LIE DERIVATIONS OF $\mathcal{J}$ -SUBSPACE LATTICE ALGEBRAS

FANGYAN LU

(Communicated by Joseph A. Ball)

ABSTRACT. We describe the structure of Lie derivations of $\mathcal{J}$ -subspace lattice algebras. The results can apply to atomic Boolean subspace lattice algebras and pentagon subspace lattice algebras, respectively.

1. INTRODUCTION

Let $\mathcal{A}$ be an associative algebra and $\mathcal{M}$ be an $\mathcal{A}$ -bimodule. By $Z(\mathcal{M}, \mathcal{A})$ we denote the center of $\mathcal{M}$ relative to $\mathcal{A}$, that is, $Z(\mathcal{M}, \mathcal{A}) = \{M \in \mathcal{M} : AM = MA \text{ for all } A \in \mathcal{A}\}$. A linear mapping $\delta : \mathcal{A} \rightarrow \mathcal{M}$ is called a *Lie derivation* if $\delta([A, B]) = [\delta(A), B] + [A, \delta(B)]$ for all $A, B \in \mathcal{A}$, where $[A, B] = AB - BA$ is the usual Lie product. We say that a Lie derivation δ is *standard* if it can be decomposed as $\delta = d + \tau$, where d is an ordinary derivation from $\mathcal{A}$ into $\mathcal{M}$ and τ is a linear mapping from $\mathcal{A}$ into the center $Z(\mathcal{M}, \mathcal{A})$ of $\mathcal{M}$ relative to $\mathcal{A}$ vanishing on each commutator. The standard problem, which has been studied for many years, is to find conditions on $\mathcal{A}$ under which each Lie derivation is standard or standard-like. This problem has been investigated for prime rings in [2, 3, 21, 25, 26], for C^* -algebras and for more general semisimple Banach algebras in [1, 9, 22, 23, 27], and for triangular algebras in [5]. In the present note, we pursue this line of investigation for $\mathcal{J}$ -subspace lattice algebras.

2. PRELIMINARIES

In this note, all algebras and vector spaces will be over the complex field $\mathbb{C}$. Given a Banach space X with topological dual X^* , by $B(X)$ and I we mean the algebra of all bounded linear operators on X and the identity operator on X , respectively. The terms *operator* on X and *subspace* of X will mean ‘bounded linear map of X into itself’ and ‘norm closed linear manifold of X ’, respectively. For $A \in B(X)$, denote by A^* the adjoint of A . For $x \in X$ and $f \in X^*$, the rank-one operator $x \otimes f$ is defined by $y \mapsto f(y)x$ for $y \in X$. For any non-empty subset $L \subseteq X$, $L^\perp$ denotes its annihilator, that is, $L^\perp = \{f \in X^* : f(x) = 0 \text{ for all } x \in L\}$. A family $\mathcal{L}$ of subspaces of X is a *subspace lattice* on X if it contains (0) and X , and is complete in the sense that it is closed under the operations $\vee$ (closed linear span) and $\cap$ (set-theoretic intersection). For a subspace lattice $\mathcal{L}$ on X , the associated

Received by the editors October 14, 2005 and, in revised form, April 25, 2006.

2000 *Mathematics Subject Classification*. Primary 47L35; Secondary 17B40, 17B60.

Key words and phrases. Lie derivations, $\mathcal{J}$ -subspace lattice algebras, finite rank operators.

The author was supported by NNSFC (No. 10571054) and a grant (No. 04KJB110116) from the government of Jiangsu Province of China.

subspace lattice algebra $\text{Alg}\mathcal{L}$ is the set of operators on X leaving every subspace in $\mathcal{L}$ invariant, that is,

$$\text{Alg}\mathcal{L} = \{A \in B(X) : Ax \in L \text{ for every } x \in L \text{ and for every } L \in \mathcal{L}\}.$$

Obviously, $\text{Alg}\mathcal{L}$ is a unital weakly closed operator algebra. A subalgebra of $\text{Alg}\mathcal{L}$ is called a *standard subalgebra* if it contains all finite-rank operators in $\text{Alg}\mathcal{L}$.

Subspace lattice algebras are important and mainly consist of non-selfadjoint operator algebras. Completely distributive subspace lattice algebras, commutative subspace lattice algebras, atomic Boolean subspace lattice algebras, pentagon subspace lattice algebras, etc., have been widely studied. See, for example, [6, 12, 13, 14, 15, 16]. Recently, Panaia in [24] introduced a new class of subspace lattices — $\mathcal{J}$ -subspace lattices, which cover atomic Boolean subspace lattices and pentagon subspace lattices. Several authors have studied $\mathcal{J}$ -subspace lattices as well as $\mathcal{J}$ -subspace lattice algebras; see, for example, [7, 17, 18, 19, 20].

Given a subspace lattice $\mathcal{L}$ on a Banach space X , put

$$\mathcal{J}(\mathcal{L}) = \{K \in \mathcal{L} : K \neq (0) \text{ and } K_- \neq X\},$$

where $K_- = \bigvee\{L \in \mathcal{L} : K \not\supseteq L\}$. Call $\mathcal{L}$ a $\mathcal{J}$ -subspace lattice on X if

- (1) $\bigvee\{K : K \in \mathcal{J}(\mathcal{L})\} = X$,
- (2) $\bigcap\{K_- : K \in \mathcal{J}(\mathcal{L})\} = (0)$,
- (3) $K \vee K_- = X$ for every $K \in \mathcal{J}(\mathcal{L})$,
- (4) $K \cap K_- = (0)$ for every $K \in \mathcal{J}(\mathcal{L})$.

