ON CLARKSON-MCCARTHY INEQUALITIES FOR n -TUPLES OF OPERATORS

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ABSTRACT. In this paper we obtain analogues of Clarkson-McCarthy inequalities for n -tuples of operators from Schatten ideals C_p . Using them, we extend the results of Bhatia and Kittaneh on inequalities for partitioned operators and for Cartesian decomposition of operators from C_p .

1. INTRODUCTION

Noncommutative analogues of Clarkson inequalities for pairs of operators in Schatten ideals obtained by McCarthy in [M] play an important role in analysis and operator theory. They were generalized for general symmetric norms by Bhatia and Holbrook [BH] and Hirzallah and Kittaneh [HK]. In [BK3] Bhatia and Kittaneh proved analogues of Clarkson-McCarthy inequalities for n -tuples of operators of special type. In this paper we obtain analogues of Clarkson-McCarthy inequalities for all n -tuples of operators. Using them, we extend the results of Bhatia and Kittaneh [BK1, BK2] on inequalities for partitioned operators and for Cartesian decompositions of operators.

Let $(B(H), \|\cdot\|)$ be the algebra of all bounded operators on a separable Hilbert space H and let $C(H)$ be the ideal of all compact operators in $B(H)$. A two-sided ideal J of $B(H)$ is symmetrically normed (s. n.) if (see [GK]) it is a Banach space in its own norm $\|\cdot\|_J$ and

$$\|AXB\|_J \leq \|A\| \|X\|_J \|B\| \text{ for } A, B \in B(H) \text{ and } X \in J.$$

By Calkin's theorem, all s. n. ideals lie in $C(H)$.

Let $A \in C(H)$ and let $\{s_j\}$ be all eigenvalues of the operator $|A| = (A^*A)^{1/2}$. For $0 < p < \infty$, all

$$C_p = \{A \in C(H): \|A\|_p = \left(\sum_j s_j^p \right)^{1/p} < \infty\}$$

are linear spaces. For $1 \leq p < \infty$, each C_p is a Schatten s. n. ideal of $B(H)$ with norm $\|\cdot\|_p$.

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Set $q = \frac{p}{p-1}$, if $1 < p < \infty$. McCarthy [M] (see also [S]) obtained the following noncommutative analogues of Clarkson estimates for L_p -spaces:

$$(1) \quad 2(\|A\|_p^p + \|B\|_p^p) \leq \|A + B\|_p^p + \|A - B\|_p^p \leq 2^{p-1}(\|A\|_p^p + \|B\|_p^p),$$

$$(2) \quad 2(\|A\|_p^p + \|B\|_p^p)^{q/p} \leq \|A + B\|_p^q + \|A - B\|_p^q,$$

for $A, B \in C_p$, if $2 \leq p < \infty$. For $1 < p < 2$ the above inequalities are reversed.

Hirzallah and Kittaneh [HK] generalized (1) and proved that, for any s. n. ideal J and $A, B \in J$,

$$(3) \quad 2(\| |A|^p + |B|^p \|_J) \leq \| |A + B|^p + |A - B|^p \|_J \leq 2^{p-1}(\| |A|^p + |B|^p \|_J),$$

if $2 \leq p < \infty$. For $0 < p < 2$, these inequalities are reversed.

Denote by $C_p(n)$ the linear space of all columns $\bar{A} = (A_j)$, $1 \leq j \leq n$, with $A_j \in C_p$. Let H^n be the orthogonal sum of n copies of the Hilbert space H . Each operator R in $B(H^n)$ can be represented as an $n \times n$ block-matrix operator $R = (R_{jk})$ with $R_{jk} \in B(H)$. Bhatia and Kittaneh in [BK3] obtained analogues of inequalities (1)-(3) for n -tuples of operators $\bar{A} \in C_p(n)$. Let $\bar{B} = R\bar{A}$, where the R_{jk} have the particular form

$$(4) \quad R_{jk} = \exp\left(i \frac{2\pi(j-1)(k-1)}{n}\right) \mathbf{1}.$$

They proved that if $2 \leq p < \infty$, then

$$(5) \quad n^{\frac{2}{p}} \sum_{j=1}^n \|A_j\|_p^2 \leq \sum_{j=1}^n \|B_j\|_p^2 \leq n^{2-\frac{2}{p}} \sum_{j=1}^n \|A_j\|_p^2,$$

$$(6) \quad n \sum_{j=1}^n \|A_j\|_p^p \leq \sum_{j=1}^n \|B_j\|_p^p \leq n^{p-1} \sum_{j=1}^n \|A_j\|_p^p,$$

$$(7) \quad n \left(\sum_{j=1}^n \|A_j\|_p^p \right)^{q/p} \leq \sum_{j=1}^n \|B_j\|_p^q, \text{ where } q = \frac{p}{p-1}.$$

For $0 < p < 2$, inequalities (5) and (6) are reversed; for $1 < p < 2$, inequality (7) is reversed.

For a s. n. ideal J , denote by $J(n)$ the linear space of all columns $\bar{A} = (A_j)$, $1 \leq j \leq n$, with all $A_j \in J$, and set $\bar{B} = R\bar{A}$. It was proved in [BK3] that if the R_{jk} have the form given in (4), then

$$(8) \quad n \left\| \sum_{j=1}^n |A_j|^p \right\|_J \leq \left\| \sum_{j=1}^n |B_j|^p \right\|_J \leq n^{p-1} \left\| \sum_{j=1}^n |A_j|^p \right\|_J, \text{ for } 2 \leq p < \infty.$$

For $0 < p < 2$, the above inequalities are reversed.

For $n = 2$, inequalities (5)-(8) give inequalities (1)-(3).

