THE DISTRIBUTION FUNCTIONS OF $\sigma(n)/n$ AND $n/\varphi(n)$

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ABSTRACT. Let $\sigma(n)$ be the sum of the positive divisors of n . We show that the natural density of the set of integers n satisfying $\sigma(n)/n \geq t$ is given by $\exp\left\{-e^t e^{-\gamma} (1 + O(t^{-2}))\right\}$, where γ denotes Euler's constant. The same result holds when $\sigma(n)/n$ is replaced by $n/\varphi(n)$, where φ is Euler's totient function.

1. INTRODUCTION

Let $\sigma(n)$ be the sum of the positive divisors of the natural number n . If $\sigma(n) \geq 2n$, then n is called abundant. More generally, we say that n is t -abundant if $\sigma(n) \geq tn$. Davenport [1] showed in 1933 that the sequence of t -abundant numbers has a natural density

$$A(t) := \lim_{N \rightarrow \infty} \frac{1}{N} |\{n \leq N : \sigma(n)/n \geq t\}|,$$

and that $A(t)$ is a continuous function of t . An estimate for the density of abundant numbers is $0.2474 < A(2) < 0.2480$, due to Deléglise [2].

A close relative of $A(t)$ is the distribution function

$$B(t) := \lim_{N \rightarrow \infty} \frac{1}{N} |\{n \leq N : n/\varphi(n) \geq t\}|,$$

where φ denotes Euler's totient function. The existence of this limit and its continuity with respect to t were established by Schoenberg [5] in 1928.

Erdős [3], and recently Tenenbaum and Toulmonde [6], studied the behavior of $B(t)$ near $t = 1$. Toulmonde [7, Thm. 1] showed that the behavior of $B(t)$ in the neighborhood of $t = 1$ determines the local behavior near any t in the image of $n/\varphi(n)$. The corresponding result for $A(t)$ holds as well [7, Sec. 10]. Using the fact that the values of $n/\varphi(n)$ and $\sigma(n)/n$ are dense in $[1, \infty)$, Toulmonde [7, Thm. 2, 3, 4] obtained estimates for the infinity and L^p norms of the modulus of continuity of $A(t)$ and $B(t)$.

In this note we are concerned with the size of $A(t)$ and $B(t)$ as t tends to infinity. In that direction, Erdős [3, Thm. 1] states that

$$\log(\log(1/B(t))) = t e^{-\gamma} (1 + o(1)) \quad (t \rightarrow \infty),$$

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where γ is Euler's constant. With the help of a strong form of Mertens' theorem [4], it is not difficult to see that Erdős' argument [3, Eq.(2)] actually implies the stronger result

$$\log(1/B(t)) = e^{te^{-\gamma}}(1 + o(1)) \quad (t \rightarrow \infty).$$

We will improve on this estimate by establishing a bound for the error term.

Theorem. *As t tends to infinity, we have*

$$A(t), B(t) = \exp \left\{ -e^{te^{-\gamma}} (1 + O(t^{-2})) \right\}$$

where $\gamma = 0.5772\dots$ is Euler's constant.

Since $\sigma(n)$ and $\varphi(n)$ are multiplicative functions, we have

$$\frac{\sigma(n)}{n} = \prod_{p^\nu || n} \frac{1 + p + \dots + p^\nu}{p^\nu} = \prod_{p^\nu || n} \frac{1 - p^{-\nu-1}}{1 - p^{-1}} < \prod_{p|n} \frac{1}{1 - p^{-1}} = \frac{n}{\varphi(n)},$$

which shows that $A(t) \leq B(t)$. To demonstrate the theorem, we will provide a lower bound for $A(t)$ and an upper bound for $B(t)$. Our strategy for obtaining the lower bound is to pick a relatively small t -abundant number n , and use the estimate $A(t) \geq 1/n$, which holds since all multiples of n are again t -abundant. The derivation of the upper bound for $B(t)$ is based on the observation

$$B(t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{n \leq N \\ n \geq t\varphi(n)}} 1 \leq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \left(\frac{n}{t\varphi(n)} \right)^s \quad (s \geq 0),$$

together with a product representation of the limit on the right hand side, and a careful choice of s depending on t .

We remark without proof that, for both $A(t)$ and $B(t)$, the correct order of magnitude of the error term implicit in the theorem is t^{-2} .

2. LOWER BOUND FOR $A(t)$

Throughout we will use the notation

$$y = y(t) := e^{te^{-\gamma}}$$

and

$$R(y) := \exp(\sqrt{\log y}).$$

Let

$$h_p = \left\lfloor \frac{\log y}{\log p} \right\rfloor,$$

and define the natural number n by

$$n = n(t) = \prod_{p \leq y} p^{h_p}.$$

To obtain a lower bound for $A(t)$, we will estimate $\frac{\sigma(n)}{n}$ from below. We write

$$(1) \quad \frac{\sigma(n)}{n} = \prod_{p \leq y} \frac{\sigma(p^{h_p})}{p^{h_p}} = \prod_{p \leq y} \frac{1 + p + p^2 + \dots + p^{h_p}}{p^{h_p}} = \prod_{p \leq y} \frac{1 - p^{-h_p-1}}{1 - p^{-1}}.$$

Note that $p^{h_p+1} \geq y$, and, if $\sqrt{y} < p \leq y$, then $h_p = 1$. Thus

$$(2) \quad \prod_{p \leq y} (1 - p^{-h_p-1}) > \prod_{p \leq \sqrt{y}} (1 - y^{-1}) \prod_{\sqrt{y} < p \leq y} (1 - p^{-2}) = 1 + O\left(\frac{1}{\sqrt{y} \log y}\right).$$

