ON LOCAL IRREDUCIBILITY OF THE SPECTRUM

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ABSTRACT. Let $\mathcal{M}_n$ be the algebra of $n \times n$ complex matrices, and for $x \in \mathcal{M}_n$ denote by $\sigma(x)$ and $\rho(x)$ the spectrum and spectral radius of x respectively. Let D be a domain in $\mathcal{M}_n$ containing 0, and let $F : D \rightarrow \mathcal{M}_n$ be a holomorphic map. We prove: (1) if $\sigma(F(x)) \cap \sigma(x) \neq \emptyset$ for $x \in D$, then $\sigma(F(x)) = \sigma(x)$ for $x \in D$; (2) if $\rho(F(x)) = \rho(x)$ for $x \in D$, then again $\sigma(F(x)) = \sigma(x)$ for $x \in D$. Both results are special cases of theorems expressing the irreducibility of the spectrum $\sigma(x)$ near $x = 0$.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let U be an open subset of $\mathbb{C}^N$. An *algebroid multifunction of degree m on U* is a set-valued function K of the form

$$(1) \quad K(x) = \left\{ w \in \mathbb{C} : w^m + \sum_{j=1}^m a_j(x) w^{m-j} = 0 \right\} \quad (x \in U),$$

where $a_1, \dots, a_m : U \rightarrow \mathbb{C}$ are holomorphic functions. It is understood that the roots are counted according to multiplicity.

An important example of an algebroid multifunction is obtained by taking U to be an open subset of $\mathcal{M}_n$, the space of $n \times n$ complex matrices (viewed as $\mathbb{C}^{n^2}$), and $K(x) := \sigma(x)$, the spectrum of x . Indeed, $\sigma(x)$ is just the zero set of the characteristic polynomial of x , so it is algebroid of degree n .

The purpose of this note is to establish two results which express, in different ways, the irreducibility of the multifunction $\sigma(x)$ near $x = 0$.

Theorem 1.1. *Let $m, n \geq 1$, let D be a connected open set in $\mathcal{M}_n$ such that $0 \in D$, and let K be an algebroid multifunction on D of degree m such that*

$$K(x) \cap \sigma(x) \neq \emptyset \quad (x \in D).$$

Then $m \geq n$, and

$$\sigma(x) \subset K(x) \quad (x \in D).$$

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This theorem and its proof were inspired in part by a recent result of Akbari and Aryapoor [1], who showed that, if $T : \mathcal{M}_n \rightarrow \mathcal{M}_n$ is a linear map such that Tx and x always have a common eigenvalue, then $\sigma(Tx) = \sigma(x)$ for all x , and consequently, using a result from [7], T has the form $Tx = v^{-1}xv$ or $Tx = v^{-1}x^t v$ for some invertible $v \in \mathcal{M}_n$. Theorem 1.1 leads to a non-linear version of their result.

Corollary 1.2. *Let $n \geq 1$, let D be a connected open set in $\mathcal{M}_n$ such that $0 \in D$, and let $F : D \rightarrow \mathcal{M}_n$ be a holomorphic map such that*

$$\sigma(F(x)) \cap \sigma(x) \neq \emptyset \quad (x \in D).$$

Then

$$\sigma(F(x)) = \sigma(x) \quad (x \in D).$$

If, further, F is injective, then $F(x)$ is conjugate to x for each $x \in D$.

Proof. Applying Theorem 1.1 with $K(x) := \sigma(F(x))$, we get $\sigma(x) \subset \sigma(F(x))$ for all $x \in D$. If $x \in D$ has distinct eigenvalues, then this inclusion must be equality. Since such x are dense in D , we deduce that $\sigma(F(x)) = \sigma(x)$ for all $x \in D$. The last part follows by a result of Baribeau and Ransford [2, Corollary 1.2]. $\square$

Our second main result is a characterization of the spectrum as an algebroid multifunction preserving the spectral radius. For $x \in \mathcal{M}_n$, we denote the spectral radius of x by $\rho(x)$.