In [BK3] Bhatia and Kittaneh further extended inequalities (5)-(8) to the case when

$$(9) \quad R_{jk} = r_{jk} \mathbf{1} \text{ with } r_{jk} \in \mathbb{C}, R^*R = |z| \mathbf{1}_n \text{ and } R^2 = zP,$$

where $z \in \mathbb{C}$ and P is a permutation matrix. For $\overline{A} \in C_p(n)$ and $\overline{B} = R\overline{A}$, they proved that

$$(10) \quad n^{\frac{\lambda}{p}-\frac{\lambda}{2}} |z|^{\lambda/2} \sum_{j=1}^n \|A_j\|_p^\lambda \leq \sum_{j=1}^n \|B_j\|_p^\lambda \leq n^{-(\frac{\lambda}{p}-\frac{\lambda}{2})} |z|^{\lambda/2} \sum_{j=1}^n \|A_j\|_p^\lambda,$$

if $2 \leq \lambda \leq p < \infty$,

$$(11) \quad \sum_{j=1}^n \|B_j\|_p^q \leq r^{q-2} |z| \left(\sum_{j=1}^n \|A_j\|_p^p \right)^{q/p}, \text{ if } 1 < p \leq 2,$$

where $r = \max |r_{jk}|$. For $\overline{A} \in J(n)$ and $\overline{B} = R\overline{A}$, they showed that, for $2 \leq p < \infty$,

$$(12) \quad n^{1-\frac{p}{2}} |z|^{p/2} \left\| \sum_{j=1}^n |A_j|^p \right\|_J \leq \left\| \sum_{j=1}^n |B_j|^p \right\|_J \leq n^{-(1-\frac{p}{2})} |z|^{p/2} \left\| \sum_{j=1}^n |A_j|^p \right\|_J.$$

In this paper we obtain the analogues of inequalities (5)-(12) for all invertible operators R in $B(H^n)$. Denote $\alpha = \|R^{-1}\|$, $\beta = \|R\|$. We will prove the following theorems.

Theorem 1. Let $\overline{A} \in C_p(n)$ and $\overline{B} = R\overline{A}$, where R is invertible. If $2 \leq p$ and $\lambda, \mu \in [2, p]$, or if $0 < p \leq 2$ and $\lambda, \mu \in [p, 2]$, then

$$n^{-|\frac{1}{p}-\frac{1}{2}|} \alpha^{-1} \left(\frac{1}{n} \sum_{j=1}^n \|A_j\|_p^\mu \right)^{\frac{1}{\mu}} \leq \left(\frac{1}{n} \sum_{j=1}^n \|B_j\|_p^\lambda \right)^{\frac{1}{\lambda}} \leq n^{|\frac{1}{p}-\frac{1}{2}|} \beta \left(\frac{1}{n} \sum_{j=1}^n \|A_j\|_p^\mu \right)^{\frac{1}{\mu}}.$$

Theorem 2. Let $\overline{A} \in C_p(n)$ and $\overline{B} = R\overline{A}$ where $R = (R_{jk})$ is invertible. Let $r = \max \|R_{jk}\|$, $\rho = \max \|(R^{-1})_{jk}\|$ and $q = \frac{p}{p-1}$. Then

$$\left(\sum_{j=1}^n \|A_j\|_p^p \right)^{\frac{1}{p}} \leq \rho^{1-\frac{2}{p}} \alpha^{\frac{2}{p}} \left(\sum_{j=1}^n \|B_j\|_p^q \right)^{\frac{1}{q}} \text{ for } 2 \leq p;$$

$$\left(\sum_{j=1}^n \|B_j\|_p^q \right)^{\frac{1}{q}} \leq r^{\frac{2}{p}-1} \beta^{\frac{2}{q}} \left(\sum_{j=1}^n \|A_j\|_p^p \right)^{\frac{1}{p}} \text{ for } 1 < p \leq 2.$$

Theorem 3. Let J be an s. n. ideal. For $\overline{A} \in J(n)$, let $\overline{B} = R\overline{A}$, where R is invertible. Then

$$n^{-|\frac{p}{2}-1|} \alpha^{-p} \left\| \sum_{j=1}^n |A_j|^p \right\|_J \leq \left\| \sum_{j=1}^n |B_j|^p \right\|_J \leq n^{|\frac{p}{2}-1|} \beta^p \left\| \sum_{j=1}^n |A_j|^p \right\|_J \text{ for } 0 < p < \infty.$$

We will prove Theorems 1 and 3 by obtaining some inequalities for the form

$$(\overline{A}, \overline{A}) = A_1^* A_1 + \cdots + A_n^* A_n \text{ with } \overline{A} \in J(n).$$

Denote by $\mathcal{P}_n$ the set of all n -tuples $P = \{P_j\}_{j=1}^n$ of mutually orthogonal infinite-dimensional projections in $B(H)$ with $\sum_{j=1}^n P_j = \mathbf{1}$. We will use the inequalities for $(\overline{A}, \overline{A})$ to prove the following result for partitioned operators.

Theorem 4. Let $\{P_i\}_{i=1}^n \in \mathcal{P}_n$, $\{Q_j\}_{j=1}^m \in \mathcal{P}_m$ and let $A \in C_p$. For $2 \leq \lambda \leq p < \infty$,

$$(13) \quad (nm)^{\frac{\lambda}{p}-1} \sum_{i,j} \|P_i A Q_j\|_p^\lambda \leq \|A\|_p^\lambda \leq (nm)^{\frac{\lambda}{2}-1} \sum_{i,j} \|P_i A Q_j\|_p^\lambda.$$

For $0 < p \leq \lambda \leq 2$, the inequalities are reversed.

Theorem 4 extends the results of [BK1] proved there for $n = m$, $P_i = Q_i$, $\lambda = 2$ and $\lambda = p$.

Let $\bar{A} = (A_j) \in C_p(n)$ and let $A_j = X_j + iY_j$ be their “Cartesian decompositions,” where X_j, Y_j are selfadjoint operators. Using Theorem 1, we will prove the following theorem.

Theorem 5. (i) Let $2 \leq p$. Then, for $0 < \lambda \leq p$ and $2 \leq \mu$,

$$n^{\frac{1}{p}} 2^{\frac{1}{p}-\frac{1}{2}} \left(\frac{1}{n} \sum_{j=1}^n \|A_j\|_p^\lambda \right)^{\frac{1}{\lambda}} \leq \left\| \sum_{j=1}^n (X_j^2 + Y_j^2) \right\|_{p/2}^{1/2} \leq n^{\frac{1}{2}} \left(\frac{1}{n} \sum_{j=1}^n \|A_j\|_p^\mu \right)^{\frac{1}{\mu}}.$$

If $0 < p \leq 2$, then, for $p \leq \lambda$ and $0 < \mu \leq 2$, the inequalities are reversed.

