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LOG-LOG CONVEXITY AND BACKWARD UNIQUENESS

IGOR KUKAVICA

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ABSTRACT. We study backward uniqueness properties for equations of the form

$$u' + Au = f.$$

Under mild regularity assumptions on A and f, it is shown that $u(0) = 0$ implies $u(t) = 0$ for $t < 0$. The argument is based on α-log and log-log convexity. The results apply to mildly nonlinear parabolic equations and systems with rough coefficients and the 2D Navier-Stokes system.

1. INTRODUCTION

Backward uniqueness for evolution partial differential equations is a classical problem initiated by Lax [L], and minimal regularity requirements under which backward uniqueness holds are not known for many important partial differential equations and systems. A basic question for an evolution equation written in a form

$$u' + Au = f(u)$$

is under which conditions $u(T) = 0$ implies $u(t) = 0$ for $t < T$. Backward uniqueness is substantially more difficult than forward uniqueness due to ill-posedness (in general) of the backward evolution problem. Since it is impossible to survey the large literature on this topic, we provide a short description of relevant work. There are basically two methods addressing this problem. The first is based on logarithmic convexity [AN1, AN2, A, BT, G, O] and the second on time-weighted inequalities [P, LP, S]. The approach, based on logarithmic convexity and second-order inequalities, was developed in [AN1]. The approach was substantially simplified by Ogawa [O], who reduced a proof of backward uniqueness to establishing upper bounds on the Dirichlet quotient $Q(t) = (Au, u)/\|u\|^2$. Further simplifications and applications were given in [BT, CFNT, G, CFKM]; in particular, the important identity (5) was established in [CFNT, G]. In summary, the most general known situation for the backward uniqueness property is

$$\|f\|^2 \leq K(Au, u) + K\|u\|^2$$

with a certain integrability assumption on K if K depends on t [G]. In our previous paper [K] we have shown the connection between backward uniqueness and unique

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continuation. The paper [CFKM] used Dirichlet quotients extensively to study backward behavior and Eulerian dynamics for the 2D Navier-Stokes equations.

In this paper, we introduce a log-Dirichlet quotient

$$\widetilde{Q}(t) = \frac{(Au(t), u(t))}{\|u(t)\|^2 \left(\log \frac{M_0^2}{\|u(t)\|^2}\right)^{\alpha}},$$

where M_0 is a suitably large constant. While the classical Dirichlet quotient measures exponential decay of $\|u(t)\|$, the log-Dirichlet quotient quantifies $\exp(-C|t|^{1/(1-\alpha)})$ type decay of $\|u(t)\|$ if $\alpha \in (0,1)$ and $\exp(-Ce^{|t|})$ type decay of $\|u(t)\|$ if $\alpha = 1$. The advantage of this quotient is that the differential inequality for $\widetilde{Q}$ contains an extra positive term on the left-hand side which is even quadratic in $\widetilde{Q}$ (see (6) below). Exploring this fact, we are able to treat nonlinear equations with much rougher coefficients than allowed before (see Section 3 below for applications). Moreover, we are able to obtain a sharper result even in the classical range (that is, a slight sublinearity is allowed when coefficients are not too irregular). We emphasize that the generality of the presentations allows applications to higher-order evolution equations as well as systems. Section 2 contains the statement and the proof of the main results. Section 3 contains three applications—a parabolic nonlinear equation, a parabolic system and a theorem on boundedness of log-Dirichlet quotients for differences of solutions of the 2D periodic Navier-Stokes equations on the global attractor.