The simplest example of a $\mathcal{J}$ -subspace lattice is any pentagon subspace lattice $\mathcal{P} = \{(0), K, L, M, X\}$. Here K, L and M are subspaces of X satisfying $K \vee L = X$, $K \cap M = (0)$ and $L \subset M$. In this case, $K_- = M$, $L_- = K$, and $\mathcal{J}(\mathcal{P}) = \{K, L\}$. For further discussion of pentagon subspace lattices see [12, 15]. Another important element of the class of $\mathcal{J}$ -subspace lattices is the atomic Boolean subspace lattice. It follows from [18, Theorem 2.1] that every commutative $\mathcal{J}$ -subspace lattice on a Hilbert space is an atomic Boolean subspace lattice. However, most $\mathcal{J}$ -subspace lattices on a Hilbert space are non-commutative [18].

Therefore, $\mathcal{J}$ -subspace lattices as well as $\mathcal{J}$ -subspace lattice algebras deserve some attention. We mention in passing that $\mathcal{J}$ -subspace lattice algebras are semi-simple Banach algebras. In the previous papers [19, 20], we studied algebraic isomorphisms, Jordan isomorphisms and Jordan derivations. Here we study Lie derivations of $\mathcal{J}$ -subspace lattice algebras. Even for pentagon subspace lattice algebras and atomic Boolean subspace lattice algebras, our results are new.

For a subspace lattice $\mathcal{L}$, the relevance of $\mathcal{J}(\mathcal{L})$ is due to the following lemma, which is crucial to what follows.

Lemma 2.1 (Longstaff [16] (see also [13])). *If $\mathcal{L}$ is a subspace lattice on X , then the rank-one operator $x \otimes f \in \text{Alg}\mathcal{L}$ if and only if there exists some $K \in \mathcal{J}(\mathcal{L})$ such that $x \in K$ and $f \in K_-^\perp$, where $K_-^\perp$ means $(K_-)^\perp$.*

From Lemma 2.1 we can see that if $\mathcal{L}$ is a $\mathcal{J}$ -subspace lattice, then $\text{Alg}\mathcal{L}$ is rich in rank-one operators. Moreover, finite-rank operators in a $\mathcal{J}$ -subspace lattice algebra have nice properties. Given a subspace lattice $\mathcal{L}$, by $\mathcal{F}(\mathcal{L})$ we denote the algebra of all finite-rank operators in $\text{Alg}\mathcal{L}$. If $K \in \mathcal{J}(\mathcal{L})$, then we write $\mathcal{F}_1(K) = \{x \otimes f : x \in K, f \in K_-^\perp\}$ and $\mathcal{F}(K) = \langle \mathcal{F}_1(K) \rangle$, the linear manifold spanned by $\mathcal{F}_1(K)$.

Lemma 2.2 ([19, Proposition 3.2] (see also [24, Proposition 3.3.1])). *Let $\mathcal{L}$ be a $\mathcal{J}$ -subspace lattice on a Banach space X . Suppose that A is an operator of rank n in $\mathcal{F}(\mathcal{L})$. Then A can be written as a sum of n rank-1 operators in $\text{Alg}\mathcal{L}$.*

Lemma 2.3 ([19, Lemma 3.6]). *Let $\mathcal{L}$ be a $\mathcal{J}$ -subspace lattice on a Banach space X . Let A and B be in $\mathcal{F}(K)$ for some $K \in \mathcal{J}(\mathcal{L})$. Then there is an idempotent $P \in \mathcal{F}(K)$ such that $A = PAP$ and $B = PBP$.*

Lemma 2.4. *Let $\mathcal{L}$ be a $\mathcal{J}$ -subspace lattice on a Banach space X . Let K, E be in $\mathcal{J}(\mathcal{L})$ with $K \neq E$ and suppose $A \in \mathcal{F}(K)$, $B \in \mathcal{F}(E)$.*

- (i) $\mathcal{F}(K)$ is an ideal of $\text{Alg}\mathcal{L}$.
- (ii) For all $C \in \text{Alg}\mathcal{L}$, $ACB = 0$.
- (iii) If $A + B = 0$, then $A = B = 0$.

Proof. (i) Let T be in $\text{Alg}\mathcal{L}$. Then for each $x \otimes f \in \mathcal{F}_1(K)$, both $T(x \otimes f)$ and $(x \otimes f)T$ are in $\mathcal{F}_1(K)$ since $Tx \in K$ and $T^*f \in K^\perp$. Therefore $\mathcal{F}(K)$ is an ideal of $\text{Alg}\mathcal{L}$.

(ii) If $E \supset K$, then $K \subseteq E_-$ and therefore $K = E \cap K \subseteq E \cap E_- = (0)$. This contradiction shows $E \not\supset K$. So $E \subseteq K_-$. From this relation and (i) we see the range of CB is contained in K_- for all $C \in \text{Alg}\mathcal{L}$. Therefore $ACB = 0$ since the restriction of A to K_- is zero.

(iii) For $x \in X$, we have $Ax = -Bx$. So both the ranges of A and B are contained in $K \cap E$. However $K \cap E = (0)$ since $E \subseteq K_-$ and $K \cap K_- = (0)$. So $A = B = 0$. $\square$

We need a deep understand of the centers of $\mathcal{J}$ -subspace lattice algebras.