(ii) Set $\tau(p) = \left\lfloor \frac{1}{p} - \frac{1}{2} \right\rfloor$. If $p \geq 2$ and $\lambda, \mu \in [2, p]$, or if $0 < p < 2$ and $\lambda, \mu \in [p, 2]$, then

$$\begin{aligned} (2n)^{-\tau(p)} \left(\frac{1}{n} \sum_{j=1}^n (\|X_j\|_p^\mu + \|Y_j\|_p^\mu) \right)^{\frac{1}{\mu}} &\leq 2^{\frac{1}{\mu}-\frac{1}{2}} \left(\frac{1}{n} \sum_{j=1}^n \|A_j\|_p^\lambda \right)^{\frac{1}{\lambda}} \\ &\leq (2n)^{\tau(p)} \left(\frac{1}{n} \sum_{j=1}^n (\|X_j\|_p^\mu + \|Y_j\|_p^\mu) \right)^{\frac{1}{\mu}}. \end{aligned}$$

For $n = 1$, part (i) of Theorem 5 was proved in [BK2, Theorem 1]; part (ii) was proved, for $\mu = 2$, in [BK2, Corollary 1].

2. PROOFS OF THEOREMS 1, 4 AND 5

It follows from the definition of the norm $\|\cdot\|_p$ that

$$(14) \quad \|A\|_p^2 = \|A^* A\|_{\frac{p}{2}} = \|A A^*\|_{\frac{p}{2}} = \|A^*\|_p^2, \text{ for } A \in C_p \text{ and } 0 < p < \infty.$$

Let $\{A_j\}_{j=1}^n$ be positive operators in C_p . McCarthy in [M] (see also [S] and [BK1]) proved that

$$(15) \quad \|A_1\|_p^p + \cdots + \|A_n\|_p^p \leq \|A_1 + \cdots + A_n\|_p^p \text{ for } 1 \leq p < \infty.$$

Bhatia and Kittaneh showed in Lemma 1 and in formula (7) of [BK2] that

$$(16) \quad \left(\sum_{j=1}^n \|A_j\|_p^p \right)^{\frac{1}{p}} \geq \left\| \sum_{j=1}^n A_j \right\|_p \geq \sum_{j=1}^n \|A_j\|_p \text{ for } 0 < p < 1.$$

The function $f(x) = x^\lambda$ is concave on $[0, \infty)$ for $0 \leq \lambda \leq 1$, and convex for $1 \leq \lambda < \infty$. Hence

$$(17) \quad n^{\lambda-1} (x_1^\lambda + \cdots + x_n^\lambda) \leq (x_1 + \cdots + x_n)^\lambda \text{ for } 0 \leq \lambda \leq 1,$$

$$(18) \quad n^{\lambda-1} (x_1^\lambda + \cdots + x_n^\lambda) \geq (x_1 + \cdots + x_n)^\lambda \text{ for } 1 \leq \lambda < \infty,$$

when $x_i \geq 0$. For $\overline{A} \in C_p(n)$, the operator $(\overline{A}, \overline{A}) = A_1^* A_1 + \cdots + A_n^* A_n$ belongs to $C_{p/2}$.

Lemma 6. (i) Let $2 \leq p$ and $\overline{A} \in C_p(n)$. For $0 < \lambda \leq p$ and $2 \leq \mu$,

$$n^{\frac{1}{p}} \left(\frac{1}{n} \sum_{j=1}^n \|A_j\|_p^\lambda \right)^{\frac{1}{\lambda}} \leq \left(\sum_{j=1}^n \|A_j\|_p^p \right)^{\frac{1}{p}} \leq \left\| \sum_{j=1}^n A_j^* A_j \right\|_{p/2}^{1/2} \leq n^{\frac{1}{2}} \left(\frac{1}{n} \sum_{j=1}^n \|A_j\|_p^\mu \right)^{\frac{1}{\mu}}.$$

(ii) If $0 < p \leq 2$, then, for $p \leq \lambda$ and $0 < \mu \leq 2$, all inequalities are reversed.

Proof. Let $2 \leq p$ and let $0 < \lambda \leq p$ and $2 \leq \mu$. Making use of the triangle inequality (ti), we prove part (i):

$$\begin{aligned} n^{\frac{1}{p}} \left(\frac{1}{n} \sum_{j=1}^n \|A_j\|_p^\lambda \right)^{\frac{1}{\lambda}} &= \left(n^{\frac{\lambda}{p}-1} \sum_{j=1}^n (\|A_j\|_p^p)^{\lambda/p} \right)^{\frac{1}{\lambda}} \stackrel{(17)}{\leq} \left(\sum_{j=1}^n \|A_j\|_p^p \right)^{\frac{1}{p}} \\ &\stackrel{(14)}{=} \left(\sum_{j=1}^n \|A_j^* A_j\|_{p/2}^{p/2} \right)^{\frac{1}{p}} \stackrel{(15)}{\leq} \left(\left\| \sum_{j=1}^n A_j^* A_j \right\|_{p/2}^{p/2} \right)^{\frac{1}{p}} = \left\| \sum_{j=1}^n A_j^* A_j \right\|_{p/2}^{1/2} \\ &\stackrel{(ti)}{\leq} \left(\left(\sum_{j=1}^n \|A_j^* A_j\|_{\frac{p}{2}} \right)^{\frac{\mu}{2}} \right)^{\frac{1}{\mu}} \stackrel{(18)}{\leq} \left(n^{\frac{\mu}{2}-1} \sum_{j=1}^n \|A_j^* A_j\|_{p/2}^{\mu/2} \right)^{\frac{1}{\mu}} \\ &\stackrel{(14)}{=} n^{\frac{1}{2}} \left(\frac{1}{n} \sum_{j=1}^n \|A_j\|_p^\mu \right)^{\frac{1}{\mu}}. \end{aligned}$$

One proves (ii), by replacing $\leq$ by $\geq$, (15) by (16), (ti) by (16) and exchanging (17) and (18). $\square$

Let $A_j \in C_p$ have mutually orthogonal ranges: $A_k^* A_j = 0$ if $j \neq k$. Then

$$\begin{aligned} \|A_1^* A_1 + \cdots + A_n^* A_n\|_{p/2} &= \|(A_1 + \cdots + A_n)^*(A_1 + \cdots + A_n)\|_{p/2} \\ &\stackrel{(14)}{=} \|A_1 + \cdots + A_n\|_p^2 \end{aligned}$$

and Lemma 6 gives us the following estimates for $\|\sum_{j=1}^n A_j\|_p$, when $2 \leq \mu = \lambda \leq p$:

$$(19) \quad n^{\frac{\lambda}{p}-1} (\|A_1\|_p^\lambda + \cdots + \|A_n\|_p^\lambda) \leq \|A_1 + \cdots + A_n\|_p^\lambda \leq n^{\frac{\lambda}{2}-1} (\|A_1\|_p^\lambda + \cdots + \|A_n\|_p^\lambda).$$

For $0 < p \leq \mu = \lambda \leq 2$, the inequalities are reversed.

