), \\ |\eta_1| &< |\eta_2| = 1/\sqrt{q} = |\zeta_1| < |\zeta_2| && \text{if } \gamma \in E \text{ and } \theta > 1/(q+1), \\ |\zeta_1| &\leq |\zeta_2| < 1/\sqrt{q} < |\eta_1| \leq |\eta_2| && \text{if } \gamma \in E^{\text{int}} \text{ and } \theta < 1/(q+1), \\ |\eta_1| &\leq |\eta_2| < 1/\sqrt{q} < |\zeta_1| \leq |\zeta_2| && \text{if } \gamma \in E^{\text{int}} \text{ and } \theta > 1/(q+1). \end{aligned}$$

Proof. Because ϕ maps circles to ellipses, the values of $|\zeta_1|, |\zeta_2|, |\eta_1|, |\eta_2|$ only depend on which ellipse of the family $\mathcal{E}$ contains γ , and such an ellipse is uniquely identified by the length σ of its shorter semi-axis. Indeed $|\zeta_j| = \rho_j(\sigma)$ and $|\eta_j| = \rho_j(\sigma)/c_\theta$ for $j = 1, 2$.

If $\gamma \in E$, then $\sigma = \hat{\sigma}$, therefore $\{\rho_1(\sigma), \rho_2(\sigma)\} = \{1/\sqrt{q}, c_\theta/\sqrt{q}\}$, and (3.1) gives the required inequalities for this case.

If $\gamma \in E^{\text{ext}}$, then $\sigma > \hat{\sigma}$, therefore $\rho_1(\sigma) < \rho_1(\hat{\sigma}) \leq 1/\sqrt{q} \leq \rho_2(\hat{\sigma}) < \rho_2(\sigma)$, and similarly $\rho_1(\sigma)/c_\theta < \rho_1(\hat{\sigma})/c_\theta \leq 1/\sqrt{q} \leq \rho_2(\hat{\sigma})/c_\theta < \rho_2(\sigma)/c_\theta$, which, together with (3.1), completes the proof for this case.

Finally, if $\gamma \in E^{\text{int}}$, then $\sigma < \hat{\sigma}$. If, in addition, $\theta < 1/(q+1)$, then $\rho_2(\sigma) < \rho_2(\hat{\sigma}) = 1/\sqrt{q} = \rho_1(\hat{\sigma})/c_\theta < \rho_1(\sigma)/c_\theta$, again by (3.1). On the other hand, if $\theta > 1/(q+1)$ we conclude in a similar fashion. $\square$

Theorem 3.2. *For $0 < \theta < 1$, the spectrum of the operator L_θ is the filled ellipse $E \cup E^{\text{int}}$. For γ in its complement E^{ext} , the resolvent $R = (\gamma - L_\theta)^{-1}$ is the G -invariant operator defined by the kernel*

$$R(x, y) = \sum_{j,k=0}^{\infty} r_{j,k} \Lambda_{j,k}(x, y) \quad \text{for every } x, y \in \mathfrak{X}$$

with

$$(3.2) \quad r_{j,k} = \beta \zeta_1^j \eta_1^k \quad \text{for every } (j, k) \in \mathbf{N} \times \mathbf{N},$$

where ζ_1 is the root of smallest modulus of the quadratic equation $\phi(\zeta) = \gamma$, where $\eta_1 = \zeta_1/c_\theta$, and where the normalization constant β is given by

$$\gamma = \left(1 + \frac{1}{q}\right)(1 - \theta)\zeta_1 + \frac{1}{\beta}.$$

This includes the case $\theta = 1/(q+1)$, where $\zeta_1 = \eta_1$ and the spectrum reduces to the real segment between the points $F_\pm = \pm 2\sqrt{q}/(q+1)$; cf. [K].