Remark 2.5. Let $\mathcal{L}$ be a $\mathcal{J}$ -subspace lattice on a Banach space X .

(i) If $A \in \text{Alg}\mathcal{L}$ commutes with each rank-one operator, then A is in the center of $\text{Alg}\mathcal{L}$. Indeed, if T is in $\text{Alg}\mathcal{L}$, then $ATx \otimes f = (Tx \otimes f)A = TA x \otimes f$ for all $x \otimes f \in \mathcal{F}(\mathcal{L})$. Hence $ATx = TA x$ for all $x \in \text{span}\{z \in K : K \in \mathcal{J}(\mathcal{L})\}$, which is dense in X . Consequently, $AT = TA$ for all $T \in \text{Alg}\mathcal{L}$. Thus, A is in the center of $\text{Alg}\mathcal{L}$.

(ii) Suppose $K \in \mathcal{J}(\mathcal{L})$ is finite-dimensional and $\{x_1, x_2, \dots, x_n\}$ is a basis of K . Let $f_1, f_2, \dots, f_n$ be in $K^\perp$ such that $f_i(x_j) = \delta_{ij}$. Let $A = \sum_{i=1}^n x_i \otimes f_i$. If $x \otimes f \in \mathcal{F}(K)$, then $x \otimes f = x \otimes fA = Ax \otimes f$; if $x \otimes f \in \mathcal{F}(E)$ with $E \in \mathcal{J}(\mathcal{L})$ not being equal to K , then $Ax \otimes f = x \otimes fA = 0$. So by the preceding result (i), A is in the center of $\text{Alg}\mathcal{L}$.

(iii) Suppose that $A \in \mathcal{F}(\mathcal{L})$ is in the center of $\text{Alg}\mathcal{L}$ and $K \in \mathcal{J}(\mathcal{L})$ is infinite-dimensional. Since the range of A is finite-dimensional, there is a non-zero functional f in $K^\perp$ such that $A^*f = 0$. Thus $Ax \otimes f = x \otimes fA = 0$ for all $x \in K$. So A restricted to K is zero. Consequently, if each element in $\mathcal{J}(\mathcal{L})$ is infinite-dimensional, then any finite rank operator in $\text{Alg}\mathcal{L}$ is not in the center of $\text{Alg}\mathcal{L}$. Therefore, the center of $\mathcal{F}(\mathcal{L})$ is the trivial set $\{0\}$. Note that the pentagon subspace lattice is this case.

(iv) Suppose that $A \in \text{Alg}\mathcal{L}$ is in the center of $\text{Alg}\mathcal{L}$. Let K be an element in $\mathcal{J}(\mathcal{L})$. Take a non-zero functional f in $K^\perp$. Then for every $x \in K$, $Ax \otimes f = (x \otimes f)A$. Applying this equation to a vector y with $f(y) = 1$, we get that $Ax = f(Ay)x$ for all $x \in K$. So the restriction of A to K is a scalar multiple of the identity on K .

We will use the results about local derivations of $\mathcal{J}$ -subspace lattice algebras. A linear map δ from an algebra $\mathcal{A}$ to an $\mathcal{A}$ -bimodule $\mathcal{M}$ is called a *local derivation* if for every $A \in \mathcal{A}$ there is a derivation $\delta_A : \mathcal{A} \rightarrow \mathcal{M}$ depending on A such that $\delta(A) = \delta_A(A)$. The concept of local derivations was first introduced by Kadison [11], and there has recently been a growing interest in the study of local derivations of operator algebras. We refer to [4, 7, 10] for more information. In [7], it is shown that each local derivation of a standard subalgebra of a $\mathcal{J}$ -subspace lattice algebra $\text{Alg}\mathcal{L}$ containing the identity operator I is a derivation. Slightly modifying the proof thereof, one can show that it is also true for $\mathcal{F}(\mathcal{L})$ (not necessarily containing I). Here, using Lemma 2.3, we include a (short) proof.

Proposition 2.6. *Let $\mathcal{L}$ be a $\mathcal{J}$ -subspace lattice on a Banach space X and let $\mathcal{A}$ be a standard algebra of $\text{Alg}\mathcal{L}$. Let $\delta : \mathcal{F}(\mathcal{L}) \rightarrow \mathcal{A}$ be a local derivation. Then δ is a derivation.*

Proof. It suffices to show that $\delta(AB) = \delta(A)B + A\delta(B)$ holds for all $A, B \in \mathcal{F}(\mathcal{L})$.

First we suppose that $A, B \in \mathcal{F}(K)$ with $K \in \mathcal{J}(\mathcal{L})$. By Lemma 2.3, there is an idempotent operator P in $\mathcal{F}(K)$ such that $A = PAP$ and $B = PBP$. Let $\{x_1, x_2, \dots, x_n\}$ be a basis of the range of P . Define linear functionals $f_1, f_2, \dots, f_n$ on X such that

$$\begin{aligned} f_i(x_j) &= \delta_{ij} \quad \text{for } 1 \leq i, j \leq n; \\ f_i(z) &= 0 \quad \text{for } z \in \ker P \text{ and } 1 \leq i \leq n. \end{aligned}$$