Proof of Theorem 4. Let $2 \leq \lambda \leq p$. Set $A_i = P_i A$. Then $A = \sum_i A_i$ and, by (19),

$$n^{\frac{\lambda}{p}-1} (\|A_1\|_p^\lambda + \cdots + \|A_n\|_p^\lambda) \leq \|A\|_p^\lambda \leq n^{\frac{\lambda}{2}-1} (\|A_1\|_p^\lambda + \cdots + \|A_n\|_p^\lambda).$$

We have $A_i^* = \sum_j Q_j A_i^*$. Since all $Q_j A_i^*$ have mutually orthogonal ranges, we have from (19),

$$m^{\frac{\lambda}{p}-1} \sum_{j=1}^m \|Q_j A_i^*\|_p^\lambda \leq \|A_i^*\|_p^\lambda \leq m^{\frac{\lambda}{2}-1} \sum_{j=1}^m \|Q_j A_i^*\|_p^\lambda.$$

Since $\|A_i\|_p = \|A_i^*\|_p$ and $\|Q_j A_i^*\|_p = \|P_i A Q_j\|_p$, we obtain (13) by combining the above inequalities.

For $0 < p \leq \lambda \leq 2$, the inequalities are reversed. Theorem 4 is proved. $\square$

Denote by $\langle \cdot, \cdot \rangle$ the scalar product on H . Let S, T be positive compact operators on H and let f and g be increasing functions on $[0, \infty)$. If $g(\langle Sx, x \rangle) \leq f(\langle Tx, x \rangle)$, for all $x \in H$ with $\|x\| = 1$, then for each s. n. ideal J ,

$$(20) \quad f(T) \in J \text{ implies } g(S) \in J \text{ and } \|g(S)\|_J \leq \|f(T)\|_J.$$

Indeed, it follows from the Minimax principal (see [GK, II.1] or [B, p. 58]) that $s_j(g(S)) \leq s_j(f(T))$ for all j . Hence, by [GK, Lemma III.3.2], $g(S) \in J$ and $\|g(S)\|_J \leq \|f(T)\|_J$. In particular,

$$(21) \quad 0 < S \leq T \text{ and } T \in J \text{ imply } S \in J \text{ and } \|S\|_J \leq \|T\|_J.$$

Corollary 7. Let $R \in B(H^n)$ and $\beta = \|R\|$. For $\bar{A} \in C_p(n)$, set $\bar{B} = R\bar{A}$. Let

either 1) $2 \leq p$, $0 < \lambda \leq p$ and $2 \leq \mu$; or 2) $0 < p \leq 2$, $0 < \lambda \leq 2$ and $p \leq \mu$.

Then

$$\left(\frac{1}{n} (\|B_1\|_p^\lambda + \cdots + \|B_n\|_p^\lambda) \right)^{\frac{1}{\lambda}} \leq \beta n^{\frac{1}{p} - \frac{1}{2}} \left(\frac{1}{n} (\|A_1\|_p^\mu + \cdots + \|A_n\|_p^\mu) \right)^{\frac{1}{\mu}}.$$

Proof. Since $K = \beta^2 \mathbf{1} - R^*R$ is a positive operator, we have

$$(22) \quad \beta^2 (\bar{A}, \bar{A}) = (\bar{A}, R^*R\bar{A}) + (\bar{A}, K\bar{A}) = (R\bar{A}, R\bar{A}) + (K^{1/2}\bar{A}, K^{1/2}\bar{A}) \geq (\bar{B}, \bar{B}).$$

Let $2 \leq p$, $0 < \lambda \leq p$ and $2 \leq \mu$. It follows from Lemma 6(i) and (21) that

$$n^{\frac{1}{p}} \left(\frac{1}{n} \sum_{j=1}^n \|B_j\|_p^\lambda \right)^{\frac{1}{\lambda}} \leq \|(\bar{B}, \bar{B})\|_{p/2}^{1/2} \leq \beta \|(\bar{A}, \bar{A})\|_{p/2}^{1/2} \leq \beta n^{\frac{1}{2}} \left(\frac{1}{n} \sum_{j=1}^n \|A_j\|_p^\mu \right)^{\frac{1}{\mu}}.$$

Let $0 < p \leq 2$, $0 < \lambda \leq 2$ and $p \leq \mu$. It follows from Lemma 6(ii) and (16) that

$$n^{\frac{1}{2}} \left(\frac{1}{n} \sum_{j=1}^n \|B_j\|_p^\lambda \right)^{\frac{1}{\lambda}} \leq \|(\bar{B}, \bar{B})\|_{p/2}^{1/2} \leq \beta \|(\bar{A}, \bar{A})\|_{p/2}^{1/2} \leq \beta n^{\frac{1}{p}} \left(\frac{1}{n} \sum_{j=1}^n \|A_j\|_p^\mu \right)^{\frac{1}{\mu}},$$

which completes the proof. $\square$

Proof of Theorem 1. Let $2 \leq p$ and $\lambda, \mu \in [2, p]$. By Corollary 7,

$$n^{\frac{1}{p}} \left(\frac{1}{n} (\|B_1\|_p^\lambda + \cdots + \|B_n\|_p^\lambda) \right)^{\frac{1}{\lambda}} \leq \beta n^{\frac{1}{2}} \left(\frac{1}{n} (\|A_1\|_p^\mu + \cdots + \|A_n\|_p^\mu) \right)^{\frac{1}{\mu}}.$$

Since $\bar{A} = R^{-1}\bar{B}$, interchanging in the above inequality λ and μ , A_j and B_j , and replacing β by α , we have

$$n^{\frac{1}{p}} \left(\frac{1}{n} (\|A_1\|_p^\mu + \cdots + \|A_n\|_p^\mu) \right)^{\frac{1}{\mu}} \leq \alpha n^{\frac{1}{2}} \left(\frac{1}{n} (\|B_1\|_p^\lambda + \cdots + \|B_n\|_p^\lambda) \right)^{\frac{1}{\lambda}}.$$

Combining the above inequalities, we complete the proof of this case. The case when $0 < p \leq 2$ and $\lambda, \mu \in [p, 2]$ is proved in the same way. $\square$

Remarks. 1) Let $R = (R_{jk})$ with R_{jk} satisfying (4). Then $\beta = \alpha^{-1} = n^{\frac{1}{2}}$.

