ON COMPLEMENTED VERSIONS OF JAMES'S DISTORTION THEOREMS

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ABSTRACT. Examples are given to show that two natural questions asked in Dowling, Randrianantoanina, and Turett, 1999, about complemented versions of James's distortion theorems have negative answers.

1. INTRODUCTION

The James's distortion theorem for ℓ_1 (respectively, for c_0) states that whenever a Banach space contains a subspace isomorphic to ℓ_1 (respectively, c_0) then the Banach space contains subspaces that are almost isometric to ℓ_1 (respectively, c_0). In [5], complemented versions of James's distortion theorems were considered in the following senses:

Theorem 1. *Let X be a Banach space whose dual unit ball is weak*-sequentially compact and $\varepsilon > 0$. If X contains a subspace isomorphic to c_0 , then there exists a subspace Z of X and a projection P from X onto Z such that Z is $(1+\varepsilon)$ -isometric to c_0 and $\|P\| \leq 1 + \varepsilon$. Moreover, if X contains a subspace isometric to c_0 , then there exists a subspace Z of X and a projection P from X onto Z such that Z is isometric to c_0 and $\|P\| = 1$.*

Theorem 2. *Let X be a Banach space which contains a complemented subspace isomorphic to ℓ_1 and $\varepsilon > 0$. Then there exists a subspace Y of X and a projection P from X onto Y such that Y is $(1+\varepsilon)$ -isometric to ℓ_1 and $\|P\| \leq 1 + \varepsilon$.*

While Theorem 2 can be viewed as the exact analogue of James's distortion theorem for complemented copies of ℓ_1 , Theorem 1 may be interpreted as a combination of the James's distortion theorem for c_0 and the classical Sobczyk Theorem. These led to the following natural questions (see [5, Question 1, Question 2]):

Question 1. If a Banach space X contains a complemented copy of c_0 and if $\varepsilon > 0$, does there exist a subspace Z of X and a projection P from X onto Z such that Z is $(1+\varepsilon)$ -isometric to c_0 and $\|P\| \leq 1 + \varepsilon$?

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Question 2. If a Banach space X contains a complemented subspace isometric to ℓ_1 , does there exist a subspace Z of X and a projection P from X onto Z such that Z is isometric to ℓ^1 and $\|P\| = 1$?

The aim of this note is to provide examples showing that, as expected, the answers to both questions are negative.

2. THE EXAMPLES

2.1. The c_0 -case. In this subsection, we exhibit a Banach space X with the property that any complemented subspace of X that is almost isometric to c_0 has a large projection constant, thus answering Question 1 negatively. The space X is a renorming of $\ell_\infty \oplus_\infty c_0$.

We denote by $\|\cdot\|$ the usual norm on $\ell_\infty \oplus_\infty c_0$. In order to define the new norm on X , let $J: \ell_1 \rightarrow \ell_\infty$ be an isometric embedding of ℓ_1 into ℓ_∞ and $Q: \ell_1 \rightarrow c_0$ be a quotient map. For $\delta > 0$, a norm $|||\cdot|||$ on X is defined by fixing its unit ball:

$$B_{(X, |||\cdot|||)} := \{(Jf, Qf); f \in \ell_1, \|f\|_1 \leq 1\} + \delta B_{(X, \|\cdot\|)}.$$

It is clear that $|||\cdot|||$ and $\|\cdot\|$ are equivalent norms on X and X contains a complemented subspace isomorphic to c_0 .

Proposition 3. Let $\varepsilon > 0$ and Z be a subspace of $(X, |||\cdot|||)$ that is $(1+\varepsilon)$ -isometric to c_0 and is complemented in X . If P is a projection from X onto Z , then

$$|||P||| \geq \frac{1+\delta}{4\delta(1+\varepsilon)^3}.$$

Proof. Throughout, we also denote by $|||\cdot|||$ the corresponding dual norm on X^* . Let $(V_n)_{n \geq 1}$ be a basic sequence $(1+\varepsilon)$ -equivalent to the unit vector basis of c_0 and whose closed linear span is Z . Let P be a projection from X onto Z . Then P is of the form