Theorem 1.3. *Let $n \geq 1$, let D be a connected open set in $\mathcal{M}_n$ such that $0 \in D$, and let K be an algebroid multifunction on D of degree at most n such that*

$$\max_{w \in K(x)} |w| = \rho(x) \quad (x \in D).$$

Then K is of degree exactly n , and there exists $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ such that

$$K(x) = \gamma\sigma(x) \quad (x \in D).$$

Once again, taking $K(x) := \sigma(F(x))$, we can deduce a non-linear version of a result about maps preserving spectral radius that was already well known for linear maps (see e.g. [3]).

Corollary 1.4. *Let $n \geq 1$, let D be a connected open set in $\mathcal{M}_n$ such that $0 \in D$, and let $F : D \rightarrow \mathcal{M}_n$ be a holomorphic map such that*

$$\rho(F(x)) = \rho(x) \quad (x \in D).$$

Then there exists $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ such that

$$\sigma(F(x)) = \gamma\sigma(x) \quad (x \in D).$$

If, further, F is injective, then $F(x)$ is conjugate to γx for all $x \in D$. $\square$

Remarks. (i) These results break down if we do not assume that $0 \in D$. Consider the following example. Let x_0 be any $n \times n$ matrix with a unique peripheral eigenvalue (i.e. a unique eigenvalue λ_0 such that $|\lambda_0| = \rho(x_0)$). Then there exists an open neighborhood D of x_0 such that each $x \in D$ has a unique peripheral eigenvalue $\lambda(x)$, and the map $x \mapsto \lambda(x)$ is holomorphic on D . Define $F : D \rightarrow \mathcal{M}_n$ by $F(x) := \lambda(x)e$, where e is the $n \times n$ identity matrix. Then F satisfies the hypotheses of Corollaries 1.2 and 1.4, but not the conclusions (except in the trivial case $n = 1$).

This example explains why the theorems above can be considered as results about irreducibility of the spectrum near to 0.

(ii) Let $\mathcal{A}$ be a semisimple, unital Banach algebra. We continue to denote spectrum and spectral radius of $x \in \mathcal{A}$ by $\sigma(x)$ and $\rho(x)$ respectively. Define $\Omega_{\mathcal{A}} := \{x \in \mathcal{A} : \rho(x) < 1\}$, the so-called spectral unit ball, and let $F : \Omega_{\mathcal{A}} \rightarrow \Omega_{\mathcal{A}}$ be a holomorphic map such that $F(0) = 0$ and $F'(0) = I$, the identity map on $\mathcal{A}$. It was shown in [8] that F fixes every central element of $\Omega_{\mathcal{A}}$, in particular all multiples of the identity. It was also asked in [8] whether, under the same hypotheses, we can conclude that $\sigma(F(x)) = \sigma(x)$ for all $x \in \Omega_{\mathcal{A}}$. It was shown that a weaker result is true, namely: $\sigma(F(x)) \cap \sigma(x) \neq \emptyset$ for all $x \in \Omega_{\mathcal{A}}$. Corollary 1.2 permits us to deduce the stronger result, at least in the special case $\mathcal{A} = \mathcal{M}_n$. This case was already known [9, Theorem 3], but the new method also applies in some more general algebras. This is pursued in [4].

2. PROOF OF THEOREM 1.1

Let $\mathcal{O}_0$ be the algebra of germs of holomorphic functions at $0 \in \mathcal{M}_n$. It is easily seen to be an integral domain, and, using the Weierstrass preparation theorem, one can show that in fact $\mathcal{O}_0$ is a unique factorization domain (see e.g. [6, Proposition 6.4.9]). We write $\mathcal{O}_0[t]$ to denote the ring of polynomials in t with coefficients in $\mathcal{O}_0$. We shall be particularly interested in the polynomial

$$P(t) := \det(te - x) = t^n - s_1(x)t^{n-1} + \cdots + (-1)^n s_n(x),$$

where $s_1(x), \dots, s_n(x)$ are (the germs at 0 of) the elementary symmetric functions of the eigenvalues of x .