Then

$$P = x_1 \otimes f_1 + x_2 \otimes f_2 + \dots + x_n \otimes f_n.$$

Clearly, $P(X) \subseteq K$ and $P^*(X^*) \subseteq K^\perp$. It follows that each $x_i \otimes f_i$ belongs to $\mathcal{F}(K)$. Thus the subalgebra $\mathfrak{D}$ of all operators C of the form $C = \sum_{i,j=1}^n \lambda_{ij} x_i \otimes f_j$, $\lambda_{ij} \in \mathbb{C}$, is contained in $\mathcal{F}(K)$. Note that $\mathfrak{D}$ is isomorphic to $M_n(\mathbb{C})$ via $C \mapsto [\lambda_{ij}]_{n \times n}$. By [4, Theorem 2.3], the restriction of δ to $\mathfrak{D}$ is a derivation. Hence $\delta(AB) = \delta(A)B + A\delta(B)$ since $A, B \in \mathfrak{D}$.

Next we suppose that $A \in \mathcal{F}(K)$ and $B \in \mathcal{F}(E)$ with K and E being distinct elements in $\mathcal{J}(\mathcal{L})$. Let δ_A be a derivation from $\mathcal{F}(\mathcal{L})$ into $\mathcal{A}$ such that $\delta(A) = \delta_A(A)$. Then by Lemma 2.4(ii) we have

$$0 = \delta_A(AB) = \delta_A(A)B + A\delta_A(B) = \delta(A)B + A\delta_A(B).$$

Hence by Lemma 2.4 (i) and (iii), we can conclude that $\delta(A)B = 0$. Similarly, we have that $A\delta(B) = 0$. Consequently, $\delta(AB) = \delta(A)B + A\delta(B)$.

Now suppose in general that $A, B \in \mathcal{F}(\mathcal{L})$. By Lemma 2.2, we can write

$$A = A_1 + A_2 + \dots + A_n \quad \text{and} \quad B = B_1 + B_2 + \dots + B_m,$$

where $A_i \in \mathcal{F}(K_i)$, $B_j \in \mathcal{F}(E_j)$, $K_1, K_2, \dots, K_n$ are distinct elements in $\mathcal{J}(\mathcal{L})$, $E_1, E_2, \dots, E_m$ are distinct elements in $\mathcal{J}(\mathcal{L})$. Thus by the preceding results, we have that

$$\begin{aligned} \delta(AB) &= \delta\left(\sum_{i=1}^n \sum_{j=1}^m A_i B_j\right) = \sum_{i=1}^n \sum_{j=1}^m \delta(A_i B_j) \\ &= \sum_{i=1}^n \sum_{j=1}^m (\delta(A_i)B_j + A_i \delta(B_j)) \\ &= \delta(A)B + A\delta(B). \end{aligned}$$

This completes the proof. $\square$

3. LIE DERIVATIONS OF $\mathcal{F}(\mathcal{L})$

The main result in this section reads as follows.

Theorem 3.1. *Let $\mathcal{L}$ be a $\mathcal{J}$ -subspace lattice and $\mathcal{A}$ be a standard subalgebra of $\text{Alg}\mathcal{L}$. Let δ be a Lie derivation from $\mathcal{F}(\mathcal{L})$ into $\mathcal{A}$. Then δ is standard.*

For the proof of the theorem, we need some lemmas. In the following, we keep the notation as in the statement of the theorem. Recall that the statement that δ is standard means that δ is the sum of a derivation d from $\mathcal{F}(\mathcal{L})$ into $\mathcal{A}$ and a linear mapping from $\mathcal{F}(\mathcal{L})$ into the center $Z(\mathcal{A}, \mathcal{F}(\mathcal{L}))$ of $\mathcal{A}$ relative to $\mathcal{F}(\mathcal{L})$ vanishing on each commutator. Here $Z(\mathcal{A}, \mathcal{F}(\mathcal{L})) = \{A \in \mathcal{A} : AB = BA \text{ for all } B \in \mathcal{F}(\mathcal{L})\}$. From Remark 2.5(i), we know that $Z(\mathcal{A}, \mathcal{F}(\mathcal{L}))$ is equal to $Z(\mathcal{A})$, the center of $\mathcal{A}$.

Lemma 3.2. *Suppose that A is in $\mathcal{F}(\mathcal{L})$, S in $\mathcal{A}$ and C in $Z(\mathcal{A})$. If $[A, S] = C$, then $C = 0$.*

Proof. Since C commutes with A , it follows from the Kleinecke-Shirokov theorem, see [8, Problem 233], for example, that C is quasi-nilpotent. If $Cx \neq 0$ for some $x \in K$ with $K \in \mathcal{J}(\mathcal{L})$, we choose $f \in K^\perp$ such that $f(Cx) = 1$. Then $C(x \otimes f)$ is idempotent. However, since C is quasi-nilpotent and commutes with $x \otimes f$, $C(x \otimes f)$ should be nilpotent. This contradiction shows that $Cx = 0$ for all $x \in K$ and $K \in \mathcal{J}(\mathcal{L})$. Consequently, $C = 0$. $\square$

Lemma 3.3. *Let K be in $\mathcal{J}(\mathcal{L})$ and P be an idempotent operator in $\mathcal{F}(K)$. Then there is an operator S in $\mathcal{F}(K)$ and a unique operator $\tau(P)$ in $Z(\mathcal{A})$ such that $\delta(P) = [P, S] + \tau(P)$.*

Proof. Set $P_1 = P$ and $P_2 = I - P$. Note that $\mathcal{A}$ does not necessarily contain I . Note that $P_2B = B - P_1B$ and $BP_2 = B - BP_1$ for $B \in \mathcal{A}$.