Proof. The kernel $R(x, y)$ represents the required resolvent for some $\gamma \in \mathbf{C}$ if and only if it satisfies the resolvent relations $R(\gamma - L_\theta) = \Lambda_{0,0} = (\gamma - L_\theta)R$ and defines a bounded operator on $\ell^2(\mathfrak{X})$. (These relations may in principle be satisfied by G -invariant kernels that do not represent bounded operators.) In view of relations (2.5) and expression (2.6) for the Laplace operator, the resolvent relations

respectively expand to

$$(3.3) \quad \gamma r_{j,k} = \begin{cases} \frac{1-\theta}{q} r_{1,0} + q\theta r_{0,1} + 1 & \text{if } j = k = 0, \\ \frac{1-\theta}{q} r_{j+1,0} + \theta r_{j-1,0} + (q-1)\theta r_{j,1} & \text{if } j > k = 0, \\ \frac{1-\theta}{q} r_{j,k-1} + q\theta r_{j,k+1} & \text{if } k > 0, \end{cases}$$

$$(3.4) \quad \gamma r_{j,k} = \begin{cases} (1-\theta)r_{1,0} + \theta r_{0,1} + 1 & \text{if } j = k = 0, \\ \frac{q-1}{q}(1-\theta)r_{1,k} + \frac{1-\theta}{q}r_{0,k-1} + \theta r_{0,k+1} & \text{if } j = 0 < k, \\ (1-\theta)r_{j+1,k} + \theta r_{j-1,k} & \text{if } j > 0. \end{cases}$$

By (3.3), if $r_{j,k}$ vanished for all $k > 0$ and all j , then so would $r_{j,0}$ for all j . Thus $r_{j,k} \neq 0$ for some j and some positive k . For fixed j , the general solution of the difference equation (3.3) in the variable k is

$$(3.5) \quad r_{j,k} = \begin{cases} c_j \eta_1^k + c'_j \eta_2^k & \text{if } \eta_1, \eta_2 \text{ are distinct roots of } \phi(c_\theta \eta) = \gamma, \\ c_j \eta^k + c'_j k \eta^k & \text{if } \eta \text{ is a double root of } \phi(c_\theta \eta) = \gamma, \end{cases}$$

where c_j, c'_j are constants depending on j . Likewise the general solution of the difference equation (3.4) in the variable j is

$$(3.6) \quad r_{j,k} = \begin{cases} d_k \zeta_1^j + d'_k \zeta_2^j & \text{if } \zeta_1, \zeta_2 \text{ are distinct roots of } \phi(\zeta) = \gamma, \\ d_k \zeta^j + d'_k j \zeta^j & \text{if } \zeta \text{ is a double root of } \phi(\zeta) = \gamma, \end{cases}$$

with d_k, d'_k constants.

If $\gamma \in E^{\text{ext}}$, each of the quadratic equations $\phi(\zeta) = \gamma$ and $\phi(c_\theta \eta) = \gamma$ has distinct roots, because double roots occur only at $\gamma = F_\pm$, the foci of E . The kernel given by (3.2) satisfies (3.5), (3.6). To prove that it defines a bounded operator, in view of Corollary 2.2 it suffices to show that $|\zeta_1|, |\eta_1| < 1/\sqrt{q}$. This is done in Lemma 3.1.

Assume instead that $\gamma \in E \cup E^{\text{int}}$. If R is an inverse of $\gamma - L_\theta$, then its coefficients $r_{j,k}$ must be expressed as in (3.5), (3.6). On the other hand, R is bounded, which implies that $R\delta_x, R^*\delta_x$ belong to ℓ^2 . Since $R^*\delta_x(y) = \overline{R(x, y)}$, this yields

$$\begin{aligned} \sum_{j=0}^{\infty} |r_{j,k}|^2 q^j &< \infty & \text{for every } k \in \mathbf{N}, \\ \sum_{k=0}^{\infty} |r_{j,k}|^2 q^k &< \infty & \text{for every } j \in \mathbf{N}. \end{aligned}$$

By Lemma 3.1, if $\theta < 1/(q+1)$, then $|\eta_1|, |\eta_2| \geq 1/\sqrt{q}$, and the latter series is divergent; while if $\theta > 1/(q+1)$, then $|\zeta_1|, |\zeta_2| \geq 1/\sqrt{q}$, and the former series is divergent. $\square$

Notation 3.3. In the sequel, in order to avoid exceptions for double roots, we set

$$\frac{\alpha_1^j - \alpha_2^j}{\alpha_1 - \alpha_2} = j\alpha^{j-1} \quad \text{if } \alpha_1 = \alpha_2 = \alpha, \text{ and } j \in \mathbf{N}.$$

Proposition 3.4. Assume that γ lies in the interior E^{int} of the spectrum of L_θ . If $\theta < 1/(q+1)$, then $\gamma - L_\theta$ has a right inverse R^r given by

$$R^r(x, y) = \sum_{j,k=0}^{\infty} r_{j,k}^r \Lambda_{j,k},$$

where (with Notation 3.3)

$$r_{j,k}^r = \begin{cases} -\frac{1}{1-\theta} \frac{\zeta_1^j - \zeta_2^j}{\zeta_1 - \zeta_2} & \text{if } j > k = 0, \\ 0 & \text{if } j = 0 \text{ or } k > 0, \end{cases}$$

and where ζ_1, ζ_2 are the solutions of $\phi(\zeta) = \gamma$.