(i) Let $\lambda = \mu$ be 2 or p . If $2 \leq p$, then Theorem 1 gives formula (5) and, if $0 < p < 2$, its reverse.

(ii) Let $2 \leq p = \mu$ in Corollary 7. For $0 < \lambda \leq p$, we have an inequality complementary to (7):

$$n^{\frac{1}{p}} \left(\frac{1}{n} (\|B_1\|_p^\lambda + \cdots + \|B_n\|_p^\lambda) \right)^{\frac{1}{\lambda}} \leq n \left(\frac{1}{n} (\|A_1\|_p^p + \cdots + \|A_n\|_p^p) \right)^{\frac{1}{p}}.$$

2) If R satisfies (9), then $\beta = \alpha^{-1} = |z|^{1/2}$. For $\lambda = \mu$, Theorem 1 gives formula (10).

For $1 \leq \mu = p < 2$, Corollary 7 gives a formula complementary to (11):

$$n^{\frac{1}{2}} |z|^{\frac{1}{2}} \left(\frac{1}{n} (\|A_1\|_p^p + \cdots + \|A_n\|_p^p) \right)^{\frac{1}{p}} \leq n^{1/p} \left(\frac{1}{n} (\|B_1\|_p^\lambda + \cdots + \|B_n\|_p^\lambda) \right)^{\frac{1}{\lambda}}.$$

3) Let R be a unitary operator. If either $2 \leq p$ and $\lambda, \mu \in [2, p]$, or $0 < p \leq 2$ and $\lambda, \mu \in [p, 2]$, then Theorem 1 gives

$$n^{-|\frac{1}{p}-\frac{1}{2}|} \left(\frac{1}{n} \sum_{j=1}^n \|A_j\|_p^\mu \right)^{\frac{1}{\mu}} \leq \left(\frac{1}{n} \sum_{j=1}^n \|B_j\|_p^\lambda \right)^{\frac{1}{\lambda}} \leq n^{|\frac{1}{p}-\frac{1}{2}|} \left(\frac{1}{n} \sum_{j=1}^n \|A_j\|_p^\mu \right)^{\frac{1}{\mu}}.$$

Let $\{P_i\}_{i=1}^n \in \mathcal{P}_n$ be a set of mutually orthogonal projections and $\sum_i P_i = \mathbf{1}$. The operator $R = (R_{jk})$, with $R_{jk} = P_{(k+j-1) \bmod(n)}$, is unitary. In particular, for $n = 2$, for $\mu = \lambda$ between 2 and p and for any projection P , we have

$$\begin{aligned} 2^{-\lambda|\frac{1}{p}-\frac{1}{2}|} (\|A_1\|_p^\lambda + \|A_2\|_p^\lambda) &\leq \|PA_1 + (\mathbf{1} - P)A_2\|_p^\lambda + \|(\mathbf{1} - P)A_1 + PA_2\|_p^\lambda \\ &\leq 2^{\lambda|\frac{1}{p}-\frac{1}{2}|} (\|A_1\|_p^\lambda + \|A_2\|_p^\lambda). \end{aligned}$$

4) Let $\{a_i\}_{i=1}^n$ be positive numbers. The function $g(\lambda) = \left(\frac{1}{n} \sum_{j=1}^n a_j^\lambda \right)^{\frac{1}{\lambda}}$ is monotone increasing. Therefore we get the sharpest inequalities in Theorem 1 if, for $0 < p \leq 2$, we set $\lambda = 2$ and $\mu = p$; and if, for $2 \leq p < \infty$, we set $\lambda = p$ and $\mu = 2$:

$$\begin{aligned} \|B_1\|_p^2 + \cdots + \|B_n\|_p^2 &\leq \beta^2 (\|A_1\|_p^p + \cdots + \|A_n\|_p^p)^{\frac{2}{p}}, \text{ for } 0 < p \leq 2, \\ (\|B_1\|_p^p + \cdots + \|B_n\|_p^p)^{\frac{2}{p}} &\leq \beta^2 (\|A_1\|_p^2 + \cdots + \|A_n\|_p^2), \text{ for } 2 \leq p < \infty. \end{aligned}$$

Let $n = 2$ and $R = (r_{ij}\mathbf{1})$, $r_{11} = r_{12} = r_{21} = -r_{22} = 1$. Then $\beta = \alpha^{-1} = 2^{1/2}$ and we have

$$(23) \quad \|A_1\|_p^2 + \|A_2\|_p^2 \leq \frac{1}{2} (\|A_1 + A_2\|_p^p + \|A_1 - A_2\|_p^p)^{\frac{2}{p}}, \text{ if } 0 < p \leq 2.$$

For $2 \leq p < \infty$, the inequality is reversed.

Compare (23) to the following inequalities proved by Ball, Carlen and Lieb in [BCL]:

$$(24) \quad \|A_1\|_p^2 + (p-1)\|A_2\|_p^2 \leq 2^{-\frac{2}{p}} (\|A_1 + A_2\|_p^p + \|A_1 - A_2\|_p^p)^{\frac{2}{p}}, \text{ for } 1 \leq p \leq 2,$$

and the reverse for $2 \leq p < \infty$. If we symmetrize (24), by exchanging A_1 and A_2 and adding the result to (24), we get

$$(25) \quad \|A_1\|_p^2 + \|A_2\|_p^2 \leq \frac{2^{1-\frac{2}{p}}}{p} (\|A_1 + A_2\|_p^p + \|A_1 - A_2\|_p^p)^{\frac{2}{p}}.$$

Since $\frac{1}{2} \leq \frac{2^{1-\frac{2}{p}}}{p}$, for $1 \leq p \leq 2$, (23) is sharper than (25).