For $A_{11} \in P_1\mathcal{F}(\mathcal{L})P_1$, we have that

$$0 = \delta([P_1, A_{11}]) = \delta(P_1)A_{11} - A_{11}\delta(P_1) + P_1\delta(A_{11}) - \delta(A_{11})P_1.$$

Multiplying this equation by P_1 from both sides, we get

$$(3.1) \quad P_1\delta(P_1)P_1A_{11} = A_{11}P_1\delta(P_1)P_1.$$

Similarly, for $A_{22} \in P_2\mathcal{F}(\mathcal{L})P_2$, considering $\delta([P_1, A_{22}])$, we have that

$$(3.2) \quad P_2\delta(P_1)P_2A_{22} = A_{22}P_2\delta(P_1)P_2.$$

Now for $A_{12} \in P_1\mathcal{F}(\mathcal{L})P_2$, we have that

$$\delta(A_{12}) = \delta([P_1, A_{12}]) = \delta(P_1)A_{12} - A_{12}\delta(P_1) + P_1\delta(A_{12}) - \delta(A_{12})P_1.$$

Multiplying this equation by P_1 from the left side and by P_2 from the right side, we get

$$(3.3) \quad P_1\delta(P_1)P_1A_{12} = A_{12}P_2\delta(P_1)P_2.$$

Similarly, for $A_{21} \in P_2\mathcal{F}(\mathcal{L})P_1$, we have that

$$(3.4) \quad P_2\delta(P_1)P_2A_{21} = A_{21}P_1\delta(P_1)P_1.$$

Using (3.1)-(3.4), it is easy to verify that $\tau(P) = P_1\delta(P_1)P_1 + P_2\delta(P_1)P_2 \in Z(\mathcal{A})$. Now let $S = P_1\delta(P_1)P_2 - P_2\delta(P_1)P_1$. Then $S \in \mathcal{F}(K)$ by Lemma 2.4(i) and $\delta(P) = [P, S] + \tau(P)$. Moreover, by Lemma 3.2, such a $\tau(P)$ is unique. $\square$

Lemma 3.4. *Let $A = x \otimes f$ be in $\mathcal{F}(K)$ with $K \in \mathcal{J}(\mathcal{L})$. Then there is an operator S in $\mathcal{F}(K)$ and a unique operator $\tau(A)$ in $Z(\mathcal{A})$ such that $\delta(A) = [A, S] + \tau(A)$.*

Proof. If $f(x) \neq 0$, then the result follows from the linearity and Lemma 3.3.

We now suppose that $f(x) = 0$. Take $y \in K$ such that $f(y) = 1$. Set $P = y \otimes f$. Then by Lemma 3.3, $\delta(P) = [P, S_1] + \tau(P)$, where $S_1 \in \mathcal{F}(K)$ and $\tau(P) \in Z(\mathcal{A})$. We associate a new Lie derivation as follows:

$$\Delta(T) = \delta(T) - [T, S_1], \quad T \in \mathcal{F}(\mathcal{L}).$$

Then $\Delta(P) = \tau(P) \in Z(\mathcal{A})$ and therefore,

$$\Delta(A) = \Delta([A, P]) = [\Delta(A), P] = \Delta(A)P - P\Delta(A).$$

From this, we obtain that $\Delta(A) = (I - P)\Delta(A)P$. So we can suppose that $\Delta(A) = z \otimes f$, where $z \in K$ and $f(z) = 0$. Now choose $g \in K^\perp$ such that $g(x) = 1$. Then we have

$$\Delta(A) = z \otimes f = (z \otimes g)(x \otimes f) - (x \otimes f)(z \otimes g).$$

Thus $S = S_1 - z \otimes g$ and $\tau(A) = 0$ are desired. $\square$

Lemma 3.5. *Suppose $A = \sum_{k=1}^n x_k \otimes f_k \in \mathcal{F}(K)$ with $K \in \mathcal{J}(\mathcal{L})$. Then there is an operator S in $\mathcal{F}(K)$ and a unique operator $\tau(A)$ in $Z(\mathcal{A})$ such that $\delta(A) = [A, S] + \tau(A)$. Moreover, $\tau(A) = \sum_{k=1}^n \tau(x_k \otimes f_k)$.*

Proof. Arguing as in the proof of Proposition 2.6, there is a matrix unit $\{y_i \otimes g_j\}_{i,j=1}^m$ such that each $x_k \otimes f_k$ belongs to the algebra $\mathfrak{D}$ of all operators of the form $\sum_{i,j=1}^m \lambda_{ij} y_i \otimes g_j$. Since $\mathfrak{D}$ is a finite-dimensional semi-simple Banach algebra, the restriction $\delta|_{\mathfrak{D}}$ of δ to $\mathfrak{D}$ is standard. See the proof of [9, Theorem 9.4]. So $\delta|_{\mathfrak{D}} = d + h$, where d is a derivation from $\mathfrak{D}$ into $\mathcal{A}$ and h is a mapping from $\mathfrak{D}$ into $Z(\mathcal{A}, \mathfrak{D}) = \{B \in \mathcal{A} : BD = DB \text{ for all } D \in \mathfrak{D}\}$. By [6, Lemma 10.7], there is an operator T in $\mathcal{A}$ such that $d(D) = [D, T]$ for all $D \in \mathfrak{D}$. Consequently, d is a derivation from $\mathfrak{D}$ into $\mathcal{F}(K)$. By [6, Lemma 10.7] again, there is an operator S in $\mathcal{F}(K)$ such that $d(D) = [D, S]$ for all $D \in \mathfrak{D}$. Thus for each k , it follows that

$$(3.5) \quad \delta(x_k \otimes f_k) = [x_k \otimes f_k, S] + h(x_k \otimes f_k).$$