2. The main result

Let H be a real or complex Hilbert space with the scalar product $(\cdot,\cdot)$ and the norm $\|\cdot\|$. Let A be a symmetric operator with the domain $D(A) \subseteq H$. Assume that $(Au, u) \geq 0$ for $u \in D(A)$. Let $u \in C([T_0, 0], D(A)) \cap C^1([T_0, 0], H)$ be a solution of

$$u' + Au = f$$

with $f \in C([T_0, 0], H)$, where the requirement $u \in C([T_0, 0], D(A))$ means $Au \in C([T_0, 0], H)$. Denote

$$L(\|u\|) = \log \frac{M_0^2}{\|u\|^2}$$

(that is, $L(x) = \log(M_0^2/x^2)$), where M_0 is any constant such that

$$M_0 \geq 2 \sup_{t \in [T_0, 0]} \|u(t)\|.$$

Note that $L(\|u(t)\|) \geq 1$ for all $t \in [T_0, 0]$. Our assumptions do not imply existence of $A^{1/2}$; however, it will be convenient to use the notation

$$\|A^{1/2}v\| = (Av, v)^{1/2}, \qquad v \in D(A).$$

On f, which in applications depends on u, we assume

$$\|f\| \leq \frac{K_1}{L(\|u\|)^{\beta/2}} \|A^{1/2}u\|^{1-\beta} \|Au\|^{\beta} + K_2 L(\|u\|)^{\alpha/2} \|u\| \tag{1}$$

and

$$\mathbb{R}\mathrm{e}(f, u) \geq -K_3 L(\|u\|)^{\alpha(2-\beta_0)/2} \|u\|^{2-\beta_0} \|A^{1/2}u\|^{\beta_0} - K_4 L(\|u\|)^{\alpha} \|u\|^2 \tag{2}$$

for some $\alpha, \beta \in [0,1]$, $\beta_0 \in [0,2]$, and $K_1, K_2, K_3, K_4 \geq 0$. Additionally, we assume

$$K_1^2 \leq \frac{\alpha}{8} \tag{3}$$

if $\beta = 1$. Note that the classical case corresponds to $\beta = 0$, $\beta_0 = 1$, $\alpha = 0$.

Theorem 2.1. *Let $u\colon [T_0, 0] \to H$ be as above. Then $u(0) = 0$ implies $u(t) = 0$ for all $t \in [T_0, 0]$.*

Proof. For $t \in [T_0, 0]$, denote $\widetilde{L}(t) = L(\|u(t)\|)$. By continuity, it is sufficient to assume $\|u(t)\| \neq 0$ for $t \in [T_0, 0)$ and prove that $\|u(0)\| \neq 0$. For this, we introduce the log-Dirichlet quotient

$$\widetilde{Q}(t) = \frac{Q(t)}{L(\|u\|)^\alpha} = \frac{\|A^{1/2}u\|^2}{\|u\|^2 L(\|u\|)^\alpha} = \frac{\|A^{1/2}u(t)\|^2}{\|u(t)\|^2 \widetilde{L}(t)^\alpha},$$

where $Q(t) = \|A^{1/2}u\|^2/\|u\|^2$ is the classical Dirichlet quotient [O, BT]. Note that while $Q(t)$ controls the exponential decay of $\|u(t)\|$, we shall show that the log-Dirichlet quotient controls $\exp(-C|t|^{1/(1-\alpha)})$ type decay if $\alpha \in (0,1)$, and $\exp(-Ce^{|t|})$ type decay if $\alpha = 1$.

Note that our assumptions imply

$$\frac{1}{2}\frac{d}{dt}(u,u) = \mathbb{R}\mathrm{e}(u_t, u) = -(Au, u) + \mathbb{R}\mathrm{e}(f, u) \tag{4}$$

and

$$\frac{1}{2}\frac{d}{dt}(Au,u) = \mathbb{R}\mathrm{e}(u_t, Au) = -(Au, Au) + \mathbb{R}\mathrm{e}(f, Au).$$

From here, we obtain the identity ([CFNT, G])

$$\frac{1}{2}Q'(t) + \big\|\big(A - Q(t)I\big)w\big\|^2 = \mathbb{R}\mathrm{e}\left(\frac{f}{\|u\|}, \big(A - Q(t)I\big)w\right), \qquad t \in [T_0, 0), \tag{5}$$

where $w = u/\|u\|$, which can be verified by a direct calculation. Using the identity

$$\widetilde{Q}'(t) = \frac{Q'(t)}{\widetilde{L}(t)^\alpha} - \alpha\frac{Q(t)\widetilde{L}'(t)}{\widetilde{L}(t)^{\alpha+1}}$$

