ON QUASIPLANES IN EUCLIDEAN SPACES

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ABSTRACT. A variational inequality for the images of k -dimensional hyperplanes under quasiconformal maps of the n -dimensional Euclidean space is proved when $1 \leq k \leq n - 2$.

1. MAIN RESULTS

Below we use the terminology and notation of [7].

Let $\mathbf{R}^n$ be the n -dimensional Euclidean space, $n \geq 2$, let $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a mapping of the class $W_{n,\text{loc}}^1(\mathbf{R}^n)$, and let $f' : \mathbf{R}^n \rightarrow \mathbf{R}^n$ denote its formal derivative. We write

$$\|f'(x)\| = \max_{|h|=1} |f'(x)h|.$$

A homeomorphism $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is called K -quasiconformal [10, p. 250-252] if $f \in W_{n,\text{loc}}^1(\mathbf{R}^n)$ and

$$(1.1) \quad \|f'(x)\|^n \leq K J(x, f) \quad \text{a.e. on } \mathbf{R}^n,$$

where $J(x, f) = \det(f'(x))$ is the Jacobian of f at the point $x \in \mathbf{R}^n$. The smallest constant K in (1.1) is called the outer dilatation of f and denoted by $K_O(f)$. The smallest constant $K \geq 1$ in the inequality

$$J(x, f) \leq K \ell(f'(x))^n$$

is called the inner dilatation of $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ and denoted by $K_I(f)$. Here, $\ell(f'(x)) = \min_{|h|=1} |f'(x)h|$. The quantity

$$K(f) = \max\{K_O(f), K_I(f)\}$$

is called the maximal dilatation of f [10, Section 14.1].

If $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is quasiconformal, then it is well known that $f(\mathbf{R}^n) = \mathbf{R}^n$ and the inverse map $f^{-1} : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is also quasiconformal in $\mathbf{R}^n$ with $K(f^{-1}) = K(f)$.

Let $d(x', x'')$ be the Euclidean distance for $x', x'' \in \mathbf{R}^n$. We write

$$B(a, t) = \{x \in \mathbf{R}^n : d(a, x) < t\}, \quad S(a, t) = \{x \in \mathbf{R}^n : d(a, x) = t\}.$$

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Let $y = (y_1, y_2, \dots, y_n)$ be a point in $\mathbf{R}^n$ and $1 \leq k \leq n - 2$. Consider a k -dimensional plane

$$\Pi_0^k = \{y = (y_1, y_2, \dots, y_n) \in \mathbf{R}^n : y_{k+1} = \dots = y_n = 0\}.$$

Let Π^k be a k -dimensional surface in $\mathbf{R}^n$, i.e., a homeomorphic image of a k -dimensional plane. Fix a point $a \in \Pi^k$ and $R, 0 < R < \infty$. Denote by $\Pi^k(a, R)$ a component of $\Pi^k \cap B(a, R)$, $a \in \Pi^k(a, R)$. We say that a k -dimensional surface Π^k is a K -quasiplane, if there exists a K -quasiconformal mapping $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $f(\Pi^k) = \Pi_0^k$.

For $n = 2$, the following result is well known [1]. A curve $\Pi^1 \subset \mathbf{R}^2$ is a K -quasiconformal line if and only if there exists a constant $C(K)$ such that

$$\frac{d(\zeta_2, \zeta_1)}{d(\zeta_3, \zeta_1)} \leq C(K)$$

for every triple of distinct points on Π^1 such that ζ_2 lies between ζ_1 and ζ_3 .

For more information on quasiconformal lines, we refer to [8], [2, Chapter 14], [9], [11], [18], [12], [17]. Quasiplanes Π^{n-1} in $\mathbf{R}^n$ with $\text{codim } \Pi^{n-1} = 1$ were considered in [13], [16] and, in the setup of Riemannian manifolds, in [14]. Quasiplanes may have a highly complicated structure; see [2, pp. xxii–xxiii]. For instance, it is well known that quasiconformal lines may have Hausdorff dimension > 1 and hence be non-rectifiable; on the other hand sufficient conditions for the $(n - 1)$ -rectifiability of quasiplanes Π^{n-1} in $\mathbf{R}^n$ were given in [13]. Below we consider the case of K -quasiplanes in $\mathbf{R}^n$ of dimension $1 \leq k \leq n - 2$. Very recently, this topic was studied by I. Prause in [19] and [20].

