A QUASICONFORMAL MAPPING CLASS GROUP ACTING TRIVIALY ON THE ASYMPTOTIC TEICHMÜLLER SPACE

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ABSTRACT. For an analytically infinite Riemann surface R , the quasiconformal mapping class group $\text{MCG}(R)$ always acts faithfully on the ordinary Teichmüller space $T(R)$. However in this paper, an example of R is constructed for which $\text{MCG}(R)$ acts trivially on its asymptotic Teichmüller space $AT(R)$.

§1. INTRODUCTION AND STATEMENT OF RESULTS

The asymptotic Teichmüller space $AT(R)$ of a Riemann surface R is a certain quotient space of the Teichmüller space $T(R)$. It was first introduced in [7] when R was the hyperbolic plane and in [3], [4] when R was an arbitrary hyperbolic Riemann surface.

A quasiconformal homeomorphism f of R is called *asymptotically conformal* if, for every $\varepsilon > 0$, there exists a compact subset V of R such that the maximal dilatation $K(f)$ of f is less than $1 + \varepsilon$ on $R - V$. Two quasiconformal homeomorphisms f_1 and f_2 of R are said to be *asymptotically equivalent* if there exists an asymptotically conformal homeomorphism $h : f_1(R) \rightarrow f_2(R)$ such that $f_2^{-1} \circ h \circ f_1 : R \rightarrow R$ is homotopic to the identity relative to the ideal boundary at infinity of R . The *asymptotic Teichmüller space* $AT(R)$ is the set of all asymptotic equivalence classes $[f]_A$ of quasiconformal homeomorphisms f of R . Since a conformal homeomorphism is asymptotically conformal, the ordinary Teichmüller equivalence is stronger than the asymptotic equivalence. Hence there exists a natural projection $\pi : T(R) \rightarrow AT(R)$ that maps each Teichmüller equivalence class $[f] \in T(R)$ to the asymptotic equivalence class $[f]_A \in AT(R)$.

For the complex conjugate R^* of R , let $B(R^*)$ be the complex Banach space of all bounded holomorphic quadratic differentials on R^* with hyperbolic supremum norm and $B_0(R^*)$ the closed subspace of $B(R^*)$ consisting of all bounded holomorphic quadratic differentials φ that vanish at infinity. Here we say that φ vanishes at infinity if for every $\epsilon > 0$ there exists a compact subset $V^* \subset R^*$ such that the norm of $\varphi|_{R^* - V^*}$ taken off V^* is less than ϵ . It is proved in [3] that the asymptotic Teichmüller space $AT(R)$ is a complex Banach manifold modeled on the quotient Banach space $B(R^*)/B_0(R^*)$. In fact, $AT(R)$ is embedded in $B(R^*)/B_0(R^*)$ as a bounded contractible domain, as is proved in [5]. This is the unique complex

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structure on $AT(R)$ such that the projection $\pi : T(R) \rightarrow AT(R)$ is holomorphic with respect to the complex structure on $T(R)$.

A pseudo-distance d_A on $AT(R)$ is induced from the Teichmüller distance d on $T(R)$ as the quotient: $d_A([f]_A, [g]_A) = \inf d([f'], [g'])$ where the infimum is taken over all $[f'] \in \pi^{-1}([f]_A)$ and all $[g'] \in \pi^{-1}([g]_A)$. It is proved in [3] that the fiber $\pi^{-1}([f]_A)$ for each $[f]_A \in AT(R)$ is a closed submanifold of $T(R)$. Also the above infimum is coincident with the infimum taken over all $[g'] \in \pi^{-1}([g]_A)$ with $[f'] \in \pi^{-1}([f]_A)$ fixed arbitrarily. Hence d_A turns out to be a complete distance which is called the *asymptotic Teichmüller distance*.

The asymptotic Teichmüller space $AT(R)$ is of interest only when R is analytically infinite. Otherwise $AT(R)$ is trivial, that is, it consists of just one point. Conversely, if R is analytically infinite, then $AT(R)$ is not trivial. This is because the Banach space $B_0(R^*)$ is a proper subspace of $B(R^*)$ whenever R is analytically infinite. See the Corollary in Section 6 of [1]. In fact, $B_0(R^*)$ is a separable Banach space whereas $B(R^*)$ is not separable. Hence $B(R^*)/B_0(R^*)$ is not separable and in particular infinite dimensional.