A strong form of Mertens' theorem [4] implies

$$(3) \quad \prod_{p \leq y} \frac{1}{1 - p^{-1}} = e^\gamma \log y + O(R^{-1}(y)) = t + O(R^{-1}(y)).$$

Combining (1), (2), and (3), we get

$$(4) \quad \frac{\sigma(n)}{n} > t - c R^{-1}(y),$$

for some constant c . If m is a multiple of n , then $\frac{\sigma(m)}{m} \geq \frac{\sigma(n)}{n}$, according to (1). Thus (4) shows that $A(t - c R^{-1}(y)) > 1/n(t)$, from which we conclude that $A(t) > 1/n(t + c R^{-1}(y))$. It remains to estimate $n(t)$. The prime number theorem in the form

$$(5) \quad \sum_{p \leq y} \log p = y (1 + O(R^{-1}(y)))$$

implies

$$\log n(t) \leq \sum_{p \leq \sqrt{y}} \log y + \sum_{\sqrt{y} < p \leq y} \log p = O(\sqrt{y}) + y (1 + O(R^{-1}(y))),$$

and therefore

$$\log(1/A(t)) \leq \log n(t + c R^{-1}(y)) \leq y (1 + O(R^{-1}(y))).$$

This completes the proof of the lower bound for $A(t)$.

3. UPPER BOUND FOR $B(t)$

A classic result (see [5]) states that the limit in

$$W(s) := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \left(\frac{n}{\varphi(n)} \right)^s$$

exists for every complex s . We will assume that s is real. Clearly, for $s \geq 0$,

$$(6) \quad B(t) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{n \leq N \\ n \geq t \varphi(n)}} 1 \leq \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n \leq N} \left(\frac{n}{t \varphi(n)} \right)^s = \frac{W(s)}{t^s}.$$

It turns out (see [5]) that $W(s)$ has the product representation

$$(7) \quad W(s) = \prod_p \left(1 + \frac{(1 - p^{-1})^{-s} - 1}{p} \right) = \prod_{p \leq y} \cdot \prod_{p > y} =: U \cdot V,$$

say. We choose s as

$$(8) \quad s = s(t) := y \log y.$$

Since $\log(1+x) \leq x$, we have

$$\begin{aligned} \log V &= \sum_{p>y} \log \left(1 + \frac{(1-p^{-1})^{-s} - 1}{p} \right) \\ &\leq \sum_{p>y} \frac{(1-p^{-1})^{-s} - 1}{p} = \sum_{p>y} \frac{1}{p} \left(e^{s(\frac{1}{p} + O(\frac{1}{p^2}))} - 1 \right). \end{aligned}$$

The contribution to the last sum from primes $p > s$ is

$$\ll \sum_{p>s} \frac{1}{p} \left(\frac{s}{p} \right) = O(1),$$

and from primes p with $y < p \leq s$ it is

$$\leq \sum_{y<p\leq s} \frac{1}{p} e^{s(\frac{1}{p} + O(\frac{1}{p^2}))} \ll \sum_{y<p\leq s} \frac{1}{p} e^{s/p}.$$

From the prime number theorem and the definition of s in (8) we obtain

$$\begin{aligned} \sum_{y<p\leq s} \frac{1}{p} e^{s/p} &\ll \int_y^s \frac{e^{s/x}}{x} \frac{dx}{\log x} \leq \frac{1}{\log y} \int_y^s \frac{e^{s/x}}{x} dx = \frac{1}{\log y} \int_1^{s/y} \frac{e^u}{u} du \\ &\leq \frac{1}{\log y} \int_1^{\frac{1}{2} \log y} e^u du + \frac{2}{\log^2 y} \int_{\frac{1}{2} \log y}^{\log y} e^u du \ll \frac{y}{\log^2 y}. \end{aligned}$$

Thus

$$(9) \quad \log V \ll \frac{y}{\log^2 y}.$$

To estimate U , we apply the bound $\log(1+x) \leq \log x + 1/x$. We have

$$\begin{aligned} (10) \quad \log U &\leq \sum_{p\leq y} \log \left(1 + \frac{(1-p^{-1})^{-s}}{p} \right) \\ &\leq - \sum_{p\leq y} \log p - s \log \prod_{p\leq y} (1-p^{-1}) + \sum_{p\leq y} p(1-p^{-1})^s. \end{aligned}$$

The last sum can be estimated with the prime number theorem as follows:

$$\begin{aligned} (11) \quad \sum_{p\leq y} p(1-p^{-1})^s &\leq \sum_{p\leq y} p e^{-s/p} \ll \int_2^y x e^{-s/x} \frac{dx}{\log x} \\ &\leq y e^{-2s/y} \int_2^{y/2} dx + \frac{1}{\log(y/2)} \int_{y/2}^y \left(\frac{y^3}{x^3} \right) x e^{-s/x} dx \\ &\leq 1 + \frac{y^3}{\log(y/2)} \frac{e^{-s/y}}{s} \ll \frac{y}{\log^2 y}. \end{aligned}$$

Now use (3), (5), (10), and (11), to conclude that

$$(12) \quad \log U \leq -y + s \log t + O\left(\frac{y}{\log^2 y}\right).$$

Combining (6), (7), (9), and (12), we obtain

$$\log B(t) \leq \log(W(s)t^{-s}) = \log U + \log V - s \log t \leq -y + O\left(\frac{y}{\log^2 y}\right),$$

which completes the proof of the upper bound for $B(t)$.

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