NON-ABELIAN LOCAL INVARIANT CYCLES

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ABSTRACT. Let f be a degeneration of Kähler manifolds. The local invariant cycle theorem states that for a smooth fiber of the degeneration, any cohomology class, invariant under the monodromy action, comes from a global cohomology class. Instead of the classical cohomology, one may consider the non-abelian cohomology. This note demonstrates that the analogous non-abelian version of the local invariant cycle theorem does not hold if the first non-abelian cohomology is the moduli space (universal categorical quotient) of the representations of the fundamental group.

A degeneration of Kähler manifolds is a proper map f from a Kähler manifold X onto the unit disk Δ such that f is of maximum rank for all $s \in \Delta$ except at the point $s = 0$. Let $\Delta^* = \Delta - \{0\}$. We call $X_t = f^{-1}(t)$ a smooth fiber or generic fiber when $t \in \Delta^*$, and $X_0 = f^{-1}(0)$ the singular or degenerated fiber. We assume X_0 has normal crossing singularities.

Fix $t \in \Delta^*$ and a base point $x \in X_t$ once and for all. There is a monodromy action (see, for example, [3])

$$\pi_1(\Delta^*) \times H^n(X_t, \mathbb{C}) \rightarrow H^n(X_t, \mathbb{C}).$$

The local invariant cycle theorem states that a cohomology class in $H^n(X_t, \mathbb{C})$, fixed by the monodromy action, is a restriction of a cohomology class in $H^n(X, \mathbb{C})$ [2, 9, 10]. Fix a generator $T \in \pi_1(\Delta^*) \cong \mathbb{Z}$. Then T determines the monodromy action and gives rise to the isomorphisms

$$T^* : H^n(X_t, \mathbb{C}) \rightarrow H^n(X_t, \mathbb{C}), \quad T_* : \pi_1(X_t, x) \rightarrow \pi_1(X_t, y).$$

These isomorphisms are actually induced from a Picard-Lefschetz diffeomorphism which we shall also denote by

$$T : X_t \rightarrow X_t,$$

where $y = T(x)$.

The map f induces a strong deformation retraction $X \rightarrow X_0$, hence, also an isomorphism $H^n(X_0, \mathbb{C}) \rightarrow H^n(X, \mathbb{C})$ [2, 3, 6]. Define

$$c : X_t \hookrightarrow X \rightarrow X_0,$$

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where the first map is the inclusion and the second the strong deformation retraction. Then c induces homomorphisms:

$$c^* : H^n(X_0, \mathbb{C}) \rightarrow H^n(X_t, \mathbb{C}), \quad c_* : \pi_1(X_t, x) \rightarrow \pi_1(X_0, c(x)).$$

The local invariant cycle theorem states that [2, 9, 10]

$$(1) \quad \text{im}(c^*) = \{\alpha \in H^n(X_t, \mathbb{C}) \mid T^*(\alpha) = \alpha\}.$$

To generalize the above statement to the non-abelian context, we choose the universal categorical quotient

$$H^1(X_s, G) = \text{Hom}(\pi_1(X_s, x), G) // G, \quad s \in \Delta,$$

as the model for the first non-abelian Betti cohomology on X_s with coefficients in the algebraic group G . Here G acts on $\text{Hom}(\pi_1(X_s, x), G)$ by conjugation [4, 5]. For any $\rho \in \text{Hom}(\pi_1(X_t, x), G)$, denote by $[\rho]$ its image in $H^1(X_t, G)$ by the universal morphism.

The map T_* induces a morphism

$$T^\# : \text{Hom}(\pi_1(X_t, T(x)), G) \rightarrow \text{Hom}(\pi_1(X_t, x), G)$$

defined by

$$T^\#(\rho)(A) = \rho \circ T_*^{-1}(A)$$

for all $A \in \pi_1(X_t, x)$. Similarly, c induces a map

$$c^\# : \text{Hom}(\pi_1(X_0, c(x)), G) \rightarrow \text{Hom}(\pi_1(X_t, x), G)$$

defined by

$$c^\#(\rho)(A) = \rho \circ c_*(A).$$

The maps $T^\#$ and $c^\#$ descend to morphisms on their respective universal categorical quotients:

$$T^\# : H^1(X_t, G) \rightarrow H^1(X_t, G), \quad c^\# : H^1(X_0, G) \rightarrow H^1(X_t, G).$$