Lemma 2.1. *The polynomial $P(t) := \det(te - x)$ is irreducible in $\mathcal{O}_0[t]$.*

Proof. Suppose that $P(t) = Q(t)R(t)$, where $Q(t), R(t) \in \mathcal{O}_0[t]$, say

$$Q(t) = q_k t^k + \cdots + q_1 t + q_0,$$

$$R(t) = r_{n-k} t^{n-k} + \cdots + r_1 t + r_0,$$

where the $q_j, r_j \in \mathcal{O}_0$. Let U be an open neighborhood of 0 in $\mathcal{M}_n$ such that all the q_j, r_j are defined and holomorphic on U . Choose $\delta > 0$ such that

$$x_\lambda := \begin{pmatrix} 0 & \delta & 0 & \cdots & 0 \\ 0 & 0 & \delta & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \delta \\ \lambda & 0 & \cdots & 0 & 0 \end{pmatrix} \in U \quad (|\lambda| < \delta).$$

Define

$$\tilde{q}_j(\lambda) := q_j(x_\lambda) \quad \text{and} \quad \tilde{r}_j(\lambda) := r_j(x_\lambda) \quad (|\lambda| < \delta).$$

Then $\tilde{q}_j, \tilde{r}_j$ are holomorphic functions such that, for $|\lambda| < \delta$,

$$\begin{aligned} t^n - \delta^{n-1} \lambda &= \det(te - x_\lambda) \\ &= (\tilde{q}_k(\lambda)t^k + \cdots + \tilde{q}_0(\lambda))(\tilde{r}_{n-k}(\lambda)t^{n-k} + \cdots + \tilde{r}_0(\lambda)). \end{aligned}$$

Equating coefficients of t^0 , we find that

$$\tilde{q}_0(\lambda)\tilde{r}_0(\lambda) = -\delta^{n-1}\lambda \quad (|\lambda| < \delta).$$

Since the right-hand side has a simple zero at $\lambda = 0$, exactly one of the functions $\tilde{q}_0, \tilde{r}_0$ vanishes at 0. Let us suppose, without loss of generality, that $\tilde{q}_0(0) = 0$ and $\tilde{r}_0(0) \neq 0$. Equating coefficients of t^1 , we obtain

$$\tilde{q}_1(\lambda)\tilde{r}_0(\lambda) + \tilde{q}_0(\lambda)\tilde{r}_1(\lambda) = 0 \quad (|\lambda| < \delta).$$

Since $\tilde{q}_0(0) = 0$ and $\tilde{r}_0(0) \neq 0$, this implies that $\tilde{q}_1(0) = 0$. Continuing in this way, we obtain successively $\tilde{q}_j(0) = 0$ for $j = 0, 1, 2, \dots, k-1$.

If it were true that $k < n$, then we would likewise obtain $\tilde{q}_k(0) = 0$. On the other hand, equating coefficients of t^n gives $\tilde{q}_k(\lambda)\tilde{r}_{n-k}(\lambda) = 1$ for $|\lambda| < \delta$. We conclude that $k = n$, and that the polynomial $R(t) = r_0$, a constant. Since $P(t)$ is monic, r_0 must be invertible in $\mathcal{O}_0$. This proves that $P(t)$ is irreducible in $\mathcal{O}_0[t]$. $\square$

Remark. Let $x_0 \in \mathcal{M}_n$ be a matrix with distinct eigenvalues. Then, for $x \in \mathcal{M}_n$ near to x_0 , the eigenvalues of x are holomorphic functions $\lambda_1(x), \dots, \lambda_n(x)$. Thus $\det(te - x)$ factors as $(t - \lambda_1(x)) \dots (t - \lambda_n(x))$. This shows that it is important in the lemma that we are working with germs at $0 \in \mathcal{M}_n$, where there is no such holomorphic splitting of the eigenvalues.

Proof of Theorem 1.1. Write K as

$$K(x) = \left\{ w \in \mathbb{C} : w^m + \sum_{j=1}^m a_j(x)w^{m-j} = 0 \right\} \quad (x \in D),$$

where $a_1, \dots, a_m : D \rightarrow \mathbb{C}$ are holomorphic functions. Define $A(t) \in \mathcal{O}_0[t]$ by

$$A(t) := t^m + a_1 t^{m-1} + \dots + a_m,$$

and let $P(t)$ be the polynomial in Lemma 2.1.