$$P = \sum_{n=1}^{\infty} V_n^* \otimes V_n$$

where $(V_n^*)_{n \geq 1}$ is a weak*-null sequence in X^* and the sum can be taken with respect to the strong operator topology. Observe that $X^* = (\ell_\infty)^* \oplus_1 \ell_1$ isomorphically and thus for every $n \geq 1$, $V_n^* = (x_n^*, a_n^*)$ where $(x_n^*)_{n \geq 1}$ (respectively, $(a_n^*)_{n \geq 1}$) is a weak*-null sequence in $(\ell_\infty)^*$ (respectively, ℓ_1). Since weak*-null sequences are weakly-null in $(\ell_\infty)^*$ (see for instance [1, Theorem 15, p.103]), we have

$$\text{weak} - \lim_{n \rightarrow \infty} x_n^* = 0.$$

There exists a convex block $(y_n^*)_{n \geq 1}$ of $(x_n^*)_{n \geq 1}$ with

$$(2.1.1) \quad \lim_{n \rightarrow \infty} \|y_n^*\| = 0.$$

There exists a strictly increasing sequence of integers $(k_n)_{n \geq 0}$ and positive scalars $\alpha_j^{(n)}$, where $k_{n-1} + 1 \leq j \leq k_n$, $\sum_{j=k_{n-1}+1}^{k_n} \alpha_j^{(n)} = 1$ for $n \geq 1$, and

$$y_n^* = \sum_{j=k_{n-1}+1}^{k_n} \alpha_j^{(n)} x_j^*.$$

For $n \geq 1$, consider the corresponding block sequences:

$$(2.1.2) \quad \begin{aligned} W_n &= \sum_{j=k_{n-1}+1}^{k_n} V_j, \\ W_n^* &= \sum_{j=k_{n-1}+1}^{k_n} \alpha_j^{(n)} V_j^*. \end{aligned}$$

Then $(W_n)_{n \geq 1}$ is equivalent to the unit vector basis of c_0 . Moreover, for every $n, k \geq 1$, $\langle W_n^*, W_k \rangle = \delta_n^k$, $W_n^* = (y_n^*, b_n^*)$ where $b_n^* = \sum_{j=k_{n-1}+1}^{k_n} \alpha_j^{(n)} a_j^*$, and $\|W_n\| \leq 1 + \varepsilon$. The latter implies that for $n \geq 1$, W_n can be decomposed as

$$(2.1.3) \quad W_n = (Jf_n, Qf_n) + (x_n, a_n)$$

with

- (i) $f_n \in \ell_1$ satisfying $\|f_n\|_1 \leq 1 + \varepsilon$;
- (ii) $\|(x_n, a_n)\| \leq \delta(1 + \varepsilon)$.

Since $(f_n)_{n \geq 1}$ is a bounded sequence in ℓ_1 , we may assume (by passing to a subsequence if necessary) that for every $n \geq 1$,

$$f_n = f_0 + g_n + h_n$$

where

- (a) $\lim_{n \rightarrow \infty} \|f_n\|_1$ exists;
- (b) $\text{weak}^* - \lim_{n \rightarrow \infty} f_n = f_0$;
- (c) $(g_n)_{n \geq 1}$ is a disjointly supported sequence in ℓ_1 ;
- (d) $\lim_{n \rightarrow \infty} \|h_n\|_1 = 0$.

We claim that

$$(2.1.4) \quad \lim_{n \rightarrow \infty} \|f_n\|_1 \leq 3\delta(1 + \varepsilon).$$

To see this claim, let $N \geq 1$; then

$$\begin{aligned} \frac{1}{N} \sum_{n=1}^N \|f_n\|_1 &\leq \|f_0\|_1 + \frac{1}{N} \sum_{n=1}^N \|g_n\|_1 + \frac{1}{N} \sum_{n=1}^N \|h_n\|_1 \\ &= \|f_0\|_1 + \frac{1}{N} \left\| \sum_{n=1}^N g_n \right\|_1 + \frac{1}{N} \sum_{n=1}^N \|h_n\|_1 \\ &\leq 2\|f_0\|_1 + \frac{1}{N} \left\| \sum_{n=1}^N f_n \right\|_1 + \frac{2}{N} \sum_{n=1}^N \|h_n\|_1. \end{aligned}$$