For an arbitrary open subset Σ on the sphere $S(a, r) \subset \mathbf{R}^n$ we define the quantity

$$(1.2) \quad \eta(\Sigma) = \sup_A \inf_{\varphi} \frac{\left(\int_{\Sigma} |\nabla_S \varphi|^n d\mathcal{H}^{n-1} \right)^{1/n}}{\left(\int_{\Sigma} |\varphi - A|^n d\mathcal{H}^{n-1} \right)^{1/n}}.$$

Here the infimum is taken over all functions

$$\varphi \in W_n^1(\Sigma), \quad \varphi|_{\partial\Sigma} = 0,$$

the supremum is taken over all constants A , and $\partial\Sigma$ is the boundary with respect to $S(a, r)$.

The quantity $\nabla_S \varphi$ is the gradient of φ on $S(a, r)$. (The most convenient for our purposes is the definition of $\nabla_S \varphi$ on a surface $S \subset \mathbf{R}^n$; see, for example, in §2 of [15].)

The zero boundary values $\varphi|_{\partial\Sigma} = 0$ are understood in the Sobolev sense, i.e., $\varphi \in W_{n,0}^1(\Sigma)$.

It is clear that

$$(1.3) \quad \eta(\Sigma) \geq \inf_{\varphi} \frac{\left(\int_{\Sigma} |\nabla_S \varphi|^n d\mathcal{H}^{n-1} \right)^{1/n}}{\left(\int_{\Sigma} |\varphi|^n d\mathcal{H}^{n-1} \right)^{1/n}}.$$

Because $\dim S = n - 1$, it follows from Sobolev's C^0 -embedding theorem that the right side of (1.3) is > 0 at least for every Σ with a smooth boundary $\partial\Sigma \neq \emptyset$. To find the best $\eta(\Sigma)$ is an open problem.

1.4. Theorem. *Let Π^k be a K -quasiplane in $\mathbf{R}^n$ with $1 \leq k \leq n - 2$. Then for every point $a \in \Pi^k$ and all $0 < r < R < \infty$, the following relation holds:*

$$(1.5) \quad \exp \left\{ \frac{1}{K_O(f)} \int_r^R \eta(\Sigma(a, \tau)) d\tau \right\} \leq D(n, K) \left(\frac{R}{r} \right)^{n\beta}.$$

Here $\beta = K^{1/(n-1)}$ and

$$D(n, K) = D_*^{2n}, \quad D_* = D_*(K) = \exp(4K(K+1)\sqrt{K-1}).$$

2. THE LANGUAGE OF DIFFERENTIAL FORMS

Let $1 \leq k \leq n - 2$ and let

$$f = (f_1, \dots, f_k, f_{k+1}, \dots, f_n) : \mathbf{R}^n \rightarrow \mathbf{R}^n$$

be a K -quasiconformal mapping.

Let Π_0^k be a k -dimensional plane of the form

$$\Pi_0^k = \{y = (y_1, \dots, y_k, y_{k+1}, \dots, y_n) : y_{k+1} = \dots = y_n = 0\}.$$

We will study some properties of the k -dimensional surface

$$\Pi^k = f^{-1}(\Pi_0^k).$$

Fix a point $a \in \Pi^k$. For $R > 0$ let $\Pi(a, R)$ be the component of $\Pi^k \cap B(a, R)$, containing the point a and $\Sigma(a, R) = S(a, R) \setminus \overline{\Pi(a, R)}$.