In this setting, the analogous statement of the local invariant cycle theorem is

$$(2) \quad \text{im}(c^\#) = \{[\rho] \in H^1(X_t, G) \mid T^\#([\rho]) = [\rho]\}.$$

Notice that when $G = \mathbb{C}$ (the addition group), $H^1(X_s, G)$ is the regular first cohomology $H^1(X_s, \mathbb{C})$ of X_s for $s \in \Delta$ and (2) reduces to the classical local invariant cycle theorem (1).

Theorem 1. *For each $g > 1$, there exists an n and a degeneration $f : X \rightarrow \Delta$ with the generic fiber X_t a Riemann surface of genus g and such that*

$$\text{im}(c^\#) \neq \{[\rho] \in H^1(X_t, G) \mid T^\#([\rho]) = [\rho]\},$$

where $G = \text{SL}(n, \mathbb{C})$. More precisely, for such an n , there exists an irreducible representation $\rho \in \text{Hom}(\pi_1(X_t, x), G)$ with $\text{im}(\rho) \subset \text{SU}(n)$ and $[\rho] \in \{[\rho] \in H^1(X_t, G) \mid T^\#([\rho]) = [\rho]\}$ but $[\rho] \notin \text{im}(c^\#)$.

Proof. For each n , let $G = \text{SL}(n, \mathbb{C})$ and $K = \text{SU}(n)$. Let Γ be the mapping class group of X_t . Then each $\gamma \in \Gamma$ induces a map

$$\gamma : \text{Hom}(\pi_1(X_t, \gamma(x)), G) \rightarrow \text{Hom}(\pi_1(X_t, x), G) : \gamma(\rho)(A) = \rho \circ \gamma^{-1}(A).$$

These maps provide a Γ -action on $H^1(X_t, G)$. In [1], Andersen proved that for $g > 1$, there exists infinitely many n such that there exists irreducible $\rho \in \text{Hom}(\pi_1(X_t), G)$ with $\text{im}(\rho) \subset K$ and $\gamma([\rho]) = [\rho]$ for all $\gamma \in \Gamma$.

Now suppose our choices of ρ and n satisfy the hypothesis and conclusions of Andersen's theorem [1]. Then since $T : X_t \rightarrow X_t$ is a diffeomorphism, $T_* \in \Gamma$. Hence $T^\#([\rho]) = [\rho]$.

The fundamental group $\pi_1(X_t, x)$ has a presentation

$$\langle A_i, 1 \leq i \leq 2g, \prod_{i=1}^g A_i A_{g+i} A_i^{-1} A_{g+i}^{-1} \rangle,$$

where the A_i 's correspond to simple loops beginning and ending at x . Since ρ is an irreducible representation, $\rho(A_i) \neq I \in G$ for some i .

Now let $f : X \rightarrow \Delta$ be a degeneration with A_i as its vanishing cycle. Then $c_*(A_i) = e \in \pi(X_0, c(x))$. Hence for every $[\alpha] \in H^1(X_0, G)$,

$$c^\#(\alpha)(A_i) = \alpha(c_*(A_i)) = I \in G.$$

Since $\rho(A_i) \neq I$, $[\rho] \neq [\alpha]$ for all $[\alpha] \in \text{im}(c^\#)$. $\square$

Final remarks. We refer to [1] for the detailed and explicit descriptions of Andersen's irreducible representation classes fixed by the Γ -action. Here we simply mention that all such representations have finite images in $\text{SU}(n)$. It will be interesting to see whether there exist examples $[\rho]$, satisfying the conclusion of Theorem 1, with $\text{im}(\rho)$ being Zariski dense in G . Unfortunately, such representations are unlikely to satisfy the overly stringent condition of being fixed by the Γ -action. In fact, it is likely that the Γ -orbit of $[\rho]$ is Zariski dense if $\text{im}(\rho)$ is Zariski dense in G . This is the case when $G = \text{SL}(2, \mathbb{C})$, $K = \text{SU}(2)$ [7, 8].

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