WEYL'S THEOREM FOR PERTURBATIONS OF PARANORMAL OPERATORS

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ABSTRACT. A bounded linear operator $T \in L(X)$ on a Banach space X is said to satisfy “Weyl’s theorem” if the complement in the spectrum of the Weyl spectrum is the set of all isolated points of the spectrum which are eigenvalues of finite multiplicity. In this paper we show that if T is a paranormal operator on a Hilbert space, then $T + K$ satisfies Weyl’s theorem for every algebraic operator K which commutes with T .

1. PRELIMINARIES AND NOTATION

Let $L(X)$ be the Banach algebra of all bounded linear operators on an infinite-dimensional complex Banach space X . If $T \in L(X)$ set $\alpha(T) := \dim \ker T$, the dimension of the kernel, and $\beta(T) := \operatorname{codim} T(X)$, the codimension of the range. The class of all *upper semi-Fredholm* operators is defined as the set $\Phi_+(X)$ of all $T \in L(X)$ such that $\alpha(T) < \infty$ and $T(X)$ is closed. The class of all *lower semi-Fredholm* operators is defined as the set $\Phi_-(X)$ of all $T \in L(X)$ such that $\beta(T) < \infty$. The class of all *semi-Fredholm operators* is denoted by $\Phi_\pm(X)$, while by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$ we shall denote the class of all *Fredholm operators*. The *index* of $T \in \Phi_\pm(X)$ is defined by $\operatorname{ind}(T) := \alpha(T) - \beta(T)$. The other two classical quantities associated with a linear operator T are the *ascent* $p := p(T)$, defined as the smallest nonnegative integer p (if it does exist) such that $\ker T^p = \ker T^{p+1}$, and the *descent* $q := q(T)$, defined as the smallest nonnegative integer q (if it does exist) such that $T^q(X) = T^{q+1}(X)$. It is well known that if $p(\lambda I - T)$ and $q(\lambda I - T)$ are both finite, then $p(\lambda I - T) = q(\lambda I - T)$ and λ is a pole of the function resolvent $\lambda \rightarrow (\lambda I - T)^{-1}$, in particular an isolated point of the spectrum $\sigma(T)$; see Proposition 38.3 and Proposition 50.2 of Heuser [14]. The class of *Weyl operators* is defined by

$$W(X) := \{T \in \Phi(X) : \operatorname{ind} T = 0\},$$

while the class of all *Browder operators* is defined by

$$B(X) := \{T \in \Phi(X) : p(T), q(T) < \infty\}.$$

Obviously $B(X) \subseteq W(X)$. The *Weyl spectrum* and the *Browder spectrum* of $T \in L(X)$ are defined by

$$\sigma_w(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\}$$

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and

$$\sigma_b(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin B(X)\}.$$

Recall that $T \in L(X)$ is said to be a *Riesz operator* if $\lambda I - T \in \Phi(X)$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Evidently, every quasi-nilpotent operator is a Riesz operator. It is well known that if Q is a quasi-nilpotent operator commuting with T , then

$$(1) \quad T \text{ invertible} \Leftrightarrow T + Q \text{ invertible},$$

so that $\sigma(T + Q) = \sigma(T)$. Moreover, if R is a Riesz operator commuting with T , then $T \in W(X)$ if and only if $T + R \in W(X)$ (see Lemma 2.2 of [17]), and this obviously implies the equality $\sigma_w(T + Q) = \sigma_w(T)$ for every quasi-nilpotent operator Q commuting with T . It is well known that a similar result holds for $\sigma_b(T)$, i.e., $\sigma_b(T + Q) = \sigma_b(T)$, for every quasi-nilpotent operator Q commuting with T ([17, Proposition 4]).

An operator $T \in L(X)$ is said to admit a *generalized Kato decomposition*, abbreviated GKD, if there exists a pair of T -invariant closed subspaces (M, N) such that $X = M \oplus N$, the restriction $T_M := T|_M$ is semi-regular (i.e., $T(M)$ is closed and $\ker T_M \subseteq (T_M)^n(M)$ for all $n \in \mathbb{N}$), while $T_N := T|_N$ is quasi-nilpotent. A relevant case is obtained if we assume in the definition above that T_N is nilpotent. In this case T is said to be of *Kato type*; see for details Chapter 1 of [1]. A classical result of Kato establishes that every semi-Fredholm operator is of Kato type (see [1, Theorem 1.62]).