We consider the differential form of degree $n - 1$

$$\omega = \sum_{i=k+1}^n (-1)^{i-k-1} f_i df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n.$$

It is clear that for a.e. $R \in (0, \infty)$, we have

$$\omega \in W^{1,1}(S(a, R)) \quad \text{and} \quad \omega|_{\partial\Sigma(a, R)} = 0.$$

We have

$$\begin{aligned} d\omega &= \sum_{i=k+1}^n (-1)^{i-k-1} df_i \wedge df_{k+1} \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n \\ &= (n-k) df_1 \wedge \dots \wedge df_n, \end{aligned}$$

and thus,

$$*d\omega = (n-k) J(x, f) \geq 0.$$

For a.e. $R \in (0, \infty)$ we write

$$\varepsilon_f(a, R) = \sup_{\omega_0} \int_{\Sigma(a, R)} (*d\omega) d\mathcal{H}^{n-1} \bigg/ \int_{\Sigma(a, R)} |\omega - \omega_0| d\mathcal{H}^{n-1},$$

where the supremum is taken over all weakly closed $(n-1)$ -forms $\omega_0 \in W^{1,1}(B(a, 2R))$ such that

$$\omega_0|_{S(a, R)} \in W^{1,1}(S(a, R)).$$

Denote

$$V(a, r) = \int_{B(a, r)} J(x, f) d\mathcal{H}^n.$$

Because $\mathcal{H}^n(\Pi_0) = 0$ we see that $\mathcal{H}^n(\Pi^k) = 0$ and for a.e. $r > 0$ we have

$$\mathcal{H}^{n-1}(\Pi^k \cap S(a, r)) = 0.$$

Thus,

$$\int_{\Sigma(a, r)} \omega_0 = \int_{S(a, r) \setminus \Pi^k} \omega_0 = \int_{S(a, r)} \omega_0.$$

For an arbitrary weakly closed form ω_0 [7] and almost all $0 < R < \infty$ the following relations hold:

$$\begin{aligned} \int_{\Sigma(a, R)} |\omega - \omega_0| d\mathcal{H}^{n-1} &\geq \int_{\Sigma(a, R)} (\omega - \omega_0) \\ (2.1) \qquad \qquad \qquad &= (n - k) \int_{B(a, R)} J(x, f) d\mathcal{H}^n = (n - k)V(a, R). \end{aligned}$$

Here we used that $f_i|_{\Pi^k} = 0$ ($i = k + 1, \dots, n$) and that for the weakly closed form ω_0

$$\begin{aligned} \int_{\Sigma(a, R)} (\omega - \omega_0) &= \int_{\Sigma(a, R)} \omega \\ &= \int_{\Sigma(a, R)} \sum_{i=k+1}^n (-1)^{i-k-1} f_i df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n \\ &= \sum_{i=k+1}^n (-1)^{i-k-1} \int_{\Sigma(a, R)} f_i df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n \\ &= \sum_{i=k+1}^n \int_{B(a, R)} df_1 \wedge \dots \wedge df_i \wedge \dots \wedge df_n \\ &= (n - k) \int_{B(a, R)} J(x, f) d\mathcal{H}^n. \end{aligned}$$

The relation

$$(2.2) \qquad \int_{\Sigma(a, R)} f_i df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n = (-1)^{i-k-1} \int_{B(a, R)} df_1 \wedge \dots \wedge df_n$$

is clear in the case of C^2 -smooth maps. In the general case we approximate the homeomorphism $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ of class $W_{\text{loc}}^{1,n}(\mathbf{R}^n)$ by smooth maps with the following properties:

- i) $f_s \rightarrow f$ locally uniformly in $\mathbf{R}^n$ as $s \rightarrow \infty$;
- ii) $\|f'_s(x) - f'(x)\|_{L^n(D)} \rightarrow 0$ as $s \rightarrow \infty$ for every subdomain $D \Subset \mathbf{R}^n$.

We easily see that such an approximation is possible if we use the familiar technique of approximating Sobolev functions by smoothed averages (see, for example, [6, §4.2.1]). Then we fix arbitrary r_1, r_2 with $0 < r_1 < R < r_2 < \infty$. Using the formula (2.2) for smooth functions f_s , we get

$$(-1)^{i-k-1} \int_{r_1}^{r_2} dr \int_{B(a,r)} J(x, f_s) d\mathcal{H}^n = \int_{r_1}^{r_2} \int_{\Sigma(a,R)} f_{si} df_{s1} \wedge \dots \wedge \widehat{df_{si}} \wedge \dots \wedge df_{sn}.$$

Since $f_s \rightarrow f$ uniformly on $B(a, r)$ and $f_k|_{\Pi^k} = 0$, passing to the limit as $s \rightarrow \infty$ we obtain

$$(-1)^{i-k-1} \int_{r_1}^{r_2} dr \int_{B(a,r)} J(x, f) d\mathcal{H}^n = \int_{r_1}^{r_2} \int_{\Sigma(a,R)} f_i df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n.$$

We divide both sides of this equation by $r_2 - r_1$. Letting $r_2 \rightarrow r_1$, we see that (2.2) holds for almost all $R > 0$.