The *quasiconformal mapping class* is a homotopy equivalence class $[g]$ of quasiconformal automorphisms g of R , and the *quasiconformal mapping class group* $MCG(R)$ is the group of all these mapping classes. Here the homotopy is also relative to the ideal boundary at infinity. Every element $[g] \in MCG(R)$ induces a biholomorphic automorphism of $T(R)$ by $[f] \mapsto [f \circ g^{-1}]$, which is also isometric with respect to the Teichmüller distance d . In this way, $MCG(R)$ acts on $T(R)$, and a representation $\iota : MCG(R) \rightarrow \text{Aut}(T(R))$ is given. Similarly, $[g] \in MCG(R)$ induces a biholomorphic isometric automorphism of $AT(R)$ by $[f]_A \mapsto [f \circ g^{-1}]_A$. Thus we have a representation $\iota_A : MCG(R) \rightarrow \text{Aut}(AT(R))$.

It is proved in [2] that the representation $\iota : MCG(R) \rightarrow \text{Aut}(T(R))$ is injective (faithful) if R is analytically infinite, or more precisely, unless R is analytically finite of either $(0, 3)$, $(0, 4)$, $(1, 1)$, $(1, 2)$ or $(2, 0)$ type. In fact, ι is bijective (see [8]). On the other hand, it is also proved in [2] that the representation $\iota_A : MCG(R) \rightarrow \text{Aut}(AT(R))$ is injective precisely when R is either the unit disk or a once-punctured disk. This fact is natural in some sense, for different actions of mapping classes of R on $T(R)$ can shrink to the same action on $AT(R)$ under the projection $\pi : T(R) \rightarrow AT(R)$. Concerning this matter, we will see in this paper that even the following extreme case can occur.

Theorem 1. *There exists an analytically infinite Riemann surface R° such that the quasiconformal mapping class group $MCG(R^\circ)$ acts trivially on the asymptotic Teichmüller space $AT(R^\circ)$, namely, the kernel of the homomorphism ι_A is entirely $MCG(R^\circ)$.*

§2. CONSTRUCTION OF THE RIEMANN SURFACE R°

In [9], we have obtained an analytically infinite Riemann surface R such that the quasiconformal mapping class group $MCG(R)$ consists only of a countable number of elements. This mapping class group also holds a property that every mapping class has an asymptotically conformal representative as an automorphism of the base surface R . However, this is weaker than the statement of Theorem 1 which is equivalent to saying that every mapping class has an asymptotically conformal representative as an automorphism of every surface R' quasiconformally equivalent

to R . For this property, it suffices to prove that every mapping class has an eventually trivial representative as an automorphism of the base surface R . Here we say that a quasiconformal automorphism f of R is *eventually trivial* if there exists a topologically finite subsurface V of R such that f is the identity on $R - V$. The Riemann surface R° as in Theorem 1 is constructed from R by removing a countable number of points appropriately so that every quasiconformal mapping class should be eventually trivial.

A pair of pants P is a hyperbolic surface with geodesic boundaries c whose interior is homeomorphic to a three-punctured sphere. Every pair of pants admits a canonical orientation reversing isometric involution. The fixed point loci of this involution consist of three geodesic segments σ , which we call the symmetry axes. Cutting along the symmetry axes, we have two congruent right-angled hexagons D . Let P_0 be a pair of pants the lengths of whose geodesic boundary components are $0!$ and $1!$ and $1!$. Let P_1 be a pair of pants with the lengths $1!$ and $2!$ and $2!$. In the same way, for every non-negative integer n , let P_n be a pair of pants with the lengths $n!$ and $(n+1)!$ and $(n+1)!$. The three symmetry axes divide P_n into two congruent right-angled hexagons D_n . The geodesic boundary components of length $n!$ and $(n+1)!$ in P_n are denoted by c_n and c_{n+1} (right and left) respectively.

The Riemann surface R is made of 2^{n+1} copies of P_n for all $n \geq 0$ as follows. We take 2 copies of P_0 and glue the geodesic boundary component c_0 of each P_0 together. The resulting hyperbolic surface with 4 geodesic boundary components c_1 is denoted by R_1 . Next take 4 copies of P_1 and glue the geodesic boundary component c_1 of each P_1 with the 4 boundary components of R_1 . The resulting hyperbolic surface with 8 geodesic boundary components c_2 is denoted by R_2 . Continuing this process, we obtain, for every $n \geq 1$, a hyperbolic surface R_n with 2^{n+1} geodesic boundary components c_n made of R_{n-1} and 2^n copies of P_{n-1} . Then we take the exhaustion of these surfaces R_n , which is $R = \bigcup_{n=1}^{\infty} R_n$. Each connected component of $R - R_n$ is a neighborhood of an end, which is denoted by E_n . At each step of gluing, we give an appropriate amount of twist along the geodesic boundaries so that R is a complete hyperbolic surface without ideal boundary at infinity.