The single-valued extension property was introduced in the early years of local spectral theory by Dunford ([10], [11]) and plays an important role in spectral theory; see the recent monographs by Laursen and Neumann [15], or by Aiena [1]. We shall consider the following local version of this property introduced by Finch [12].

Definition 1.1. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have the *single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0 , the only analytic function $f : U \rightarrow X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$.

An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Evidently, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic functions it easily follows that $T \in L(X)$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. This implies, in particular, that both T and T^* have SVEP at every isolated point of $\sigma(T) = \sigma(T^*)$. The SVEP is inherited by the restrictions to closed invariant subspaces, i.e., if $T \in L(X)$ has the SVEP at λ_0 and M is a closed T -invariant subspace, then $T|_M$ has SVEP at λ_0 . Furthermore, if T has SVEP, then $\mu I - T$ has SVEP for every $\mu \in \mathbb{C}$.

The proof of the following result may be found in [1, Corollary 2.12].

Theorem 1.2. Suppose that T has SVEP. If Q is a quasi-nilpotent operator which commutes with T , then also $T + Q$ has SVEP.

The *quasi-nilpotent part* of T is defined by

$$H_0(T) := \{x \in X : \lim_{n \rightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

It is easily seen that $\ker T^m \subseteq H_0(T)$ for every $m \in \mathbb{N}$. Moreover, T is quasi-nilpotent if and only if $H_0(T) = X$ (see [1, Theorem 1.68]), while if T is invertible, then $H_0(T) = \{0\}$.

If $T \in L(X)$, the *analytic core* $K(T)$ is the set of all $x \in X$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in X$ such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$. Note that both $H_0(T)$ and $K(T)$ generally are not closed linear subspaces (for details we refer to [1]) and $T(K(T)) = K(T)$. We have

$$(2) \quad H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda;$$

see [3].

A bounded operator $T \in L(X)$ on a Banach space X is said to be *paranormal* if

$$\|Tx\|^2 \leq \|T^2x\|\|x\| \quad \text{holds for all } x \in X.$$

There is a growing interest concerning paranormal operators [4], [14, §54], [6], [8] and subclasses of paranormal operators [13], [17], [2]. In fact, the class of paranormal operators properly contains a relevant number of Hilbert space operators, among them p -hyponormal operators, log-hyponormal operators, quasi-hyponormal operators ($(A^*A)^2 \leq A^{*2}A^2$), and the class of operators satisfying the condition $|T|^2 \leq |T^2|$; see [13] and [9]. Every paranormal operator on a Hilbert space has SVEP. This may be easily seen as follows: if $\lambda \neq 0$ and $\lambda \neq \mu$, then, by Theorem 2.6 of [6], we have $\|x + y\| \geq \|y\|$ whenever $x \in \ker(\mu I - T)$ and $y \in \ker(\lambda I - T)$. It then follows that if U is an open disc and $f : U \rightarrow X$ is an analytic function such that $0 \neq f(z) \in \ker(zI - T)$ for all $z \in U$, then f fails to be continuous at every $0 \neq \lambda \in U$.

Note that every paranormal operator T is normaloid (i.e., $\|T\| = r(T)$, $r(T)$ the spectral radius of T ; see [14, Proposition 54.6]), so, if T is a quasi-nilpotent paranormal operator, then $T = 0$.

An operator $T \in L(X)$ for which there exists a complex nonconstant polynomial h such that $h(T)$ is paranormal is said to be *algebraically paranormal*. Note that algebraic paranormality is preserved under translation by scalars and under restriction to closed invariant subspaces. Every algebraically paranormal operator has SVEP, since $h(T)$ has SVEP ([1, Theorem 2.40]).

2. WEYL'S THEOREMS

Let us write $\text{iso } K$ for the set of all isolated points of $K \subseteq \mathbb{C}$. For a bounded operator $T \in L(X)$ define

$$\pi_{00}(T) := \{\lambda \in \text{iso } \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}$$

and

$$p_{00}(T) := \sigma(T) \setminus \sigma_b(T) = \{\lambda \in \sigma(T) : \lambda I - T \in \mathcal{B}(X)\},$$

to be the set of all *Riesz points* in $\sigma(T)$. Clearly, $p_{00}(T) \subseteq \pi_{00}(T)$ for every $T \in L(X)$ and $p_{00}(T) = p_{00}(T + Q)$ for every quasi-nilpotent operator Q commuting with T .