Thus, (2.1) is proved completely.

Fix $0 < \alpha < 1$ and choose ω_0 such that

$$\alpha \varepsilon_f(a, R) \leq \int_{\Sigma(a,R)} (*d\omega) d\mathcal{H}^{n-1} \bigg/ \int_{\Sigma(a,R)} |\omega - \omega_0| d\mathcal{H}^{n-1}.$$

By (2.1) we have

$$\alpha \varepsilon_f(a, R) \leq \int_{\Sigma(a,R)} J(x, f) d\mathcal{H}^n \bigg/ V(a, R)$$

and setting $\alpha \rightarrow 1$ we obtain

$$\varepsilon_f(a, R) \leq \int_{\Sigma(a,R)} J(x, f) d\mathcal{H}^n \bigg/ V(a, R).$$

Since $|\nabla d(a, x)| \equiv 1$ we arrive at

$$\int_{\Sigma(a,R)} J(x, f) d\mathcal{H}^{n-1} = \int_{S(a,R)} J(x, f) d\mathcal{H}^{n-1} = V'(a, R) \geq \varepsilon_f(a, R) V(a, R),$$

which is true for a.e. $R \in (0, \infty)$.

The function $t \mapsto V(a, t)$ is absolutely continuous on $[r, R]$ and solving this differential inequality we have

$$(2.3) \quad \exp \left\{ \int_r^R \varepsilon_f(\tau) d\tau \right\} \leq \frac{V(a, R)}{V(a, r)}.$$

We put

$$l(a, t) = \min_{x \in S(a,t)} |f(x) - f(a)|, \quad L(a, t) = \max_{x \in S(a,t)} |f(x) - f(a)|.$$

Now,

$$V(a, r) \geq \Omega_n l(a, r)^n, \quad V(a, R) \leq \Omega_n L(a, R)^n,$$

where $\Omega_n = \mathcal{H}^n(B(0, 1))$. Hence,

$$(2.4) \quad \frac{V(a, R)}{V(a, r)} \leq \frac{L(a, R)^n}{l(a, r)^n}.$$

We use the following bound for quasiconformal maps.

2.5. Theorem. *Let $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a K -quasiconformal map. Then for every point $a \in \mathbf{R}^n$ and an arbitrary ρ , $0 < \rho < \infty$ the following estimate holds:*

$$(2.6) \quad \frac{L(a, \rho)}{l(a, \rho)} \leq D_*, \quad D_* = D_*(K) = \exp(4K(K+1)\sqrt{K-1}).$$

For a proof see [2, Theorem 14.8].

Using (2.6), we see that

$$(2.7) \quad \frac{L(a, R)}{l(a, r)} = \frac{L(a, R)}{l(a, R)} \frac{l(a, R)}{L(a, r)} \frac{L(a, r)}{l(a, r)} \leq D_*^2 \frac{l(a, R)}{L(a, r)}, \quad D_* = D_*(K).$$

We next show that if $\beta = K^{1/(n-1)}$, then for $0 < r < R$

$$(2.8) \quad \frac{l(a, R)}{L(a, r)} \leq \left(\frac{R}{r}\right)^\beta.$$

Indeed, because the case $l(a, R) \leq L(a, r)$ is clear, we may assume that $l(a, R) > L(a, r)$. Applying the well-known formula for the n -capacity of a spherical condenser in $\mathbf{R}^n$, we obtain

$$\begin{aligned} \omega_{n-1} \left(\ln \frac{l(a, R)}{L(a, r)} \right)^{1-n} &= \text{cap}(\overline{B}(f(a), L(a, r)), B(f(a), l(a, R))) \\ &\geq \text{cap}(f\overline{B}(a, r), fB(a, R)) \\ &\geq \frac{1}{K} \text{cap}(\overline{B}(a, r), B(a, R)) \\ &= \frac{1}{K} \omega_{n-1} \left(\ln \frac{R}{r} \right)^{1-n}, \end{aligned}$$

where ω_{n-1} is the $(n-1)$ -dimensional area of the boundary of $B(0, 1)$.