Note that $\|f_0\|_1 \leq \varliminf_{N \rightarrow \infty} N^{-1} \left\| \sum_{n=1}^N f_n \right\|_1$ and $\lim_{N \rightarrow \infty} N^{-1} \sum_{n=1}^N \|h_n\|_1 = 0$. We deduce that

$$\lim_{n \rightarrow \infty} \|f_n\|_1 \leq 3 \varlimsup_{N \rightarrow \infty} \frac{1}{N} \left\| \sum_{n=1}^N f_n \right\|_1.$$

We observe that

$$\begin{aligned}
 \frac{1}{N} \left\| \sum_{n=1}^N f_n \right\|_1 &= \frac{1}{N} \left\| \sum_{n=1}^N Jf_n \right\|_\infty \\
 &\leq \frac{1}{N} \left\| \sum_{n=1}^N (Jf_n, Qf_n) \right\| \\
 &\leq \frac{1}{N} \left\| \sum_{n=1}^N W_n \right\| + \frac{1}{N} \left\| \sum_{n=1}^N (x_n, a_n) \right\| \\
 &\leq \frac{1}{N} \left\| \sum_{n=1}^N W_n \right\| + \delta(1 + \varepsilon).
 \end{aligned}$$

Since (W_n) is equivalent to the unit vector basis of c_0 , we have

$$\lim_{N \rightarrow \infty} N^{-1} \left\| \sum_{n=1}^N W_n \right\| = 0.$$

Combining all the above estimates, we get inequality (2.1.4).

We now show that if Π is the projection from X onto the closed linear span of $(W_n)_{n \geq 1}$ defined by $\Pi = \sum_{n=1}^\infty W_n^* \otimes W_n$, then

$$(2.1.5) \quad |||\Pi||| \geq \frac{1 + \delta}{4\delta(1 + \varepsilon)^2}.$$

To see this, we first observe that for every $y^* \in \ell_1$,

$$(2.1.6) \quad |||(0, y^*)||| = \|y^*\|_1(1 + \delta).$$

For every $n \geq 1$, we have

$$\begin{aligned}
 1 &= \langle W_n^*, W_n \rangle \\
 &= \langle (y_n^*, b_n^*), (Jf_n + x_n, Qf_n + a_n) \rangle \\
 &= \langle y_n^*, Jf_n + x_n \rangle + \langle b_n^*, Qf_n + a_n \rangle \\
 &\leq (\|y_n^*\| + \|b_n^*\|)(\|f_n\|_1 + \delta(1 + \varepsilon)).
 \end{aligned}$$

Taking limits as $n \rightarrow \infty$, we deduce that

$$(2.1.7) \quad \varliminf_{n \rightarrow \infty} \|b_n^*\|_1 \geq \frac{1}{4\delta(1 + \varepsilon)}.$$

We can estimate $|||\Pi|||$ as follows:

$$\begin{aligned}
 |||\Pi||| &\geq \sup_{n \geq 1} \frac{|||W_n^*|||}{1 + \varepsilon} \\
 &= \sup_{n \geq 1} \frac{|||(y_n^*, b_n^*)|||}{1 + \varepsilon} \\
 &\geq \varliminf_{n \rightarrow \infty} \frac{|||(y_n^*, b_n^*)|||}{1 + \varepsilon} \\
 &= \varliminf_{n \rightarrow \infty} \frac{|||(0, b_n^*)|||}{1 + \varepsilon}.
 \end{aligned}$$

Thus (2.1.5) follows by combining (2.1.7) and (2.1.6). We conclude the proof by observing that $|||\Pi||| \leq (1 + \varepsilon)|||P|||$. $\square$

2.2. The ℓ_1 -case. Now we provide an example showing that Theorem 2 does not extend to the isometric case. In particular, the answer to Question 2 is negative.