Following Coburn [7], we say that *Weyl's theorem holds* for $T \in L(X)$ if

$$(3) \quad \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T).$$

Theorem 2.1 ([2]). *If $T \in L(X)$, then the following assertions are equivalent:*

- (i) *Weyl's theorem holds for T .*
- (ii) *T has SVEP at every point $\lambda \notin \sigma_w(T)$ and $\pi_{00}(T) = p_{00}(T)$.*

Let $\mathcal{P}_0(X)$, X a Banach space, denote the class of all operators $T \in L(X)$ such that there exists $p := p(\lambda) \in \mathbb{N}$ for which

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \pi_{00}(T).$$

Theorem 2.2. *$T \in \mathcal{P}_0(X)$ if and only if $p_{00}(T) = \pi_{00}(T)$. In particular, if T has SVEP, then Weyl's theorem holds for T if and only if $T \in \mathcal{P}_0(X)$.*

Proof. Suppose $T \in \mathcal{P}_0(X)$ and $\lambda \in \pi_{00}(T)$. Then there exists $p \in \mathbb{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$. Since λ is isolated in $\sigma(T)$ then, by [1, Theorem 3.74],

$$X = H_0(\lambda I - T) \oplus K(\lambda I - T) = \ker(\lambda I - T)^p \oplus K(\lambda I - T),$$

from which we obtain

$$(\lambda I - T)^p(X) = (\lambda I - T)^p(K(\lambda I - T)) = K(\lambda I - T),$$

so $X = \ker(\lambda I - T)^p \oplus (\lambda I - T)^p(X)$, which implies, by [1, Theorem 3.6], that $p(\lambda I - T) = q(\lambda I - T) \leq p$. By definition of $\pi_{00}(T)$ we know that $\alpha(\lambda I - T) < \infty$ and this implies by Theorem 3.4 of [1] that $\beta(\lambda I - T)$ is also finite. Therefore $\lambda \in p_{00}(T)$ and hence $\pi_{00}(T) \subseteq p_{00}(T)$. Since the opposite inclusion holds for every operator we then conclude that $p_{00}(T) = \pi_{00}(T)$.

Conversely, if $p_{00}(T) = \pi_{00}(T)$ and $\lambda \in \pi_{00}(T)$, then $p := p(\lambda I - T) = q(\lambda I - T) < \infty$. By Theorem 3.16 of [1] it then follows that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$.

The last assertion is clear from Theorem 2.1. $\square$

In [18] Oudghiri studied the class $\mathcal{P}(X)$ of operators on Banach spaces for which there exists $p := p(\lambda) \in \mathbb{N}$ such that

$$(4) \quad H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathbb{C}.$$

Evidently, if $T \in \mathcal{P}(X)$, then $T \in \mathcal{P}_0(H)$. By (2) we also see that condition (4) entails that T has SVEP. Consequently, by Theorem 2.2, if $T \in \mathcal{P}(X)$, then Weyl's theorem holds for T ([18]). Note that $\mathcal{P}(X)$ is a rather large class of operators, for instance it contains every generalized scalar operator [18], and this implies that Weyl's theorem holds for p-hyponormal, log-hyponormal, and M-hyponormal operators. The next example shows that paranormal operators on Hilbert spaces may fail property (4).

Example 2.3. Let T be a quasi-hyponormal operator on a Hilbert space H . Then $p(\lambda I - T) \leq 1$ for all $\lambda \in \mathbb{C}$, every eigenvalue is an isolated point of $\sigma(T)$, 0 is an eigenvalue and the corresponding eigenspace $\ker T$ is not reducing; see [5] for details. Clearly, $p(T) = 1$. Suppose that T satisfies property (4). Then $H_0(T) = \ker T$, and since $0 \in \text{iso } \sigma(T)$ by Theorem 3.74 of [1] we have $H = \ker T \oplus K(T)$, from which we obtain $T(H) = T(K(T)) = K(T)$. Hence $H = \ker T \oplus T(H)$. Since $T(H) = K(T)$ is closed (it coincides with the range of the spectral projection associated with $\{0\}$, see [1, Theorem 3.74]), it then follows that $T(H) = (\ker T^*)^\perp$, so $\ker T$ is reducing, a contradiction.