To make R° from R , we puncture infinitely many pairs of pants P_n in R . The removed point q_n in P_n is chosen at the middle of the symmetric axis connecting the right and left c_{n+1} in P_n . In order to choose the punctured pairs of pants, we use a family of streams as is defined in the next two paragraphs. A pair of pants P_n in R is punctured if no stream in the family intersects it. Otherwise, it is not punctured. In this manner, we have a punctured Riemann surface R° .

An n -stream ($n \geq 0$) is a path starting from a pair of pants P_n and going through $P_{n+1}, P_{n+2}, \dots$ in order towards infinity. Each n -stream takes a right as its initial direction at P_n and then continues to keep right until it reaches the turning point P_i ($i > n$). The stream switches its direction from right to left once it is here at P_i . We call this integer i the *turning number* for the stream. After P_i , it takes a left throughout.

We make a family of the streams satisfying the following conditions:

- (1) Two 0-streams start from both the pairs of pants P_0 .
- (2) If a stream passes or starts from a pair of pants P_n ($n \geq 0$), then, among the two adjacent pairs of pants P_{n+1} to the P_n , the one where the stream

does not pass has the two adjacent pairs of pants P_{n+2} from which two $(n+2)$ -streams start.

(3) Every pair of pants P_n intersects at most one stream.

(4) All the turning numbers of the streams are mutually distinct.

Then we use this family to determine the punctured pairs of pants as explained above.

§3. GEOMETRY OF THE RIEMANN SURFACE R°

The punctured Riemann surface R° has no ideal boundary at infinity. This can be seen from a general fact that if R has no ideal boundary at infinity and R° is obtained by removing a totally disconnected closed set from R , then R° has no ideal boundary at infinity. Indeed, a Riemann surface has ideal boundary at infinity if and only if it has a simply connected subdomain W such that the relative boundary ∂W is connected and that the harmonic measure of ∂W in W is not full; if R° has this property then so does R . Remark that, since R° has no ideal boundary at infinity, homotopy in the definition of the Teichmüller equivalence for $T(R^\circ)$ can be regarded as free homotopy.

A boundary component of $P_n \cap R^\circ$ is not geodesic with respect to the intrinsic hyperbolic metric on R° . For each of the three boundary components of $P_n \cap R^\circ$, there exists a unique simple closed geodesic in R° in its free homotopy class, which is denoted by c_n° or c_{n+1}° . A subsurface of R° bounded by these three simple closed geodesics, c_n° , c_{n+1}° right and c_{n+1}° left, is denoted by P_n° , which is either a punctured or a non-punctured pair of pants. Similarly, R_n° is defined to be the subsurface of R° bounded by all c_n° , and E_n° is defined to be a connected component of $R^\circ - R_n^\circ$.

The complete hyperbolic metric on R° is not less than the hyperbolic metric on R . This is because of a monotone property of the hyperbolic densities with respect to the inclusion relation between domains. This in particular implies that the length $\ell(c_n^\circ)$ of the geodesic c_n° in R° is not less than the length $\ell(c_n)$ of the geodesic c_n freely homotopic to c_n° in R . On the other hand, for our particular surface R° , $\ell(c_n^\circ)$ has the following upper bound in terms of $\ell(c_n)$.

Lemma 2. *The length $\ell(c_n^\circ)$ is not greater than $4\ell(c_n) = 4n!$.*

Proof. In the original surface R , consider an embedded annulus

$$H_\omega(c_n) = \{p \in E_n \mid 0 < d_R(c_n, p) < \omega\}$$

called a half-collar for the simple closed geodesic c_n , where d_R denotes the hyperbolic distance on R . By Proposition 5 and Lemma 3 in [9], we see that the length of the shortest geodesic arc $\delta_n(\not\subset c_n)$ connecting c_n to itself in E_n is

$$\ell(\delta_n) = 2 \operatorname{arcsinh} \left\{ \frac{\cosh((n+1)!/2)}{\sinh(n!/4)} \right\} > n! \times n.$$

Hence uniform width ω of the half-collar $H_\omega(c_n)$ can be taken as

$$\operatorname{arcsinh} \left\{ \frac{\cosh((n+1)!/2)}{\sinh(n!/4)} \right\} > \operatorname{arcsinh} 1 =: \omega$$

for every $n \geq 1$.