We claim that $P(t)$ divides $A(t)$ in $\mathcal{O}_0[t]$. Indeed, suppose otherwise. Then, since $P(t)$ is irreducible in $\mathcal{O}_0[t]$ and $\mathcal{O}_0$ is a unique factorization domain, there exist $Q(t), R(t) \in \mathcal{O}_0[t]$ such that

$$A(t)Q(t) + P(t)R(t) = c,$$

where $c \in \mathcal{O}_0 \setminus \{0\}$ (see e.g. [1, Theorem C]). Choose a connected open subset V of D containing 0 such that the coefficients of $A(t), P(t), Q(t), R(t)$ and c are all defined and holomorphic on V . Thus A, P, Q, R can be regarded as holomorphic functions on $\mathbb{C} \times V$, and by the identity principle they satisfy

$$A(\lambda, x)Q(\lambda, x) + P(\lambda, x)R(\lambda, x) = c(x) \quad (\lambda \in \mathbb{C}, x \in V).$$

Now, by hypothesis, given $x \in V$, we have $K(x) \cap \sigma(x) \neq \emptyset$. Pick $\lambda_x \in K(x) \cap \sigma(x)$. Then $A(\lambda_x, x) = 0$ and $P(\lambda_x, x) = 0$, so $c(x) = 0$. Thus $c \equiv 0$ on V . This contradicts the fact that $c \in \mathcal{O}_0 \setminus \{0\}$. So $P(t)$ divides $A(t)$, as claimed.

From this, it follows that $n \leq m$, and also that $\sigma(x) \subset K(x)$ for all x in a neighborhood of 0. Finally, as D is connected, we deduce that in fact $\sigma(x) \subset K(x)$ for all $x \in D$ (see e.g. Lemma 3.1(i) below). $\square$

3. PROOF OF THEOREM 1.3

It will be convenient to introduce a little notation. Given an algebroid multi-function $K(x)$, we write

$$\rho(K(x)) := \max_{w \in K(x)} |w|.$$

Thus, for example, the hypothesis in Theorem 1.3 becomes

$$(2) \quad \rho(K(x)) = \rho(x) \quad (x \in D).$$

Also, we denote by C_r the circle with center 0 and radius r .

The proof of Theorem 1.3 uses a number of well-known properties of algebroid multifunctions, which, for convenience, we gather together in the following lemma.

Lemma 3.1. *Let $N \geq 1$, let D be a connected open subset of $\mathbb{C}^N$, and let K be an algebroid multifunction on D .*

- (i) *(Identity principle) If L is another algebroid multifunction on D and if $L(x) \subset K(x)$ for x in a non-empty open subset of D , then $L(x) \subset K(x)$ for all $x \in D$.*
- (ii) *(Maximum principle) If the function $x \mapsto \rho(K(x))$ attains a maximum ρ_0 in D , then the set function $x \mapsto K(x) \cap C_{\rho_0}$ is constant.*
- (iii) *(Liouville's theorem) If $D = \mathbb{C}^N$ and K is bounded, then K is constant.*
- (iv) *(Removable singularity theorem) If $D = \{\lambda \in \mathbb{C} : 0 < |\lambda| < r\}$ and K is bounded, then 0 is a removable singularity of K .*

Proof. (i) There is a dense subdomain D' of D on which L is locally the union of holomorphic functions. Let E be the interior of the set $\{x \in D' : L(x) \subset K(x)\}$. Using the usual identity principle for holomorphic functions, we see that E is open and closed in D' , and by hypothesis $E \neq \emptyset$. It follows that $L(x) \subset K(x)$ for all $x \in D'$. By continuity, $L(x) \subset K(x)$ for all $x \in D$.

(ii) The function $x \mapsto \rho(K(x))$ is plurisubharmonic on D , so if it attains a maximum ρ_0 , then by the maximum principle $\rho(K(x)) = \rho_0$ for all $x \in D$. We apply this, not to K , but rather to the algebroid multifunction $\tilde{K}(x) := \frac{1}{2}(\rho_0 e^{i\theta_0} + K(x))$. Note that $\rho(\tilde{K}(x)) \leq \rho_0$, with equality iff $\rho_0 e^{i\theta_0} \in K(x)$. Thus either $\rho_0 e^{i\theta_0} \in K(x)$ for all $x \in D$ or $\rho_0 e^{i\theta_0} \notin K(x)$ for all $x \in D$. This gives the result.

(iii) Write K as in (1). If K is bounded on $\mathbb{C}^N$, then so are the coefficient functions $a_1, \dots, a_m$. By Liouville's theorem, each a_j is constant. Hence so is K .