On the other hand, by Lemma 3.4, for each k , there is an operator S_k in $\mathcal{F}(K)$ such that $\delta(x_k \otimes f_k) = [x_k \otimes f_k, S_k] + \tau(x_k \otimes f_k)$. This together with (3.5) yields

$$(3.6) \quad [x_k \otimes f_k, T_k] = C_k,$$

where $T_k = S - S_k$ and $C_k = \tau(x_k \otimes f_k) - h(x_k \otimes f_k)$. Since C_k commutes with $x_k \otimes f_k$, it follows from the Kleinecke-Shirokov theorem that C_k is quasi-nilpotent. Moreover, arguing as in the proof of Lemma 3.2, one shows that $C_k y_i = 0$ for all $i = 1, 2, \dots, m$.

Since $x_k \otimes f_k = (\sum_{i=1}^m y_i \otimes g_i)(x_k \otimes f_k)(\sum_{i=1}^m y_i \otimes g_i)$, there are i, j such that $f_k(y_i) \neq 0$ and $g_j(x_k) \neq 0$. Applying equation (3.6) to y_i , we get $f_k(T y_i) x_k - f_k(y_i) T x_k = C_k y_i = 0$. So $C_k = x_k \otimes g$, where $g = T_k^* f_k - \frac{f_k(T_k y_i)}{f_k(y_i)} f_k$. Thus $g_j(x_k) y_i \otimes g = (y_i \otimes g_j)(x_k \otimes g) = C_k(y_i \otimes g_j) = 0$. So $g = 0$ and therefore $C_k = 0$ and $[x_k \otimes f_k, T_k] = 0$. Consequently, $[x_k \otimes f_k, S_k] = [x_k \otimes f_k, S]$ and $\tau(x_k \otimes f_k) = h(x_k \otimes f_k)$ for each $k = 1, 2, \dots, n$. Thus

$$\delta(A) = \sum_{k=1}^n ([x_k \otimes f_k, S] + \tau(x_k \otimes f_k)) = [A, S] + \sum_{k=1}^n \tau(x_k \otimes f_k).$$

By Lemma 3.2, the sum $\sum_{k=1}^n \tau(x_k \otimes f_k)$ is independent of the representation of A . Let $\tau(A) = \sum_{k=1}^n \tau(x_k \otimes f_k)$. The proof is complete. $\square$

Proof of Theorem 3.1. Let A be in $\mathcal{F}(\mathcal{L})$. Then there uniquely exist finitely many distinct $K_1, K_2, \dots, K_n$ in $\mathcal{J}(\mathcal{L})$ such that $A = A_1 + A_2 + \dots + A_n$ with $A_i \in \mathcal{F}(K_i)$. By Lemma 3.5, for each i , there is an operator S_i in $\mathcal{F}(K_i)$ and an operator $\tau(A_i)$ in $Z(\mathcal{A})$ such that $\delta(A_i) = [A_i, S_i] + \tau(A_i)$. Let $S = S_1 + S_2 + \dots + S_n$. Since $S_i A_j = A_j S_i = 0$ if $i \neq j$, it follows that

$$\delta(A) = \sum_{i=1}^n \delta(A_i) = \sum_{i=1}^n ([A_i, S_i] + \tau(A_i)) = [A, S] + \sum_{i=1}^n \tau(A_i).$$

Define $\tau(A)$ to be an operator in $Z(\mathcal{A})$ such that $\delta(A) = [A, T] + \tau(A)$ for some T in $\mathcal{F}(\mathcal{L})$. By Lemma 3.2 and the equation above, we see that $\tau(A)$ is well-defined and $\tau(A) = \sum_{i=1}^n \tau(A_i)$.

Now define a mapping from $\mathcal{F}(\mathcal{L})$ into $\mathcal{A}$ as $\Delta(A) = \delta(A) - \tau(A)$ for $A \in \mathcal{F}(\mathcal{L})$. Then Δ is linear since τ is linear. Moreover, for each $A \in \mathcal{F}(\mathcal{L})$ there is an operator S such that $\Delta(A) = [A, S]$. Consequently, Δ is a local derivation from $\mathcal{F}(\mathcal{L})$ into $\mathcal{A}$. By Proposition 2.6, Δ is a derivation. Thus δ is standard. This completes the proof. $\square$

The following corollary can be applied to pentagon subspace lattice algebras.

Corollary 3.6. *Let $\mathcal{L}$ be a $\mathcal{J}$ -subspace lattice on a Banach space X and suppose that the dimension of each element in $\mathcal{J}(\mathcal{L})$ is infinite. Let δ be a Lie derivation from $\mathcal{F}(\mathcal{L})$ into itself. Then δ is a derivation.*

Proof. By Remark 2.5 (iii), $Z(\mathcal{F}(\mathcal{L})) = \{0\}$ in this case. Hence by Theorem 3.1, δ is a derivation. $\square$

4. LIE DERIVATIONS OF $\mathcal{J}$ -SUBSPACE LATTICE ALGEBRAS

In this section we describe the structure of Lie derivations of $\mathcal{J}$ -subspace lattice algebras as follows.