From this inequality we arrive at (2.8). Thus, by (2.7) and (2.8) we obtain

$$(2.9) \quad \frac{L(a, R)}{l(a, r)} \leq D_*^2 \left(\frac{R}{r}\right)^\beta.$$

Using (2.7), we now have

$$\frac{V(a, R)}{V(a, r)} \leq D_*^{2n} \left(\frac{l(a, R)}{L(a, r)}\right)^n,$$

and further by (2.8),

$$(2.10) \quad \frac{V(a, R)}{V(a, r)} \leq D_*^{2n} \left(\frac{R}{r}\right)^{n\beta}.$$

The estimates (2.3) and (2.10) imply the following statement.

2.11. Theorem. *Let Π^k be a K -quasiplane in $\mathbf{R}^n$ with $1 \leq k \leq n-2$. Then for every point $a \in \Pi^k$ and arbitrary numbers $0 < r < R < \infty$, we have*

$$(2.12) \quad \exp \left\{ \int_r^R \varepsilon_f(a, \tau) d\tau \right\} \leq D_*^{2n}(K) \left(\frac{R}{r} \right)^{n\beta}, \quad \beta = K^{1/(n-1)},$$

for every K -quasiconformal mapping $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that $\Pi^k = f^{-1}(\Pi_0^k)$.

3. PROOF OF THE MAIN THEOREM

Fix a K -quasiconformal map $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ with $f(\Pi^k) = \Pi_0^k$. We shall estimate $\varepsilon_f(a, R)$ by $\eta(\Sigma(a, R))$. Setting

$$\begin{aligned} \omega_0 &= \sum_{i=k+1}^n (-1)^{i-k-1} \alpha_i df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n, \\ \varphi &= \sum_{i=k+1}^n \left| df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n \right|^2, \end{aligned}$$

where α_i are arbitrary constants, we have

$$(3.1) \quad \varepsilon_f(a, R) \geq \int_{\Sigma(a, R)} *d\omega \bigg/ \int_{\Sigma(a, R)} |\omega - \omega_0| d\mathcal{H}^{n-1}.$$

Observe that

$$\begin{aligned} & \int_{\Sigma(a, R)} |\omega - \omega_0| d\mathcal{H}^{n-1} \\ & \leq \int_{\Sigma(a, R)} \sqrt{\sum_{i=k+1}^n (f_i - \alpha_i)^2} \sqrt{\sum_{i=k+1}^n \left| df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n \right|^2} d\mathcal{H}^{n-1} \\ & \leq \left(\int_{\Sigma(a, R)} \left(\sum_{i=k+1}^n (f_i - \alpha_i)^2 \right)^{n/2} d\mathcal{H}^{n-1} \right)^{1/n} \left(\int_{\Sigma(a, R)} (\varphi)^{n/(2(n-1))} d\mathcal{H}^{n-1} \right)^{(n-1)/n}. \end{aligned}$$

It is not difficult to check that

$$(3.2) \quad (\varphi)^{n/(2(n-1))} \leq c_1(n, k) \|f'(x)\|^n,$$

where $c_1(n, k)$ is a constant and, in fact,

$$(3.3) \quad c_1(n, k) = (n - k)^{n/(2(n-1))}.$$

Indeed, since the form $df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n$ is simple,

$$\begin{aligned} & |df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n|^2 \\ &= \langle df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n, df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n \rangle \\ &\leq \prod_{\substack{s=1 \\ s \neq i}}^n |df_s|^2. \end{aligned}$$

Using the inequality between geometric and arithmetic means we obtain

$$\left(\prod_{\substack{s=1 \\ s \neq i}}^n |df_s|^2 \right)^{1/(n-1)} \leq \frac{1}{n-1} \sum_{\substack{s=1 \\ s \neq i}}^n |df_s|^2$$

and hence,

$$(3.4) \quad |df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n|^{2/(n-1)} \leq \frac{1}{n-1} \sum_{\substack{s=1 \\ s \neq i}}^n |df_s|^2.$$