The space $C_p(n)$ is a Banach space with norm $\|\bar{A}\|_p = \left(\sum_{j=1}^n \|A_j\|_p^p\right)^{1/p}$. Each operator R in $B(H^n)$ defines a bounded operator on $C_p(n)$: $\bar{A} \rightarrow R\bar{A}$, which we also denote by R . It follows from the second inequality of Theorem 1 that $\|R\|_{C_p(n)} \leq n^{|\frac{1}{p}-\frac{1}{2}|} \|R\|$. Each $P \in \mathcal{P}_n$ generates an isomorphism θ_P of $B(H)$ onto $B(H^n)$: $\theta_P(R) = (R_{jk})$, where $R_{jk} = P_j R P_k$. Set

$$\|R\|_{p,n} = \sup\{\|\theta_P(R)\|_{C_p(n)} : P \in \mathcal{P}_n\}.$$

The above argument yields

Corollary 8. $\|R\|_{p,n} \leq n^{|\frac{1}{p}-\frac{1}{2}|} \|R\|$ for $0 < p < \infty$.

Proof of Theorem 5. For $\bar{A} = (A_j) \in C_p(n)$, let $A_j = X_j + iY_j$, where X_j, Y_j are selfadjoint operators. Set $T_{2j} = A_j$ and $T_{2j-1} = A_j^*$. Then $\bar{T} \in C_p(2n)$,

$$\|T_{2j-1}\|_p = \|T_{2j}\|_p = \|A_j\|_p \text{ and } T_{2j-1}^* T_{2j-1} + T_{2j}^* T_{2j} = A_j A_j^* + A_j^* A_j = 2(X_j^2 + Y_j^2).$$

Taking this into account and using Lemma 6 for $\bar{T}$, we obtain the proof of (i).

Now set $S_{2j-1} = X_j$ and $S_{2j} = Y_j$. Then $\bar{S} \in C_p(2n)$. Consider the $2n \times 2n$ block-matrix operator $R = (R_{km})$ such that all $R_{kk} = \mathbf{1}$, $R_{2j-1, 2j} = R_{2j, 2j-1} = i\mathbf{1}$, for $j = 1, \dots, n$, and all the other entries $R_{km} = 0$. It can be considered as an $n \times n$ block-matrix diagonal operator with each diagonal 2×2 block $\begin{pmatrix} \mathbf{1} & i\mathbf{1} \\ i\mathbf{1} & \mathbf{1} \end{pmatrix}$.

Then $\beta = \|R\| = \|R^* R\|^{1/2} = 2^{1/2}$ and $\alpha = \|R^{-1}\| = 2^{-1/2}$. Set $\bar{B} = R\bar{S}$. Then $B_{2j-1} = X_j + iY_j = A_j$ and $B_{2j} = iX_j + Y_j = iA_j^*$. Replacing $\bar{A}$ by $\bar{S}$ and n by $2n$ in Theorem 1, we obtain the proof of (ii). $\square$

3. PROOF OF THEOREM 2

The proof is similar to the proof of Theorem 4 of [BK3] and should be read together with it. It is based on a refinement of the lemma in Section 5 of [BK3]. Note first that, for $A \in C_p$,

$$(26) \quad |A|^\lambda \in C_{p/\lambda} \text{ and } \||A|^\lambda\|_{p/\lambda} = \|A\|_p^\lambda, \text{ for } \lambda > 0.$$

Lemma 9. Let $1 < p \leq 2$ and $\bar{A} \in C_p(n)$. For $R = (R_{jk}) \in B(H^n)$, set $\bar{B} = R\bar{A}$ and $r = \max \|R_{jk}\|$. Then, for $\bar{Y} \in C_q(n)$,

$$\left| \text{Tr} \sum_{j=1}^n Y_j^* B_j \right| \leq r^{1-\frac{2}{q}} \beta^{2/q} \left(\sum_{k=1}^n \|A_k\|_p^p \right)^{\frac{1}{p}} \left(\sum_{j=1}^n \|Y_j\|_q^p \right)^{\frac{1}{p}}, \text{ where } q = \frac{p}{p-1}.$$

Proof. Let $A_j = |A_j| U_j$ and $Y_j = |Y_j| V_j$ be the polar decompositions. For $z = x + iy$, $\frac{1}{2} \leq x \leq 1$, set

$$A_j(z) = |A_j|^{pz} U_j \text{ and } Y_j(z) = \|Y_j\|_q^{pz-q(1-z)} |Y_j|^{q(1-z)} V_j.$$

By (26), $A_j(z) \in C_{x^{-1}}$ and $Y_j(z) \in C_{(1-x)^{-1}}$,

$$\begin{aligned} \|A_j(z)\|_{x^{-1}} &= \||A_j|^{pz}\|_{x^{-1}} = \|A_j\|_p^{px} \\ \text{and } \|Y_j(z)\|_{(1-x)^{-1}} &= \|Y_j\|_q^{px-q(1-x)} \|Y_j\|_q^{q(1-x)} \|Y_j\|_q^{qx} = \|Y_j\|_q^{px}. \end{aligned}$$

Set $\overline{B(z)} = R\overline{A(z)}$ and

$$f(z) = \text{Tr} \left(\overline{B(z)}, \overline{Y(z)} \right) = \text{Tr} \sum_{j=1}^n Y_j(z)^* B_j(z).$$

Let $z_1 = 1 + iy$. Then all $Y_j(z_1) \in B(H)$ and $A_j(z_1) \in C_1$, so $B_j(z_1) \in C_1$. Hence

$$\begin{aligned} |f(z_1)| &= \left| \text{Tr} \sum_{j=1}^n Y_j(z_1)^* B_j(z_1) \right| \leq \left\| \sum_{j=1}^n Y_j(z_1)^* B_j(z_1) \right\|_1 \leq \sum_{j=1}^n \|Y_j(z_1)\| \|B_j(z_1)\|_1 \\ &\leq \sum_{j=1}^n \|Y_j\|_q^p \left(\sum_{k=1}^n \|R_{jk}\| \|A_k\|_p^p \right) \leq r \left(\sum_{j=1}^n \|Y_j\|_q^p \right) \left(\sum_{k=1}^n \|A_k\|_p^p \right) = M_1. \end{aligned}$$