with

$$\widetilde{L}'(t) = -\frac{\frac{d}{dt}\|u\|^2}{\|u\|^2} = 2Q(t) - 2\frac{\mathbb{R}\mathrm{e}(f,u)}{\|u\|^2}$$

we get

$$\frac{1}{2}\widetilde{Q}'(t) + \frac{\alpha\widetilde{Q}(t)^2}{\widetilde{L}(t)^{1-\alpha}} + \frac{\big\|\big(A - Q(t)I\big)w\big\|^2}{\widetilde{L}(t)^\alpha} = \frac{\mathbb{R}\mathrm{e}(f, (A - Q(t)I)w)}{\|u\|\widetilde{L}(t)^\alpha} + \frac{\alpha\widetilde{Q}(t)\,\mathbb{R}\mathrm{e}(f,u)}{\|u\|^2\widetilde{L}(t)}, \qquad t \in [T_0, 0).$$

Using $(f, (A - Q(t)I)w) \leq \|f\|\|(A - Q(t)I)w\|$ on the first term on the right-hand side and

$$\frac{\alpha\widetilde{Q}(t)\,\mathbb{R}\mathrm{e}(f,u)}{\|u\|^2\widetilde{L}(t)} \leq \frac{\alpha\widetilde{Q}(t)\|f\|}{\|u\|\widetilde{L}(t)} \leq \frac{\alpha\widetilde{Q}(t)^2}{2\widetilde{L}(t)^{1-\alpha}} + \frac{\alpha\|f\|^2}{2\widetilde{L}(t)^{1+\alpha}\|u\|^2} \leq \frac{\alpha\widetilde{Q}(t)^2}{2\widetilde{L}(t)^{1-\alpha}} + \frac{\|f\|^2}{2\widetilde{L}(t)^\alpha\|u\|^2}$$

on the second, we obtain

$$\widetilde{Q}'(t) + \frac{\alpha\widetilde{Q}(t)^2}{\widetilde{L}(t)^{1-\alpha}} + \frac{\big\|\big(A - Q(t)I\big)w\big\|^2}{\widetilde{L}(t)^\alpha} \leq \frac{2\|f\|^2}{\|u\|^2\widetilde{L}(t)^\alpha}. \tag{6}$$

Therefore, by squaring (1) and applying $\|Au\| \leq \|(A-Q(t)I)u\| + Q(t)\|u\|$,

$$\begin{aligned}\frac{2\|f\|^2}{\|u\|^2\widetilde{L}(t)^\alpha} &\leq \frac{4K_1^2}{\widetilde{L}(t)^{\alpha+\beta}}\frac{\|A^{1/2}u\|^{2-2\beta}\|Au\|^{2\beta}}{\|u\|^2} + 4K_2^2 \\ &\leq \frac{8K_1^2}{\widetilde{L}(t)^{\alpha+\beta}}Q(t)^{1-\beta}\big\|\big(A-Q(t)I\big)w\big\|^{2\beta} + \frac{8K_1^2Q(t)^{1+\beta}}{\widetilde{L}(t)^{\alpha+\beta}} + 4K_2^2.\end{aligned}$$

If $\beta \in (0,1)$, we use $ab \leq a^p/p + b^q/q$ with $p = 1/\beta$ and $q = 1/(1-\beta)$. We obtain

$$\begin{aligned}\frac{2\|f\|^2}{\|u\|^2\widetilde{L}(t)^\alpha} \leq \frac{(1-\beta)(8K_1^2)^{1/(1-\beta)}\widetilde{Q}(t)}{\widetilde{L}(t)^{\beta/(1-\beta)}} &+ \frac{\beta\big\|\big(A-Q(t)I\big)w\big\|^2}{\widetilde{L}(t)^\alpha} \\ &+ \frac{8K_1^2Q(t)^{1+\beta}}{\widetilde{L}(t)^{\alpha+\beta}} + 4K_2^2.\end{aligned}$$