However, we have:

Theorem 2.4. *If $T \in L(H)$ is algebraically paranormal, then every isolated point of the spectrum is a pole of the resolvent. Furthermore, $T \in \mathcal{P}_0(H)$.*

Proof. Note first that every quasi-nilpotent algebraically paranormal operator T is nilpotent. In fact, suppose that h is a polynomial for which $h(T)$ is paranormal. From the spectral mapping theorem we have $\sigma(h(T)) = h(\sigma(T)) = \{h(0)\}$, so $h(0)I - h(T)$ is quasi-nilpotent. Since $h(0)I - h(T)$ is paranormal, then $h(0)I - h(T) = 0$, and hence there are some $n \in \mathbb{N}$ and $\mu \in \mathbb{C}$, such that

$$0 = h(0)I - h(T) = \mu T^m \prod_{i=1}^n (\lambda_i I - T) \quad \text{with } \lambda_i \neq 0.$$

Since all $\lambda_i I - T$ are invertible it then follows that $T^m = 0$.

We show now that for every isolated point λ of $\sigma(T)$ we have $p(\lambda I - T) = q(\lambda I - T) < \infty$, i.e., λ is a pole of the resolvent. If $\lambda \in \text{iso } \sigma(T)$, let P denote the spectral projection associated with $\{\lambda\}$, $M := K(\lambda I - T) = \ker P$ and $N := H_0(\lambda I - T) = P(X)$. Then, by the classical spectral decomposition, (M, N) is a GKD for $\lambda I - T$. Since $\lambda I - T|_N$ is quasi-nilpotent and algebraically paranormal, then $\lambda I - T|_N$ is nilpotent and hence $\lambda I - T$ is of Kato type. The SVEP for T and T^* at λ then implies by Theorem 3.16 and Theorem 3.17 of [1] that both $p(\lambda I - T)$ and $q(\lambda I - T)$ are finite; hence λ is a pole of the resolvent. To show the equality $p_{00}(T) = \pi_{00}(T)$ it suffices to prove the inclusion $\pi_{00}(T) \subseteq p_{00}(T)$. If $\lambda \in \pi_{00}(T)$, then $0 < \alpha(\lambda I - T) < \infty$, and the equality $p(\lambda I - T) = q(\lambda I - T) < \infty$ entails by [1, Theorem 3.4] that $\beta(\lambda I - T) = \alpha(\lambda I - T) < \infty$, so $\lambda \in \sigma(T) \setminus \sigma_b(T) = p_{00}(T)$. $\square$

Weyl's theorem for algebraically paranormal operators T on Hilbert spaces has been proved by Curto and Han [8] (this is also immediate from Theorem 2.2 and Theorem 2.4, once observed that T has SVEP).

A bounded operator T is said to be *algebraic* if there exists a nontrivial polynomial q such that $q(T) = 0$. From the spectral mapping theorem it easily follows that the spectrum of an algebraic operator is a finite set.

It is known that Weyl's theorem is not generally transmitted to perturbation of operators satisfying Weyl's theorem. In [19] Oudghiri proved that if T has property (4) and K is an algebraic operator commuting with T , then $T + K$ satisfies Weyl's theorem. Our main result extends Oudghiri's result to paranormal operators:

Theorem 2.5. *Suppose that $T \in L(H)$ is paranormal, K algebraic and $TK = KT$. Then Weyl's theorem holds for $T + K$.*

The proof of Theorem 2.5 proceeds through some steps, stated below as lemmas. We first begin with a useful remark.

Remark 2.6. Suppose that for a linear operator T we have $\alpha(T) < \infty$. Then $\alpha(T^n) < \infty$ for all $n \in \mathbb{N}$. This may be easily seen by an inductive argument. Suppose that $\dim \ker T^n < \infty$. Since $T(\ker T^{n+1}) \subseteq \ker T^n$, then the restriction $T_0 := T|_{\ker T^{n+1}} : \ker T^{n+1} \rightarrow \ker T^n$ has kernel equal to $\ker T$ so the canonical mapping $\hat{T} : \ker T^{n+1} / \ker T \rightarrow \ker T^n$ is injective. Therefore we have $\dim \ker T^{n+1} / \ker T \leq \dim \ker T^n < \infty$, and since $\dim \ker T < \infty$ we then conclude that $\dim \ker T^{n+1} < \infty$.