For $z_2 = \frac{1}{2} + iy$, all $B_j(z_2) \in C_2$ and $Y_j(z_2) \in C_2$. By Theorem 1,

$$\begin{aligned} |f(z_2)| &\leq \left\| \sum_{j=1}^n Y_j(z_2)^* B_j(z_2) \right\|_1 \leq \sum_{j=1}^n \|Y_j(z_2)\|_2 \|B_j(z_2)\|_2 \\ &\leq \left(\sum_{j=1}^n \|Y_j(z_2)\|_2^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^n \|B_j(z_2)\|_2^2 \right)^{\frac{1}{2}} \\ &\leq \left(\sum_{j=1}^n \|Y_j\|_q^p \right)^{\frac{1}{2}} \beta \left(\sum_{j=1}^n \|A_j(z_2)\|_2^2 \right)^{\frac{1}{2}} \\ &\leq \beta \left(\sum_{j=1}^n \|Y_j\|_q^p \right)^{\frac{1}{2}} \left(\sum_{k=1}^n \|A_k\|_p^p \right)^{\frac{1}{2}} = M_2. \end{aligned}$$

Since $\frac{1}{2} \leq \frac{1}{p} \leq 1$, we have from the three-line theorem that

$$\left| f\left(\frac{1}{p}\right) \right| = \left| \text{Tr} \sum_{j=1}^n Y_j^* B_j \right| \leq M_1^{2(\frac{1}{p}-\frac{1}{2})} M_2^{2(1-\frac{1}{p})},$$

which completes the proof of the lemma. $\square$

Proceed now as in [BK3]. Consider the polar decomposition $B_j = W_j |B_j|$ and set $Y_j = \|B_j\|_p^{q-p} W_j |B_j|^{p-1}$. By (26), $Y_j \in C_q$, where $q = \frac{p}{p-1}$,

$$\|Y_j\|_q = \|B_j\|_p^{q-p} \| |B_j|^{p-1} \|_q = \|B_j\|_p^{q-1}$$

and

$$\text{Tr}(Y_j^* B_j) = \|B_j\|_p^{q-p} \text{Tr}(|B_j|^p) = \|B_j\|_p^q.$$

Using Lemma 9, we complete the proof of Theorem 2 for $1 < p \leq 2$.

To prove the theorem for $2 \leq p < \infty$, we use a duality argument. For $\overline{A} \in C_p(n)$, the $n \times n$ block-diagonal operator $A = (A_{jk})$ with $A_{jj} = A_j$ belongs to the Schatten ideal $C_p(H^n)$ of operators on H^n and defines a bounded functional F_A on $C_q(H^n)$

by the formula

$$F_A(X) = \operatorname{Tr}(A^*X) = \operatorname{Tr}\left(\sum_{j=1}^n A_j^* X_{jj}\right) = \operatorname{Tr}((\overline{X}, \overline{A}))$$

$$\text{with } \|F_A\| = \|A\|_p = \left(\sum_{j=1}^n \|A_j\|_p^p\right)^{1/p},$$

where $\overline{X} = (X_{jj}) \in C_q(n)$. Let $\widehat{X}$ be the block-diagonal operator with diagonal blocks X_{jj} . Then $\|\widehat{X}\|_q = \left(\sum_{j=1}^n \|X_{jj}\|_q^q\right)^{1/q}$ and (see [GK]) $\|\widehat{X}\|_q \leq \|X\|_q$. Hence

$$\begin{aligned} \|A\|_p &= \sup \frac{|F_A(X)|}{\|X\|_q} \leq \sup \frac{|\operatorname{Tr}((\overline{X}, \overline{A}))|}{\|\widehat{X}\|_q} = \sup \frac{|\operatorname{Tr}((\overline{X}, R^{-1}\overline{B}))|}{\|\widehat{X}\|_q} \\ &= \sup \frac{|\operatorname{Tr}(((R^{-1})^*\overline{X}, \overline{B}))|}{\|\widehat{X}\|_q}. \end{aligned}$$

Let $\rho = \max \| (R^{-1})_{ij} \|$ and $\alpha = \|R^{-1}\|$. Since $1 \leq q \leq 2$, exchanging p and q in Lemma 9, we have

$$\begin{aligned} |\operatorname{Tr}(((R^{-1})^*\overline{X}, \overline{B}))| &\leq \rho^{1-\frac{2}{p}} \alpha^{\frac{2}{p}} \left(\sum_{j=1}^n \|X_{jj}\|_q^q\right)^{\frac{1}{q}} \left(\sum_{j=1}^n \|B_j\|_p^p\right)^{\frac{1}{q}} \\ &= \rho^{1-\frac{2}{p}} \alpha^{\frac{2}{p}} \|\widehat{X}\|_q \left(\sum_{j=1}^n \|B_j\|_p^p\right)^{\frac{1}{q}}. \end{aligned}$$

Combining the above inequalities, we complete the proof of the theorem for $2 \leq p < \infty$. $\square$

4. GENERALIZATION OF HIRZALLAH-KITTANEH INEQUALITIES: THEOREM 3

Let A be a positive operator. If g is a convex function on $[0, \infty)$, then (see Lemma 1 of [HK])

$$(27) \quad g(\langle Ax, x \rangle) \leq \langle g(A)x, x \rangle, \text{ for } x \in H \text{ with } \|x\| = 1.$$

If g is concave, the inequality is reversed. A continuous function g on $[0, \infty)$ is *operator monotone* if $0 < A \leq B$ implies $g(A) \leq g(B)$ (see [B]).

To generalize Hirzallah-Kittaneh inequalities we need the following result, which gives us some useful inequalities for $(\overline{A}, \overline{A})$. Its proof essentially repeats the elegant proof of Theorem 1 in [HK] and uses Bhatia-Kittaneh's generalization (see [BK3]) of the Ando-Zhan theorem (see [AZ]).

Proposition 10. (i) Let g be an increasing function on $[0, \infty)$ such that $g(0) = 0$ and $\lim_{t \rightarrow \infty} g(t) = \infty$ and let its inverse function g^{-1} be operator monotone. Then, for each s. n. ideal J and all $\overline{A} \in J(n)$,

$$\left\| \sum_j g\left(\frac{1}{n} |A_j|^2\right) \right\|_J \leq \left\| g\left(\frac{1}{n} (\overline{A}, \overline{A})\right) \right\|_J \leq \frac{1}{n} \left\| \sum_j g(|A_j|^2) \right\|_J.$$

(ii) Let g be a nonnegative operator monotone function on $[0, \infty)$. Then, for each s. n. ideal J and each set $\overline{A} = \{A_j\}_{j=1}^n$ of operators in $B(H)$ such that $g(t(\overline{A}, \overline{A})) \in J$, for all $t > 0$, the above inequalities are reversed.