(iv) This is similar to (iii), this time using the usual removable singularity theorem on the functions a_j . $\square$

Proof of Theorem 1.3. Assume first that D is the whole of $\mathcal{M}_n$. We can suppose that $m = n$ (if $m < n$, then just consider $K(x) \cup \{0, \dots, 0\}$).

We claim that, if $x, y \in \mathcal{M}_n$ and $\sigma(x) = \sigma(y)$, then $K(x) = K(y)$. The idea is the same as in [9, Theorem 1]. Given $x \in \mathcal{M}_n$, there exists a diagonal matrix d , a strictly upper triangular matrix u and a matrix c such that $x = e^{-c}(d + u)e^c$. Consider the algebroid multifunction $\lambda \mapsto K(e^{-\lambda c}(d + \lambda u)e^{\lambda c})$. This is a bounded and entire, so by Lemma 3.1(iii) it is constant. In particular $K(x) = K(d)$. This justifies the claim.

Thus K induces a self-map $\tilde{K}$ of the set of monic polynomials of degree n via $\tilde{K} : (\text{characteristic polynomial of } x) \mapsto (\text{monic polynomial whose roots are } K(x))$.

If we identify a monic polynomial of degree n with its n -tuple of coefficients, then we can regard the induced map as a map $\tilde{K} : \mathbb{C}^n \rightarrow \mathbb{C}^n$. By considering the action of K on companion matrices, we see that $\tilde{K}$ is holomorphic. Let Γ_n be the so-called symmetrized n -disc, namely the set of n -tuples corresponding to characteristic polynomials of matrices x such that $\rho(x) < 1$. The condition (2) implies that $\tilde{K}$ maps Γ_n into Γ_n , and $\partial\Gamma_n$ into $\partial\Gamma_n$. Hence it is a proper self-map of Γ_n . By a result of Edigarian and Zwonek [5], we deduce that K must be of the form

$K(x) = B(\sigma(x))$, where B is a Blaschke product. From (2) with $x = \lambda e$, we see that $|B(\lambda)| = |\lambda|$ for all λ close to 0. Therefore $B(z) = \gamma z$, where γ is a constant of modulus one. Hence $K(x) = \gamma\sigma(x)$ for all x .

Now we turn to the case where D is an arbitrary domain in $\mathcal{M}_n$ containing 0. Choose $\delta > 0$ such that D contains the ball $\{x \in \mathcal{M}_n : \|x\| < \delta\}$.

Let $x \in \mathcal{M}_n$. The map $\lambda \mapsto K(\lambda x)/\lambda$ is a bounded algebroid multifunction on $\{\lambda \in \mathbb{C} : 0 < |\lambda| < \delta/\|x\|\}$, so, by Lemma 3.1(iv), it has a removable singularity at 0. Denote by $K_0(x)$ its value there. Note that $K_0(x)$ contains at most n points, and that $\rho(K_0(x)) = \rho(x)$. Also, since $\lim_{\lambda \rightarrow 0} K(\lambda x)/\lambda = K_0(x)$ boundedly on each bounded set of x 's, it follows that K_0 is an algebroid multifunction on $\mathcal{M}_n$. By what we have already proved, there exists $\gamma \in \mathbb{C}$ with $|\gamma| = 1$ such that $K_0(x) = \gamma\sigma(x)$ for all $x \in \mathcal{M}_n$.

Fix $x \in \mathcal{M}_n$ once again. By Lemma 3.1(ii) applied to the map $\lambda \mapsto K(\lambda x)/\lambda$, we see that

$$(K(\lambda x)/\lambda) \cap C_{\rho(x)} = K_0(x) \cap C_{\rho(x)} \quad (0 < |\lambda| < \delta/\|x\|).$$

In particular, if $\|x\| < \delta$, then $K(x) \cap K_0(x) \neq \emptyset$, in other words $\gamma^{-1}K(x) \cap \sigma(x) \neq \emptyset$. Applying Theorem 1.1, we deduce that $\gamma^{-1}K(x) = \sigma(x)$ on a neighbourhood of 0. By Lemma 3.1(i) $\gamma^{-1}K(x) = \sigma(x)$ on the whole of D . This completes the proof. $\square$

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