First, recall that a norm $\|\cdot\|$ on a Banach space E is said to be *strictly convex* if $\text{Ext}(B_E) = S_E$. This is equivalent to the following property (see for instance [6, p. 246]):

$$(2.2.1) \quad \text{If } x, y \in S_E \text{ satisfy } \|x + y\| = 2, \text{ then } x = y.$$

It is clear from (2.2.1) that if $\|\cdot\|$ is strictly convex, then $(E, \|\cdot\|)$ does not contain any ℓ_∞^2 (the two dimensional ℓ_∞) isometrically. Indeed, if $e_1 = (1, 0)$ and $e_2 = (0, 1)$ are the unit vector basis of ℓ_∞^2 , then $x = e_1$ and $y = e_1 + e_2$ fail to satisfy (2.2.1).

Define a norm $|\cdot|$ on ℓ_1 that is equivalent to the usual norm and such that its dual norm $|\cdot|^*$ is strictly convex. Such a dual norm on ℓ_∞ can be taken by setting:

$$|(a_i)|^* := (\| (a_i) \|_\infty^2 + \sum_{i=1}^{\infty} 2^{-i} |a_i|^2)^{1/2}.$$

Details on the existence of the norm $|\cdot|$ can be found in [6, pp. 241–254]. We define a Banach space Y by setting:

$$Y := (C[0, 1], \|\cdot\|_\infty) \oplus_\infty (\ell_1, |\cdot|).$$

Proposition 4. *The Banach space Y contains a complemented subspace that is isometric to ℓ_1 but ℓ_1 is not isometric to a quotient of Y .*

Let (e_n) be the unit vector basis of ℓ_1 and for $n \geq 1$, set $v_n := e_n/|e_n|$. Fix a sequence $(f_n)_{n \geq 1}$ in $C[0, 1]$ that is isometrically equivalent to the unit vector basis of ℓ_1 and for $n \geq 1$, define $U_n := (f_n, v_n) \in Y$. We claim that $(U_n)_{n \geq 1}$ is isometrically equivalent to the unit vector basis of ℓ_1 , and its closed linear span is complemented. In fact, for any finite sequence $(a_n)_{n \geq 1}$ of scalars,

$$\begin{aligned} \sum_{n \geq 1} |a_n| &= \left\| \sum_{n \geq 1} a_n f_n \right\|_\infty \\ &\leq \left\| \sum_{n \geq 1} a_n U_n \right\|_Y \\ &\leq \sum_{n \geq 1} |a_n|, \end{aligned}$$

therefore, $\sum_{n \geq 1} |a_n| = \left\| \sum_{n \geq 1} a_n U_n \right\|_Y$. Moreover, if we denote by Z the closed linear span of $(U_n)_{n \geq 1}$, then Z is a complemented subspace of Y . Indeed, let $T : (\ell_1, |\cdot|) \rightarrow Y$ be defined by setting $T(v_n) = U_n$ for all $n \geq 1$ and Π be the second projection from Y onto $(\ell_1, |\cdot|)$. Then $T \circ \Pi$ is a projection from Y onto Z .

The fact that ℓ_1 is not isometric to a quotient of Y follows from the next lemma, which we assume is well known.

Lemma 5. *Let E and F be Banach spaces and $T : c_0 \rightarrow E \oplus_1 F$ be an isometry. Then there exists $c_j \geq 0$, $j = 1, 2$ with:*

- (a) $c_1 + c_2 = 1$;
- (b) if $T = (T_1, T_2)$, then $\|T_j(e)\| = c_j \|e\|$ for $j = 1, 2$ and all $e \in c_0$.

In particular, if $E \oplus_1 F$ contains an isometric copy of c_0 , then either E or F contains an isometric copy of c_0 .