Theorem 4.1. *Let $\mathcal{L}$ be a $\mathcal{J}$ -subspace lattice on a Banach space X . Let $\delta : \text{Alg}\mathcal{L} \rightarrow \text{Alg}\mathcal{L}$ be a linear mapping. The following are equivalent.*

- (i) δ is a Lie derivation.
- (ii) For each $K \in \mathcal{J}(\mathcal{L})$, there exists an operator T_K in $B(K)$ and a linear functional $\lambda_K : \text{Alg}\mathcal{L} \rightarrow \mathbb{C}$ vanishing on every commutator such that $\delta(A)x = (T_K A - A T_K)x + \lambda_K(A)x$ for all $A \in \mathcal{A}$ and $x \in K$.

Proof. (ii) $\Rightarrow$ (i). This is a straightforward verification.

(i) $\Rightarrow$ (ii). Obviously, the restriction of δ to $\mathcal{F}(\mathcal{L})$ is a Lie derivation. Hence it is standard by Theorem 3.1. Therefore, there exists a derivation $d : \mathcal{F}(\mathcal{L}) \rightarrow \text{Alg}\mathcal{L}$ and a linear mapping $\tau : \mathcal{F}(\mathcal{L}) \rightarrow Z(\text{Alg}\mathcal{L})$ vanishing on every commutator such that

$$\delta(F) = d(F) + \tau(F)$$

for every $F \in \mathcal{F}(\mathcal{L})$.

Fix an element $K \in \mathcal{J}(\mathcal{L})$. Take vectors $x_K \in K$ and $f_K \in K^\perp$ such that $f_K(x_K) = 1$. Define a linear mapping $T_K : K \rightarrow K$ by

$$(4.1) \quad T_K x = d(x \otimes f_K)x_K$$

for $x \in K$. This is well-defined because of $x \otimes f_K \in \mathcal{F}(\mathcal{L})$ for $x \in K$.

For $F \in \mathcal{F}(\mathcal{L})$ and $x \in K$, by (4.1) we have that

$$T_K F x = d(F x \otimes f_K) x_K = d(F)(x \otimes f_K) x_K + F d(x \otimes f_K) x_K.$$

So for all $F \in \mathcal{F}(\mathcal{L})$ and for all $x \in K$,

$$(4.2) \quad d(F)x = (T_K F - F T_K)x.$$

Let A be in $\text{Alg}\mathcal{L}$. Let x be in K . Then for $F \in \mathcal{F}(\mathcal{L})$, by (4.1) and (4.2),

$$\begin{aligned} \delta([A, F])x &= d(AF - FA)x + \tau(AF - FA)x \\ &= (T_K(AF - FA) - (AF - FA)T_K)x + \tau([A, F])x \end{aligned}$$

and

$$\begin{aligned} \delta([A, F])x &= [\delta(A), F]x + [A, d(F)]x \\ &= (\delta(A)F - F\delta(A))x + (A(T_K F - F T_K) - (T_K F - F T_K)A)x. \end{aligned}$$

Comparing these two equations, we get

$$(\delta(A) - (T_K A - A T_K))F x = F(\delta(A) - (T_K A - A T_K))x + \tau([A, F])x.$$

Taking $f \in K^\perp$ with $f(x) = 1$ and then putting $F = x \otimes f$ in the last equation, we get that

$$(\delta(A) - (T_K A - A T_K))x = \lambda_1(A, x, f)x + \tau([A, x \otimes f])x.$$

Here $\lambda_1(A, x, f) = f((\delta(A) - (T_K A - A T_K))x)$. Since $\tau([A, x \otimes f])$ is in the center of $\text{Alg}\mathcal{L}$, it follows from Remark 2.5 that $\tau([A, x \otimes f])x = \lambda_2(A, x, f)x$ for some $\lambda_2(A, x, f) \in \mathbb{C}$. Consequently, $(\delta(A) - (T_K A - A T_K))x$ is a scalar multiple of x . Since this holds for each $x \in K$, it follows easily that $\delta(A) - (T_K A - A T_K)$ viewed as a linear mapping from K into K is a scalar multiple of the identity on K . Namely, there exists a scalar $\lambda_K(A)$ such that

$$(\delta(A) - (T_K A - A T_K))x = \lambda_K(A)x$$

for each $x \in K$. One can easily see that $\lambda_K : \text{Alg}\mathcal{L} \rightarrow \mathbb{C}$ is linear and vanishes on every commutator.

It remains to verify the boundedness of T_K . Suppose $\{x_n\}_{n=1}^\infty$ is a sequence of vectors in K , $\lim_{n \rightarrow \infty} x_n = 0$ and $\lim_{n \rightarrow \infty} T_K x_n = x_0$. For any $f \in K^\perp$,

$$\begin{aligned} f(x_n)d(x_K \otimes f_K)x_K &= d((x_K \otimes f)(x_n \otimes f_K))x_K \\ &= d(x_K \otimes f)x_n + (x_K \otimes f)d(x_n \otimes f_K)x_K \\ &= d(x_K \otimes f)x_n + f(T_K x_n)x_K. \end{aligned}$$

Taking the limit, we get that $f(x_0) = 0$. Since x_0 is in K and $f \in K^\perp$ is arbitrary, $x_0 = 0$. So T_K is closed. The Closed Graph Theorem gives the boundedness of T_K , completing the proof. $\square$

ACKNOWLEDGMENT

The author would like to thank the referee for the very thorough reading of the paper and valuable comments.