The third term equals

$$\begin{aligned}\frac{8K_1^2\widetilde{Q}(t)^{1+\beta}}{\widetilde{L}(t)^{\beta(1-\alpha)}} &= \left(\frac{\alpha\widetilde{Q}(t)^2}{\widetilde{L}(t)^{1-\alpha}}\right)^\beta \frac{8K_1^2\widetilde{Q}(t)^{1-\beta}}{\alpha^\beta} \\ &\leq \frac{\alpha\beta\widetilde{Q}(t)^2}{\widetilde{L}(t)^{1-\alpha}} + (1-\beta)\left(\frac{8K_1^2}{\alpha^\beta}\right)^{1/(1-\beta)}\widetilde{Q}(t).\end{aligned}$$

Since $\widetilde{L}(t) \geq 1$, we get $\widetilde{Q}'(t) \leq K_5\widetilde{Q}(t) + K_6$, where $K_5 = (1-\beta)(8K_1^2)^{1/(1-\beta)}(1+\alpha^{-\beta/(1-\beta)})$ and $K_6 = 4K_2^2$, which implies

$$\sup_{t\in[T_0,0)} \widetilde{Q}(t) < \infty. \tag{7}$$

If $\beta = 0$, all the above inequalities hold trivially. If $\beta = 1$, then we have

$$\begin{aligned}\widetilde{Q}'(t) + \frac{\alpha\widetilde{Q}(t)^2}{\widetilde{L}(t)^{1-\alpha}} &+ \frac{\big\|\big(A-Q(t)I\big)w\big\|^2}{\widetilde{L}(t)^\alpha} \\ &\leq \frac{8K_1^2}{\widetilde{L}(t)^{1+\alpha}}\big\|\big(A-Q(t)I\big)w\big\|^2 + \frac{8K_1^2\widetilde{Q}(t)^2}{\widetilde{L}(t)^{1-\alpha}} + 4K_2^2.\end{aligned}$$

Since (3) holds if $\beta = 1$, we get $\widetilde{Q}' \leq K_5\widetilde{Q} + K_6$ with $K_5 = 0$ and $K_6 = 4K_2^2$, and (7) follows also in this case.

It remains to be checked that (7) implies that $\|u(0)\|$ is nonzero. From (4), we get

$$\frac{\frac{1}{2}\frac{d}{dt}\|u\|^2}{\widetilde{L}(t)^\alpha\|u\|^2} + \widetilde{Q}(t) = \frac{\mathbb{R}\mathrm{e}(f,u)}{\widetilde{L}(t)^\alpha\|u\|^2}, \qquad t \in [T_0, 0).$$

Using (2), we get

$$\frac{\frac{1}{2}\frac{d}{dt}\|u\|^2}{\widetilde{L}(t)^\alpha\|u\|^2} + \widetilde{Q}(t) \geq -\frac{K_3Q(t)^{\beta_0/2}}{\widetilde{L}(t)^{\alpha\beta_0/2}} - K_4 = -K_3\widetilde{Q}(t)^{\beta_0/2} - K_4.$$

Therefore,

$$\frac{\frac{1}{2}\frac{d}{dt}\|u\|^2}{\widetilde{L}(t)^\alpha\|u\|^2} + K_7\widetilde{Q}(t) \geq -K_8, \tag{8}$$

where $K_7 = 1 + \beta_0K_3/2$ and $K_8 = K_4 + (2-\beta_0)K_3/2$.

If $\alpha < 1$, then we integrate (8) to obtain

$$\widetilde{L}(t_2)^{1-\alpha} \le \widetilde{L}(t_1)^{1-\alpha} + 2(1-\alpha)K_7 \int_{t_1}^{t_2} \widetilde{Q}(\tau)\, d\tau + 2(1-\alpha)K_8(t_2 - t_1)$$

if $T_0 \le t_1 \le t_2 < 0$, from which

$$\begin{aligned}&\log \|u(t_2)\|^2 \\ &\ge 2\log M_0 - \left(\widetilde{L}(t_1)^{1-\alpha} + 2(1-\alpha)K_7 \int_{t_1}^{t_2} \widetilde{Q}(\tau)\, d\tau + 2(1-\alpha)K_8(t_2 - t_1)\right)^{1/(1-\alpha)}\end{aligned}$$