Proof. Denote by c_{00} the space of finitely supported sequences of scalars and let $(e_n)_{n \geq 1}$ be the unit vector basis of c_0 . Write $T = (T_1, T_2)$ with $T_1 : c_0 \rightarrow E$ and $T_2 : c_0 \rightarrow F$. We shall verify that for every $x \in c_{00}$ with $\|x\| = 1$,

$$\|T_j(x)\| = \|T_j(e_1)\| \text{ for } j = 1, 2.$$

To see this, we will show first that if x and y are disjointly supported unit vectors, then

$$(2.2.2) \quad \|T_j(x)\| = \|T_j(y)\| \text{ for } j = 1, 2.$$

Write $2x = (x - y) + (x + y)$. Then

$$\begin{aligned} 2 &= \|T(2x)\| = \|T_1(2x)\| + \|T_2(2x)\| \\ &\leq (\|T_1(x - y)\| + \|T_1(x + y)\|) + (\|T_2(x - y)\| + \|T_2(x + y)\|) \\ &= \|T(x - y)\| + \|T(x + y)\| = 2. \end{aligned}$$

For $j = 1, 2$, set $a_j = \|T_j(x - y)\| + \|T_j(x + y)\| - 2\|T_j(x)\|$. Then (a_1, a_2) is a positive element of ℓ_1^2 whose norm is equal to zero so

$$2\|T_j(x)\| = \|T_j(x - y)\| + \|T_j(x + y)\|, \quad j = 1, 2.$$

By reversing the role of x and y , we get (2.2.2).

Now, let $x \in c_{00}$ with $\|x\| = 1$. Choose, $n > 1$ so that e_n and x are disjointly supported. From (2.2.2),

$$\|T_j(x)\| = \|T_j(e_n)\| = \|T_j(e_1)\|, \quad j = 1, 2.$$

Setting $c_j = \|T_j(e_1)\|$ for $j = 1, 2$ proves the lemma. $\square$

End of the proof of Proposition 4. If ℓ_1 is isometric to a quotient of Y , then the dual space $Y^* = C[0, 1]^* \oplus_1 (\ell_\infty, |\cdot|*)$ contains an isometric copy of $\ell_\infty = \ell_1^*$ and hence of c_0 . But since $|\cdot|*$ is strictly convex and $C[0, 1]^*$ is a L_1 -space, this is in contradiction with Lemma 5 and thus completes the proof. $\square$

3. CONCLUDING REMARKS

The notion of asymptotically isometric copies of ℓ_1 (respectively, c_0) is closely related to James's distortion theorems. We recall that a Banach space E is said to contain an asymptotically isometric copy of ℓ_1 (respectively, c_0) if there exist a null sequence $(\varepsilon_n)_{n \geq 1}$ in $(0, 1)$ and a sequence $(x_n)_{n \geq 1}$ in E such that for all finite sequence $(t_n)_n$ of scalars:

$$\sum_n (1 - \varepsilon_n) |t_n| \leq \left\| \sum_n t_n x_n \right\| \leq \sum_n |t_n|,$$

respectively,

$$\sup_n (1 - \varepsilon_n) |t_n| \leq \left\| \sum_n t_n x_n \right\| \leq \sup_n |t_n|.$$

The norm introduced in the definition of the Banach space Y can be used to provide examples confirming the optimality of James's distortion theorems. We refer to [4] for earlier examples.

Proposition 6. (a) $(\ell_1, |\cdot|)$ does not contain any subspace asymptotically isometric to ℓ_1 .

(b) $(\ell_\infty, |\cdot|*)$ does not contain any subspace asymptotically isometric to c_0 .

Proof. These statements follow from the norm $|\cdot|_*$ being strictly convex and from some known results. First, containing an asymptotically isometric copy of c_0 and containing an isometric copy of c_0 is equivalent in a dual space ([3]). Second, according to [2], a Banach space contains an asymptotically isometric copy of ℓ_1 if and only if its dual contains an isometric copy of $L_1[0, 1]$. But since $(\ell_\infty, |\cdot|_*)$ is strictly convex, it does not contain any isometric copy of ℓ_1 , and therefore it cannot contain any isometric copy of $L_1[0, 1]$. Part (a) was already observed in [2] where an explicit formula for $|\cdot|_*$ was given (see [2, Corollary 12]). $\square$

The proof of Proposition 4 yields that the last part of the conclusion of Proposition 4 can be strengthened to “ c_0 is not isometric to a subspace of Y^* ”. However, in [3] Dowling proved that ℓ_1 is a quotient of X if and only if c_0 embeds isometrically into X^* , so the more natural statement involving ℓ_1 is only formally weaker.

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