REFERENCES

1. J. Alaminos, M. Mathieu and A. R. Villena, *Symmetric amenability and Lie derivations*, Math. Proc. Cambridge Philos. Soc., 137(2004), 433-439. MR2092069 (2005g:46092)
2. K. I. Beidar and M. A. Chebotar, *On Lie derivations of Lie ideals of prime rings*, Israel J. Math., 123(2001), 131-148. MR1835291 (2002e:16052)
3. M. Brešar, *Commuting traces of biadditive mappings, commutativity-preserving mappings and Lie mappings*, Trans. Amer. Math. Soc., 335(1993), 525-546. MR1069746 (93d:16044)
4. M. Brešar and P. Semrl, *Mappings which preserve idempotents, local automorphism, and local derivations*, Can. J. Math., 45(1993), 483-496. MR1222512 (94k:47054)
5. W. Cheung, *Lie derivations of triangular algebras*, Linear Multilinear Algebra, 51(2003), 299-310. MR1995661 (2004f:16057)
6. K. Davidson, *Nest algebras*, Pitman Res. Notes, Math. Ser. 191, Longman Sci. Tech., New York, 1988. MR0972978 (90f:47062)
7. D. Hadwin and J. Li, *Local derivations and local automorphisms*, J. Math. Anal. Appl., 290(2004), 702-714. MR2033052 (2004j:47071)
8. P. R. Halmos, *A Hilbert Space Problem Book*, 2nd ed., Springer-Verlag, New York/Heidelberg/Berlin, 1982. MR0675952 (84e:47001)
9. B. E. Johnson, *Symmetric amenability and the nonexistence of Lie and Jordan derivations*, Math. Proc. Cambridge Philos. Soc., 120(1996), 455-473. MR1388200 (97m:46078)
10. B. E. Johnson, *Local derivations on C^* -algebras are derivations*, Trans. Amer. Math. Soc., 353(2001), 313-325. MR1783788 (2002c:46132)
11. R. V. Kadison, *Local derivations*, J. Algebra, 130(1990), 494-509. MR1051316 (91f:46092)
12. A. Katavolos, M. S. Lambrou and W. E. Longstaff, *Pentagon subspace lattices on Banach spaces*, J. Operator Theory, 46 (2001), 355-380. MR1870412 (2003a:47137)
13. M. S. Lambrou, *Approximants, commutants and double commutants in normed algebras*, J. London Math. Soc., 25(1982), 499-513. MR0657507 (84f:47053)
14. M. S. Lambrou, *Automatic continuity and implementation of homomorphisms*, (manuscript).
15. M. S. Lambrou and W. E. Longstaff, *Non-reflexive pentagon subspace lattices*, Studia Math., 125(1997), 187-199. MR1455633 (98f:47006)
16. W. E. Longstaff, *Strongly reflexive lattices*, J. London Math. Soc., 11 (1975), 491-498. MR0394233 (52:15036)
17. W. E. Longstaff, J. B. Nation and O. Panaia, *Abstract reflexive sublattices and completely distributive collapsibility*, Bull. Austral. Math. Soc., 58(1998), 245-260. MR1642047 (2000m:06016)
18. W. E. Longstaff and O. Panaia, *$\mathcal{J}$ -subspace lattices and subspace M -bases*, Studia Math., (3) 139 (2000), 197-212. MR1762581 (2001g:46020)
19. F. Lu and P. Li, *Algebraic isomorphisms and Jordan derivations of $\mathcal{J}$ -subspace lattice algebras*, Studia Math., 158(2003), 287-301. MR2014303 (2004m:47162)
20. F. Lu and P. Li, *Jordan isomorphisms of $\mathcal{J}$ -subspace lattice algebras*, Linear Algebra Appl., 371(2003), 255-264. MR1997374 (2004f:47104)
21. W. S. Martindale, *Lie derivations of primitive rings*, Michigan J. Math., 11(1964), 183-187. MR0166234 (29:3511)
22. M. Mathieu and A. R. Villena, *The structure of Lie derivations on C^* -algebras*, J. Funct. Anal., 202(2003), 504-525. MR1990536 (2004k:46125)
23. C. R. Miers, *Lie derivations of von Neumann algebras*, Duke Math. J., 40(1973), 403-409. MR0315466 (47:4015)
24. O. Panaia, *Quasi-spatiality of isomorphisms for certain classes of operator algebras*, Ph. D. dissertation, University of Western Australia, 1995.
25. G. A. Swain, *Lie derivations of the skew elements of prime rings with involution*, J. Algebra, 184(1996), 679-704. MR1409235 (97f:16059)
26. G. A. Swain and P. S. Blau, *Lie derivations in prime rings with involution*, Canad. Math. Bull., 42(1999), 401-411. MR1703701 (2000g:16039)
27. R. A. Villena, *Lie derivations on Banach algebras*, J. Algebra, 226(2000), 390-409. MR1749896 (2001e:47063)

DEPARTMENT OF MATHEMATICS, SUZHOU UNIVERSITY, SUZHOU 215006, PEOPLE'S REPUBLIC
OF CHINA

E-mail address: fylu@suda.edu.cn