Further, for an arbitrary vector $h \in \mathbf{R}^n$,

$$|f'(x)h|^2 = \sum_{s=1}^n \langle \nabla f_s, h \rangle^2,$$

and setting $h = \nabla f_i / |\nabla f_i|$, we find that

$$(3.5) \quad |\nabla f_i|^2 + \frac{1}{|\nabla f_i|^2} \sum_{\substack{s=1 \\ s \neq i}}^n \langle \nabla f_s, \nabla f_i \rangle^2 \leq \|f'\|^2.$$

Thus, from (3.4) it follows that

$$|df_1 \wedge \dots \wedge \widehat{df_i} \wedge \dots \wedge df_n| \leq \|f'(x)\|^{n-1}$$

and (3.2) follows. Now (3.2) yields

$$\int_{\Sigma(a,R)} (\varphi)^{n/(2(n-1))} d\mathcal{H}^{n-1} \leq c_1(n,k) \int_{\Sigma(a,R)} \|f'(x)\|^n d\mathcal{H}^{n-1}.$$

Next we use the inequality

$$\left(\frac{1}{N} \sum_{i=1}^n |a_i|^{t_1} \right)^{1/t_1} \leq \left(\frac{1}{N} \sum_{i=1}^n |a_i|^{t_2} \right)^{1/t_2} \quad (t_1 \leq t_2)$$

(see, for example, [4, §16 Chapter I]).

We have

$$\left(\frac{1}{n-k} \sum_{i=k+1}^n |f_i - \alpha_i|^2 \right)^{1/2} \leq \left(\frac{1}{n-k} \sum_{i=k+1}^n |f_i - \alpha_i|^n \right)^{1/n}$$

and hence,

$$\left(\sum_{i=k+1}^n |f_i - \alpha_i|^2 \right)^{1/2} \leq c_2(n,k) \left(\sum_{i=k+1}^n |f_i - \alpha_i|^n \right)^{1/n}$$

with $c_2(n, k) = (n - k)^{(n-2)/(2n)}$. Now we see from definition (1.2) of $\eta(\Sigma(a, R))$ that

$$\begin{aligned} & \int_{\Sigma(a, R)} \left(\sum_{i=k+1}^n (f_i - \alpha_i)^2 \right)^{n/2} d\mathcal{H}^{n-1} \\ & \leq c_2^n(n, k) \sum_{i=k+1}^n \int_{\Sigma(a, R)} |f_i - \alpha_i|^n d\mathcal{H}^{n-1} \\ & \leq c_2^n(n, k) \eta^{-n}(\Sigma(a, R)) \sum_{i=k+1}^n \int_{\Sigma(a, R)} |\nabla_S f_i|^n d\mathcal{H}^{n-1}. \end{aligned}$$

However, (3.5) implies that $|\nabla_S f_i| \leq \|f'\|$ for $i = 1, \dots, n$ and hence,

$$\sum_{i=k+1}^n |\nabla_S f_i|^n \leq (n - k) \|f'\|^n.$$

Thus, we arrive at the estimate

$$\int_{\Sigma(a, R)} \left(\sum_{i=k+1}^n (f_i - \alpha_i)^2 \right)^{n/2} d\mathcal{H}^{n-1} \leq c_3(n, k) \eta^{-n}(\Sigma(a, R)) \int_{\Sigma(a, R)} \|f'(x)\|^n d\mathcal{H}^{n-1},$$

where

$$c_3(n, k) = (n - k) c_2(n, k) = (n - k)^{n/2}.$$

Substituting these estimates into (3.1), we have the inequality

$$\varepsilon_f(a, R) \geq (n - k)^{-1} \eta(\Sigma(a, R)) \int_{\Sigma(a, R)} J(x, f) d\mathcal{H}^{n-1} \bigg/ \int_{\Sigma(a, R)} \|f'(x)\|^n d\mathcal{H}^{n-1}.$$

Thus, we obtained

$$(3.6) \quad \varepsilon_f(a, R) \geq \frac{1}{K_O(f)} \eta(\Sigma(a, R))$$

and using Theorem 2.11, we prove Theorem 1.4.

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