Lemma 2.7. *Suppose that $T \in L(X)$ and N is nilpotent such that $TN = NT$. Then $T \in \mathcal{P}_0(X)$ if and only if $T + N \in \mathcal{P}_0(X)$.*

Proof. Suppose that $N^p = 0$. Observe first that without any assumption on T we have

$$(5) \quad \ker T \subseteq \ker (T + N)^p \quad \text{and} \quad \ker (T + N) \subseteq \ker T^p.$$

The first inclusion in (5) is clear, since for $x \in \ker T$ we have

$$(T + N)^p x = N^p x = 0.$$

To show the second inclusion in (5) observe that if $x \in \ker(T + N)$, then $T^p x = (-1)^p N^p x = 0$.

Suppose now that $T \in \mathcal{P}_0(X)$, or equivalently $p_{00}(T) = \pi_{00}(T)$. We show first that $\pi_{00}(T) = \pi_{00}(T + N)$. Let $\lambda \in \pi_{00}(T)$. There is no harm if we suppose $\lambda = 0$. From $\sigma(T + N) = \sigma(T)$ we see that $0 \in \text{iso}(T + N)$. Since $0 \in \pi_{00}(T)$, then $\alpha(T) > 0$ and hence by the first inclusion in (5) we have $\alpha(T + N)^p > 0$ and this obviously implies that $\alpha(T + N) > 0$. To show that $\alpha(T + N) < \infty$, observe that

$$\ker(T + N) \subseteq \ker T^p \subseteq H_0(T).$$

The equality $p_{00}(T) = \pi_{00}(T)$ is equivalent to saying that $H_0(\lambda I - T)$ is finite-dimensional for all $\lambda \in \pi_{00}(T)$ (see [1, Theorem 3.84]), and hence $H_0(T)$ is finite-dimensional. Therefore $\alpha(T + N) < \infty$, so $0 \in \pi_{00}(T + N)$ and the inclusion $\pi_{00}(T) \subseteq \pi_{00}(T + N)$ is proved.

To show the opposite inclusion, assume that $0 \in \pi_{00}(T + N)$. Clearly, $0 \in \text{iso } \sigma(T) = \text{iso } \sigma(T + N)$. By assumption $\alpha(T + N) > 0$, so the second inclusion in (5) entails that $\alpha(T^p) > 0$ and this trivially implies that $\alpha(T) > 0$. We also have $\alpha(T + N) < \infty$ and hence, by Remark 2.6, $\alpha(T + N)^p < \infty$. From the first inclusion in (5) we then conclude that $\alpha(T) < \infty$. This shows that $0 \in \pi_{00}(T)$, so the equality $\pi_{00}(T) = \pi_{00}(T + N)$ is proved.

Finally, if $T \in \mathcal{P}_0(X)$, then $p_{00}(T + N) = p_{00}(T) = \pi_{00}(T) = \pi_{00}(T + N)$, so $T + N \in \mathcal{P}_0(X)$. Conversely, if $T + N \in \mathcal{P}_0(X)$ by symmetry we have

$$p_{00}(T) = p_{00}(T + N) = \pi_{00}(T + N) = \pi_{00}((T + N) - N) = \pi_{00}(T),$$

so the proof is complete. $\square$

From Lemma 2.7 and from the equalities (1) we easily obtain the following Oberai's result [16]: if N is a nilpotent operator commuting with T , then

$$(6) \quad \text{Weyl's theorem holds for } T \Leftrightarrow \text{Weyl's theorem holds for } T + N.$$

Define

$$\pi_{0f}(T) := \{\lambda \in \text{iso } \sigma(T) : \alpha(\lambda I - T) < \infty\}.$$

Lemma 2.8. *Let $T \in L(X)$, X a Banach space. If R is a Riesz operator that commutes with T , then*

$$(7) \quad \pi_{0f}(T + R) \cap \sigma(T) \subseteq \text{iso } \sigma(T).$$

If $T \in L(H)$ is paranormal and N is a nilpotent operator commuting with T , then

$$(8) \quad \pi_{0f}(T + N) \cap \sigma(T) \subseteq p_{00}(T).$$

Proof. The inclusion (7) has been proved in [19, Lemma 2.3].