Let A_n be the annular cover of R with respect to the c_n . The half-collar $H_\omega(c_n)$ can be regarded as embedded in A_n . The value of the width $\omega = \operatorname{arcsinh} 1$ implies that their moduli are related as $m(H_\omega(c_n)) = m(A_n)/4$. Since ω is less than

$\ell(\delta_n)/2$, the half-collar $H_\omega(c_n)$ does not contain any q_i for $i \geq 0$. Hence $H_\omega(c_n)$ is also regarded as embedded in R° as well as in the annular cover A_n° of R° with respect to the c_n° . Then their moduli satisfy an inequality $m(A_n^\circ) \geq m(H_\omega(c_n))$. From the two inequalities above, we have $m(A_n^\circ) \geq m(A_n)/4$. Since the geodesic length and the modulus of the annular cover are inversely proportional, this implies $\ell(c_n^\circ) \leq 4\ell(c_n)$. $\square$

The measurement of $P_n \subset R$ given by Proposition 5 in [9] also yields the following estimate.

Proposition 3. *Let P_n be one of the pairs of pants in R such that $P_n \cap R^\circ$ is punctured and let c_n be the boundary component of P_n . Let η_n be a closed or non-closed curve in $P_n \cap R^\circ$. When η_n is closed, assume that it is freely homotopic to c_n in R but not in R° . When η_n is not closed, assume that both end points of η_n are on c_n , and that η_n is not homotopic to a subarc of c_n in R° fixing the end points. Then the hyperbolic length of η_n in R is greater than $n! \times n$.*

Proof. Consider the right-angled hexagon D_n , which is the symmetric half of P_n . Then the length of η_n can be estimated by double the distance between the side of D_n where q_n lies and other sides not adjacent to it, which is either $\ell(c_{n+1})$ or $\ell(\delta_n)$. Both are greater than $n! \times n$. $\square$

This proposition gives the following estimate on the hyperbolic length of a closed geodesic on R° , which corresponds to Lemma 3 in [9].

Lemma 4. *The length of a non-trivial closed geodesic c° ($\neq c_n^\circ$) in R° that intersects E_n° is greater than $n! \times n$.*

Proof. First assume that c° does not cross any c_i trivially, which means that no subarcs of c° and c_i together bound a simply connected domain. Let m be the largest integer among i for which c° has intersection with some P_i . Then the subarc $\eta_m = c^\circ \cap P_m$ satisfies the conditions as in Proposition 3. Since the length of η_m measured by the hyperbolic metric on R° is not less than that on R , it is greater than $m! \times m \geq n! \times n$. If c° crosses some c_i trivially, we replace the subarc of c° with the subarc of c_i , which together bounds a simply connected domain. By this replacement, the length measured in R does not increase. Hence we may reduce our argument to the case under the first assumption. $\square$

§4. QUASICONFORMAL AUTOMORPHISMS OF R°

In this section, we complete the proof of Theorem 1. First, we notice that the distribution of the punctures makes R° to satisfy the following property.

Lemma 5. *Let E_n and E'_n be any connected components of $R - R_n$ possibly coincident and let $h : E_n \rightarrow E'_n$ be a conformal homeomorphism that is not the identity. Then the image of $E_n \cap R^\circ$ under h is not coincident with $E'_n \cap R^\circ$.*

Proof. If the $P_n \subset \overline{E_n}$ and the $P'_n \subset \overline{E'_n}$ are of different type regarding puncture, there is nothing to prove. If P_n and P'_n are both punctured, we consider one of pairs of $(n+1)$ -streams starting from the adjacent P_{n+1} and P'_{n+1} . Since their turning numbers are distinct by construction, we have an index $i > n+1$ such that the corresponding P_i and P'_i under h are of different type. If neither P_n nor P'_n are punctured, we consider the streams intersecting P_n and P'_n respectively. Since

their turning numbers are distinct, we again have an index $i > n$ such that P_i and P'_i are of different type. Note that, in the above arguments, we mainly suppose the case where h maps the right-hand side of E_n to the right-hand side of E'_n . Indeed the claim is rather easier to prove in the other case. $\square$

By this property of R° and a property of R proved by Theorem 3 in [9], we have the following theorem concerning quasiconformal automorphisms of R° , which is an essential ingredient for proving Theorem 1.