On the other hand, if $\theta > 1/(q+1)$, then $\gamma - L_\theta$ has a left inverse R^l given by

$$R^l(x, y) = \sum_{j,k=0}^{\infty} r_{j,k}^l \Lambda_{j,k},$$

where (again with Notation 3.3)

$$r_{j,k}^l = \begin{cases} -\frac{1}{\theta q} \frac{\eta_1^k - \eta_2^k}{\eta_1 - \eta_2} & \text{if } k > j = 0, \\ 0 & \text{if } k = 0 \text{ or } j > 0, \end{cases}$$

and where η_1, η_2 are the solutions of $\phi(c_\theta \eta) = \gamma$.

Proof. If $\theta > 1/(q+1)$, then $|\eta_1|, |\eta_2| < 1/\sqrt{q}$ by Lemma 3.1. Therefore r^l satisfies (3.3) and

$$|r_{0,k}^l| \leq \frac{1}{q\theta} k |\eta_2|^{k-1} \quad \text{for every } k > 0.$$

By Corollary 2.2 we obtain

$$\|R\| \leq \sum_{j,k=0}^{\infty} \|r_{j,k}^l \Lambda_{j,k}\| \leq \sum_{k=1}^{\infty} |r_{0,k}^l| q^{k/2} \leq \frac{1}{\sqrt{q}\theta} \sum_{k=1}^{\infty} k (\sqrt{q} |\eta_1|)^{k-1} < \infty.$$

We conclude that R^l is a left inverse of $\gamma - L_\theta$.

If $\theta < 1/(q+1)$, then $|\zeta_1|, |\zeta_2| < 1/\sqrt{q}$ by Lemma 3.1. Therefore r^r satisfies (3.4). By Corollary 2.2 it defines a bounded operator which is a right inverse for $\gamma - L_\theta$. $\square$

Remark 3.5. By Theorem 3.2 and Proposition 3.4, if $0 < \theta < 1/(q+1)$ the interior E^{int} of the spectrum of L_θ is contained in the point spectrum. If $x_0, x'_0 \in \mathfrak{X}$ are such that $\mathbf{d}(x_0, x'_0) = (1, 1)$ (i.e., they are distinct children of a same parent vertex) and if ζ_1, ζ_2 are the solutions of $\phi(\zeta) = \gamma$, an $\ell^2(\mathfrak{X})$ eigenfunction f of eigenvalue $\gamma \in E^{\text{int}}$ is easily seen to be given by (always with Notation 3.3)

$$f(x) = \begin{cases} \frac{\zeta_1^{k+1} - \zeta_2^{k+1}}{\zeta_1 - \zeta_2} & \text{if } \mathbf{d}(x_0, x) = (0, k), \\ -\frac{\zeta_1^{k+1} - \zeta_2^{k+1}}{\zeta_1 - \zeta_2} & \text{if } \mathbf{d}(x'_0, x) = (0, k), \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, if $1/(q+1) < \theta < 1$, the interior of the spectrum is contained in the residual spectrum [RS, VI.3 p. 188]. Of course the spectrum has empty interior if $\theta = 1/(q+1)$.

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DIPARTIMENTO DI MATEMATICA “G. CASTELNUOVO”, UNIVERSITÀ DI ROMA “LA SAPIENZA”,
PIAZZALE A. MORO 2, 00185 ROMA, ITALY
E-mail address: `casadio@mat.uniroma1.it`

DIPARTIMENTO DI MATEMATICA “G. CASTELNUOVO”, UNIVERSITÀ DI ROMA “LA SAPIENZA”,
PIAZZALE A. MORO 2, 00185 ROMA, ITALY
E-mail address: `sandroft@mat.uniroma1.it`