Assume that T is paranormal, N is nilpotent commuting with T and let $p \in \mathbb{N}$ be such that $N^p = 0$. Let $\lambda \in \pi_{0f}(T + N) \cap \sigma(T)$. By Remark 2.6 we have $\alpha((\lambda I - (T + N))^p) < \infty$. From the first inclusion of (5) it then follows that $\alpha(\lambda I - T) \leq \alpha((\lambda I - (T + N))^p) < \infty$. Now, $\lambda \in \text{iso } \sigma(T + N) = \text{iso } \sigma(T)$ and T is paranormal, so by Theorem 2.4 λ is a pole of the resolvent, or equivalently $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$. By Theorem 3.4 of [1] we then conclude that $\alpha(\lambda I - T) = \beta(\lambda I - T) < \infty$ and hence $\lambda \in p_{00}(T)$, as required. $\square$

Now we can give the proof of our main result.

Proof of Theorem 2.5. Let $\sigma(K) = \{\mu_1, \dots, \mu_n\}$. Denote by P_i the spectral projection associated with K and the spectral set $\{\mu_i\}$. Let $Y_i := P_i(H)$ and $Z_i := \ker P_i$. Then $H = Y_i \oplus Z_i$, the closed subspaces Y_i and Z_i are invariant under T and K , and $\sigma(K|Y_i) = \{\mu_i\}$. Define $K_i := K|Y_i$ and $T_i := T|Y_i$. Clearly, the restrictions T_i and K_i commute for every $i = 1, 2, \dots, n$ and

$$(9) \quad \sigma(T + K) = \sigma(T_i + K_i) \cup \sigma((T + K)|Z_i).$$

We show first that $T_i + K_i \in \mathcal{P}_0(Y_i)$ for every $i = 1, 2, \dots, n$. Let h be a polynomial such that $h(K) = 0$. Then $h(K_i) = h(K)|Y_i = 0$, and from

$$\{0\} = \sigma(h(K_i)) = h(\sigma(K_i)) = h(\{\mu_i\})$$

we obtain that $h(\mu_i) = 0$. Write

$$h(\mu) = (\mu_i - \mu)^\nu q(\mu) \quad \text{with } q(\mu_i) \neq 0.$$

Then $0 = h(K_i) = (\mu_i I - K_i)^\nu q(K_i)$ where $q(K_i)$ is invertible. Hence $(\mu_i I - K_i)^\nu = 0$, so the operators $N_i := \mu_i I - K_i$ are nilpotent for all $i = 1, 2, \dots, n$. Note that

$$(10) \quad T_i + K_i = (\mu_i I + T_i) + (K_i - \mu_i I) = \mu_i I + T_i - N_i.$$

Since every T_i is paranormal and hence algebraically paranormal, then $\mu_i I + T_i$ is algebraically paranormal, and hence by Theorem 2.4, $\mu_i I + T_i \in \mathcal{P}_0(Y_i)$. By Lemma 2.7 it then follows that $T_i + K_i = (\mu_i I + T_i) - N_i \in \mathcal{P}_0(Y_i)$.

We show now that $T + K \in \mathcal{P}_0(H)$. Assume that $\lambda_0 \in \pi_{00}(T + K)$. Then $\lambda_0 \in \text{iso } \sigma(T + K)$ and $\alpha(\lambda_0 I - (T + K)) < \infty$. From equality (10) we have

$$(11) \quad (\lambda_0 - \mu_i)I - (T_i - N_i) = \lambda_0 I - (T_i + K_i).$$

Fix $i \in \mathbb{N}$ such that $1 \leq i \leq n$. We distinguish two cases:

- (I) $(\lambda_0 - \mu_i)I - T_i$ is invertible.
- (II) $(\lambda_0 - \mu_i)I - T_i$ is not invertible.