and using (7) we conclude that $\|u(0)\|$ cannot vanish. If $\alpha = 1$, then $\frac{1}{2}\frac{d}{dt}(-\log \widetilde{L}(t)) \ge -K_7\widetilde{Q}(t) - K_8$, and we get

$$\log \|u(t_2)\|^2 \ge \log M_0^2 - \widetilde{L}(t_1) \exp\left(2K_7 \int_{t_1}^{t_2} \widetilde{Q}(\tau)\, d\tau + 2K_8(t_2 - t_1)\right) \tag{9}$$

for $T_0 \le t_1 \le t_2 < 0$. □

Remark 2.2. It is easy to adjust the statement so that K_1, K_2, K_3, and K_4 depend on t under suitable integrability conditions. We only need to require that K_5, K_6, K_7, and K_8 are integrable on $[0, T_0]$.

Remark 2.3. Nonnegativity of A was not used in the preceding. Therefore, Theorem 2.1 still holds by letting $\beta = 1$, $\beta_0 = 2$, $\alpha = 1$ in the assumptions and substituting $\|A^{1/2}u\|^2$ with (Au, u). Other cases can also be handled similarly.

Remark 2.4. The proof of Theorem 2.1 also gives a lower bound on the long term decay of bounded solutions $\|u\|$ on $(0, \infty)$. In particular, if $\alpha = 1$, the proof implies the boundedness of $\widetilde{Q}$ for all $t \ge 0$, and the decay estimate (9) follows.

3. Applications

In this section, we present examples of partial differential equations where Theorem 2.1 applies.

3.1. **A second-order evolution equation.** First, consider a solution of

$$\frac{\partial u}{\partial t} - \Delta u + W_j(x,t)|u|^\theta \partial_j u + V(x,t)|u|^\theta u = 0,$$

where $u \in C^1([0,T], H^2(\mathbb{R}^n)) \cap C([0,T], L^2(\mathbb{R}^n)) \cap L^\infty(\mathbb{R}^n \times [0,T])$ and where $\theta > 0$ is arbitrary. By Theorem 2.1, if $u(0) \ne 0$ and $V \in L^\infty([0,T], L^p(\mathbb{R}^n))$ and $W_j \in L^\infty([0,T], L^q(\mathbb{R}^n))$ for $j = 1, \dots, n$ where $p > n/2$ and $q > n$, with an additional condition $p, q \ge 2$, then

$$u(t) \ne 0, \qquad t \in (0, T].$$

A similar statement can be obtained for higher-order parabolic equations. Let $n \ge 4$ (the cases $n = 1, 2, 3$ are similar). It is sufficient to check the assumptions under the restriction $p \in (n/2, n)$ since other cases can be covered by a standard logarithmic convexity argument. As $n \ge 4$ and $p \in (n/2, n)$, we may choose $\epsilon \in (0, \theta)$ such that

$$\frac{1}{n} < \frac{1}{p} + \frac{\epsilon}{2} < \frac{2}{n}$$

and estimate for every $t \in [0, T]$,

$$\|V|u|^{1+\theta}\|_{L^2} \le M^{\theta-\epsilon} \|V\|_{L^p} \|u\|_{L^2}^{\epsilon} \|u\|_{L^{\tilde{p}}},$$

where $M = \|u\|_{L^\infty(\mathbb{R}^n\times[0,T])}$ and $1/p + \epsilon/2 + 1/\widetilde{p} = 1/2$ and then interpolate

$$\|u\|_{L^{\tilde{p}}} \le C\|\nabla u\|_{L^2}^{1-\gamma}\|\Delta u\|_{L^2}^{\gamma},$$

where $\gamma = n/p + \epsilon n/2 - 1$, which gives (1). Other conditions and cases are checked in a similar manner. We note that the equation $W_j = 0$, $\theta = 0$ was treated in [K] by reducing the backward uniqueness to a unique continuation theorem in [E, EV] (however, the assumption $n \ge 5$ should be added to the assumptions or a requirement $p \ge 2$ added for $v \in C([T_0, 0], L^p(\mathbb{R}^n))$ in [K, Corollary 2.3]).