Theorem 6. *Let $g^\circ : R^\circ \rightarrow R^\circ$ be an $n/4$ -quasiconformal automorphism of the Riemann surface R° for any integer $n \geq 5$. Then, for each connected component E_n° of $R^\circ - R_n^\circ$, the restriction $g^\circ|_{E_n^\circ} : E_n^\circ \rightarrow R^\circ$ is homotopic to the inclusion map $id|_{E_n^\circ} : E_n^\circ \rightarrow R^\circ$ in R° .*

Proof. Since a puncture is a removable singularity for a quasiconformal map, $g^\circ : R^\circ \rightarrow R^\circ$ extends to a quasiconformal automorphism $g : R \rightarrow R$ of the original surface R . The maximal dilatation of g is bounded by $n/4 \leq n$. Then, by Theorem 3 and its proof in [9], the restriction $g|_{E_n} : E_n \rightarrow R$ is homotopic to a conformal homeomorphism $h : E_n \rightarrow E'_n$ onto a connected component E'_n of $R - R_n$.

Suppose to the contrary that $g^\circ|_{E_n^\circ}$ is not homotopic to $id|_{E_n^\circ}$. There are two cases to be considered: $h \neq id_{E_n}$ and $h = id_{E_n}$. In the former case, the image of $E_n \cap R^\circ$ under h is not coincident with $E'_n \cap R^\circ$ by Lemma 5. Hence in both cases, there exists some $c_m = \partial E_m \subset \overline{E_n}$ ($m \geq n$) such that $g(c_m)$ and $c'_m := h(c_m)$ are freely homotopic in R but not in R° . Then we see that either the geodesic $g(c_m)^\circ$ freely homotopic to $g(c_m)$ in R° has intersection with $h(E_m)^\circ$ or the geodesic $g^{-1}(c'_m)^\circ$ has intersection with E_m° . Without loss of generality, we may assume the former case. By Lemma 2, the geodesic length $\ell(c_m^\circ)$ in R° is not greater than $4m!$. By Lemma 4, the geodesic length $\ell(g(c_m)^\circ)$ is greater than $m! \times m$. Thus the maximal dilatation of g is greater than $m/4 \geq n/4$. However, this is a contradiction. $\square$

Corollary 7. *Every quasiconformal mapping class $[g] \in \text{MCG}(R^\circ)$ has an eventually trivial representative $g_0 : R^\circ \rightarrow R^\circ$.*

Proof. For a K -quasiconformal automorphism g of R° , take an integer n satisfying $n/4 > K$. Then, applying Theorem 6 to g , we see that g is homotopic to the inclusion map on $R^\circ - R_n^\circ$. We modify g to a quasiconformal automorphism g_0 of R° homotopic to g such that g_0 is precisely the identity on $R^\circ - R_n^\circ$. Since R_n° is topologically finite, such a modification is always possible. $\square$

Proof of Theorem 1. Let $f : R^\circ \rightarrow S^\circ$ be any quasiconformal homeomorphism of the Riemann surface R° onto another Riemann surface S° and let $[g] \in \text{MCG}(R^\circ)$ be any quasiconformal mapping class. By Corollary 7, there is an eventually trivial representative g_0 in $[g]$. Then $f \circ g \circ f^{-1}$ is homotopic to $f \circ g_0 \circ f^{-1}$. Here $f \circ g_0 \circ f^{-1}$ is the identity on $S^\circ - f(R_n^\circ)$ for some n , and in particular it is an asymptotically conformal automorphism of S° . This implies that the action of every $[g] \in \text{MCG}(R^\circ)$ fixes each asymptotic equivalence class $[f]_A \in \text{AT}(R^\circ)$, namely, $\text{MCG}(R^\circ)$ acts trivially on $\text{AT}(R^\circ)$. $\square$

Remark. Note that R° as well as R satisfies the property that, for every $L > 0$, the number of simple closed geodesics whose lengths are less than L is finite. Actually, it is proved in [6] that if a Riemann surface does not satisfy this property, then the quasiconformal mapping class group has no common fixed point in the asymptotic

Teichmüller space and in particular it acts non-trivially. Also Theorem 6 implies that $\text{MCG}(R^\circ)$ consists of a countable number of elements, as does $\text{MCG}(R)$. In general, we will prove in the forthcoming paper [10] that, if $\text{MCG}(R)$ has a common fixed point in $AT(R)$, then $\text{MCG}(R)$ must be countable.

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