In the first case, $N_i = \mu_i I - K_i$ being quasi-nilpotent and commuting with $(\lambda_0 - \mu_i)I - T_i$, from the equivalence (1) we deduce that $(\lambda_0 - \mu_i)I - T_i + N_i = \lambda_0 I - (T_i + K_i)$ is invertible and hence

$$(12) \quad H_0(\lambda_0 I - (T_i + K_i)) = \ker(\lambda_0 I - (T_i + K_i)) = \{0\}.$$

In the second case we have $\lambda_0 - \mu_i \in \sigma(T_i)$. We claim that $\lambda_0 \in \pi_{00}(T_i + K_i)$. From (9), taking into account equality (10), we see that

$$(13) \quad \lambda_0 \in \text{iso } \sigma(T_i + K_i) = \text{iso } \sigma(\mu_i I + (T_i - N_i)),$$

from which we obtain $\lambda_0 - \mu_i \in \text{iso } \sigma(T_i - N_i)$. Moreover, since $\alpha(\lambda_0 I - (T + K)) < \infty$, the inclusion

$$\ker(\lambda_0 I - (T_i + K_i)) \subseteq \ker(\lambda_0 I - (T + K))$$

implies that $\alpha(\lambda_0 I - (T_i + K_i)) = \alpha((\lambda_0 - \mu_i)I - (T_i - N_i)) < \infty$. Therefore $\lambda_0 - \mu_i \in \pi_{0f}(T_i - N_i) \cap \sigma(T_i)$. Since T_i is paranormal and T_i satisfies Weyl's theorem, by Lemma 2.8 and Theorem 2.1 we then obtain that

$$\lambda_0 - \mu_i \in p_{00}(T_i) = \pi_{00}(T_i) = \sigma(T_i) \setminus \sigma_w(T_i).$$

From equalities (1) we also have

$$\sigma(T_i) \setminus \sigma_w(T_i) = \sigma(T_i - N_i) \setminus \sigma_w(T_i - N_i),$$

so $(\lambda_0 - \mu_i)I - (T_i - N_i) = \lambda_0 I - (T_i + K_i)$ is Weyl.

On the other hand, $\lambda_0 - \mu_i \in \sigma(T_i) = \sigma(T_i - N_i)$; hence $(\lambda_0 - \mu_i)I - (T_i - N_i) = \lambda_0 I - (T_i + K_i)$ is not invertible. This implies that $\alpha(\lambda_0 I - (T_i + K_i)) > 0$; otherwise, $\lambda_0 I - (T_i + K_i)$ being Weyl we would have $\alpha(\lambda_0 I - (T_i + K_i)) = \beta(\lambda_0 I - (T_i + K_i)) = 0$ and hence $\lambda_0 \notin \sigma(T_i + K_i)$, contradicting (13).

Therefore, $\lambda_0 \in \pi_{00}(T_i + K_i)$. Since $T_i + K_i \in \mathcal{P}_0(Y_i)$ we then conclude that there exists $\nu_i \in \mathbb{N}$ such that

$$(14) \quad H_0(\lambda_0 I - (T_i + K_i)) = \ker(\lambda_0 I - (T_i + K_i))^{\nu_i}.$$

From the equality

$$H_0(\lambda_0 I - (T + K)) = \bigoplus_{i=1}^n H_0(\lambda_0 I - (T_i + K_i)),$$

and taking into account equalities (12) and (14), we then obtain

$$\begin{aligned} H_0(\lambda_0 I - (T + K)) &= \bigoplus_{i=1}^n \ker(\lambda_0 I - (T_i + K_i))^{\nu_i} \\ &= \ker(\lambda_0 I - (T + K))^\nu, \end{aligned}$$

where $\nu := \max\{\nu_1, \dots, \nu_n\}$. The last equalities hold for every $\lambda_0 \in \pi_{00}(T + K)$, so $T + K \in \mathcal{P}_0(H)$.

We show now that $T + K$ has SVEP. Clearly, every T_i has SVEP, since T_i is paranormal. By Theorem 1.2 it then follows that all $T_i - N_i$ have SVEP, and hence every $T_i + K_i = \mu_i I + (T_i - N_i)$ has SVEP. From Theorem 2.9 of [1] we obtain that

$$T + K = \bigoplus_{i=1}^n (T_i + K_i)$$

has SVEP. By Theorem 2.2 we then conclude that $T + K$ satisfies Weyl's theorem, as desired.

It is well known that if for an operator $F \in L(X)$ there exists a positive integer n for which F^n is finite-dimensional, then F is algebraic. $\square$

Corollary 2.9. *If $T \in L(H)$ is paranormal and F is an operator that commutes with T and such that F^n is a finite-dimensional operator for some positive integer n , then $T + F$ satisfies Weyl's theorem.*

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