Proof. As g^{-1} is operator monotone, it is concave (see p.120 of [B]). Since g is increasing, it implies that g is convex. Let $\overline{A} \in J(n)$. Then, for each unit vector x in H ,

$$\begin{aligned} \left\langle \sum_j g(|A_j|^2)x, x \right\rangle &= \sum_j \left\langle g(|A_j|^2)x, x \right\rangle \stackrel{(27)}{\geq} \sum_j g\left(\left\langle |A_j|^2 x, x \right\rangle\right) \\ &\geq ng \left(\frac{1}{n} \sum_j \left\langle |A_j|^2 x, x \right\rangle \right) = ng \left(\frac{1}{n} \langle \overline{A}, \overline{A} \rangle x, x \right) \quad (\text{by convexity of } g, \text{ cf. (18)}). \end{aligned}$$

Set $B = \frac{1}{n}(\overline{A}, \overline{A})$. Then $\text{Sp}(B) \subseteq [0, a]$ for some $a > 0$. Since g is convex and $g(0) = 0$, we have $g(x) = xh(x)$ where h is a bounded function on $(0, a]$. Hence $h(B) \in B(H)$, so $g\left(\frac{1}{n}(\overline{A}, \overline{A})\right) = g(B) = Bh(B) \in J$.

As g is an increasing function, setting $f(t) = t$ in (20), we obtain that

$$\left\| \sum_j g(|A_j|^2) \right\|_J \geq n \left\| g\left(\frac{1}{n}(\overline{A}, \overline{A})\right) \right\|_J = n \left\| g\left(\frac{1}{n} \sum_j |A_j|^2\right) \right\|_J.$$

Since g^{-1} is operator monotone, it follows from Theorem 5(ii) of [BK3] that

$$\left\| g\left(\frac{1}{n} \sum_j |A_j|^2\right) \right\|_J \geq \left\| \sum_j g\left(\frac{1}{n} |A_j|^2\right) \right\|_J.$$

Combining the above inequalities, we complete the proof of part (i).

If g is an operator monotone function, it is concave. Using this (cf. (17)) and the reverse of (27), we obtain, similar to the above, that for each unit vector x in H ,

$$\left\langle \sum_j g(|A_j|^2)x, x \right\rangle \leq ng \left(\frac{1}{n} \langle \overline{A}, \overline{A} \rangle x, x \right).$$

Since $g\left(\frac{1}{n}(\overline{A}, \overline{A})\right) \in J$ and g is an increasing function, we have from (20) that $\sum_j g(|A_j|^2) \in J$ and

$$\left\| \sum_j g(|A_j|^2) \right\|_J \leq n \left\| g\left(\frac{1}{n}(\overline{A}, \overline{A})\right) \right\|_J = n \left\| g\left(\frac{1}{n} \sum_j |A_j|^2\right) \right\|_J.$$

Since g is operator monotone, it follows from Theorem 5(i) of [BK3] that

$$\left\| g\left(\frac{1}{n} \sum_j |A_j|^2\right) \right\|_J \leq \left\| \sum_j g\left(\frac{1}{n} |A_j|^2\right) \right\|_J.$$

Combining the above inequalities, we complete the proof. $\square$

Corollary 11. Let R be an invertible $n \times n$ block-matrix operator, $\alpha = \|R^{-1}\|$, $\beta = \|R\|$. Let J be an s. n. ideal and $\overline{B} = R\overline{A}$ for $\overline{A} \in J(n)$. If a function g satisfies the conditions of Proposition 10(i), then

$$(28) \quad n \left\| \sum_j g\left(\frac{\alpha^{-2}|A_j|^2}{n}\right) \right\|_J \leq \left\| \sum_j g(|B_j|^2) \right\|_J \leq \frac{1}{n} \left\| \sum_j g(\beta^2 n |A_j|^2) \right\|_J.$$

If g satisfies the conditions of Proposition 10(ii), then

$$(29) \quad n \left\| \sum_j g\left(\frac{\beta^2 |A_j|^2}{n}\right) \right\|_J \geq \left\| \sum_j g(|B_j|^2) \right\|_J \geq \frac{1}{n} \left\| \sum_j g(\alpha^{-2} n |A_j|^2) \right\|_J.$$

Proof. As in (22), we have $(\overline{A}, \overline{A}) \leq (\alpha \overline{B}, \alpha \overline{B})$ and $(\overline{B}, \overline{B}) \leq (\beta \overline{A}, \beta \overline{A})$. Let g satisfy the conditions of Proposition 10(i). Then

$$\begin{aligned} n \left\| \sum_j g\left(\frac{|A_j|^2}{n}\right) \right\|_J &\leq n \left\| g\left(\frac{(\overline{A}, \overline{A})}{n}\right) \right\|_J \stackrel{(20)}{\leq} n \left\| g\left(\frac{(\alpha \overline{B}, \alpha \overline{B})}{n}\right) \right\|_J \\ &\leq \left\| \sum_j g(|\alpha B_j|^2) \right\|_J. \end{aligned}$$

Similarly,

$$n \left\| \sum_j g\left(\frac{1}{n} |B_j|^2\right) \right\|_J \leq \left\| \sum_j g(|\beta A_j|^2) \right\|_J.$$

Combining these inequalities yields the proof of (28).

As $(\overline{B}, \overline{B}) \leq (\beta \overline{A}, \beta \overline{A})$, it follows from (20) that if $g(t(\overline{A}, \overline{A})) \in J$, for all $t > 0$, then $g(t(\overline{B}, \overline{B})) \in J$. Taking this into account, we obtain (29) in the same way as (28). $\square$

Proof of Theorem 3. For $0 < k < 1$, the function $g(t) = t^k$ is operator monotone; for $k \geq 1$, the inverse of g is operator monotone (see [B, p. 115]). Setting $p = 2k$, we obtain Theorem 3 from (28) and (29). $\square$

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