3.2. **A second-order parabolic system.** Consider a solution of

$$\frac{\partial u_i}{\partial t} - \partial_k(a_{ijkl}(x,t)\partial_l u_j) + W_{ijk}(x,t)|u|^\theta \partial_j u_k + V_{ik}(x,t)|u|^\theta u_k = 0, \qquad i = 1,\dots,m,$$

where $u = (u_1, u_2, \dots, u_m) \in C^1([0,T], H^2(\mathbb{R}^n)^m) \cap C([0,T], L^2(\mathbb{R}^n)^m) \cap L^\infty(\mathbb{R}^n \times [0,T])$ and where $\theta > 0$ is arbitrary. We assume that the coefficient tensor $a_{ijkl}(x,t)$ is bounded, C^1, symmetric, and uniformly strictly elliptic. By Theorem 2.1, if $u(0) \ne 0$ and $V \in L^\infty([0,T], L^p(\mathbb{R}^n))$ and $W_j \in L^\infty([0,T], L^q(\mathbb{R}^n))$ for $j = 1,\dots,n$, where $p > n/2$ and $q > n$ with an additional assumption $p, q \ge 2$, then

$$u(t) \ne 0, \qquad t \in (0,T].$$

Strictly speaking, Theorem 2.1 gives the statement if a_{ijkl} depends only on x. It is easy to extend the statement to the general case as well.

3.3. **2D Navier-Stokes equation.** Consider the Navier-Stokes system

$$\begin{aligned}&\partial_t u - \Delta u + u\cdot\nabla u + \nabla p = f,\\ &\nabla\cdot u = 0\end{aligned}$$

with periodic boundary conditions on $\Omega = [0, 2\pi]^2$. Let H be the closure of

$$\Big\{ v \in L^2_{\mathrm{per}}(\Omega)^2 : v \text{ is an } \Omega\text{−periodic trigonometric polynomial}, \qquad \nabla\cdot v = 0 \text{ in } \Omega, \int_\Omega v = 0 \Big\}$$

in the (real) Hilbert space $L^2_{\mathrm{per}}(\Omega)^2$. Then, under the condition $f \in H$ (f is time-independent), the equation possesses a global attractor

$$\mathcal{A} = \Big\{ u_0 \in H : S(t)u_0 \text{ exists for all } t \in \mathbb{R},\ \sup_{t\in\mathbb{R}} \|S(t)u_0\|_{L^2_{\mathrm{per}}(\Omega)} < \infty \Big\},$$

where $S(t)u_0$ denotes a solution starting at u_0 on its maximal interval of existence (cf. [CF]). It is an open problem whether

$$\sup_{u_1,u_2\in\mathcal{A}, u_1\ne u_2} \frac{\|\nabla(u_1-u_2)\|^2_{L^2_{\mathrm{per}}(\Omega)}}{\|u_1-u_2\|^2_{L^2_{\mathrm{per}}(\Omega)}} < \infty.$$

By the proof of Theorem 2.1, we get the following statement.

Theorem 3.1. *With the above assumptions, we have*

$$\sup_{u_1,u_2\in\mathcal{A}, u_1\ne u_2} \frac{\|\nabla(u_1-u_2)\|^2_{L^2_{\mathrm{per}}(\Omega)}}{\|u_1-u_2\|^2_{L^2_{\mathrm{per}}(\Omega)} \log(M_0^2/\|u_1-u_2\|^2_{L^2_{\mathrm{per}}})} < \infty,$$

where $M_0 = 4\sup_{u_0\in\mathcal{A}} \|u_0\|_{L^2_{\mathrm{per}}(\Omega)}$.

The upper bound can be explicitly estimated in terms of $\|f\|_{L^2_{\rm per}}$ A similar statement holds for the 2D Navier-Stokes equations in a smooth bounded domain.

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Department of Mathematics, University of Southern California, Los Angeles, California 90089

E-mail address: `kukavica@usc.edu`