THE ATOMIC DECOMPOSITION IN $L^1(R^n)$

WAEEL ABU-SHAMMALA AND ALBERTO TORCHINSKY

(Communicated by Michael T. Lacey)

ABSTRACT. In this paper we present an atomic decomposition of integrable functions. As an application we compute the distance of f in $L^1(R^n)$ to the Hardy space $H^1(R^n)$.

In this paper we show the following atomic decomposition of an integrable function.

Theorem 1. $f \in L^1(R^n)$ if and only if f can be written as

$$f(x) = \sum_j \lambda_j a_j(x) + \lambda \chi_J(x),$$

where $\sum_j |\lambda_j| < \infty$, the a_j 's are q_j atoms, $1 < q_j \leq 2$, and J denotes the unit interval $[0, 1] \times \cdots \times [0, 1]$ of sidelength 1. Furthermore, $\lambda = \int_{R^n} f(x) dx$, and

$$\|f\|_1 \sim |\lambda| + \inf \sum_j |\lambda_j|,$$

where the infimum is taken over all decompositions of f as above.

Recall that a compactly supported function a with vanishing integral is an L^q atom with defining interval I if

$$\text{supp}(a) \subseteq I, \quad \int_I a(x) dx = 0, \quad \text{and} \quad |I| \left(\frac{1}{|I|} \int_I |a(x)|^q dx \right)^{1/q} \leq 1.$$

The relevance of these atoms lies in the fact that the Hardy space $H^1(R^n)$ can be characterized in terms of decompositions involving L^q atoms for each fixed $q > 1$; see [3] and [5]. Here we will allow q to vary and approach 1, thus making it possible for functions spanned by these atoms to leave $H^1(R^n)$.

More precisely, let $X_\infty(R^n) = X_\infty$ be the space

$$X_\infty = \left\{ f = \sum_1^\infty \lambda_j a_j : \sum_1^\infty |\lambda_j| < \infty \right\},$$

where the a_j 's are L^{q_j} atoms, $1 < q_j \leq 2$. Endowed with the atomic norm

$$\|f\|_{X_\infty} = \inf \left\{ \sum_1^\infty |\lambda_j| : f = \sum_1^\infty \lambda_j a_j \right\},$$

Received by the editors January 23, 2006 and, in revised form, May 23, 2006.

2000 *Mathematics Subject Classification.* Primary 42B25.

Key words and phrases. Atomic decomposition.

©2007 American Mathematical Society
 Reverts to public domain 28 years from publication

X_∞ becomes a Banach space. X_∞ can also be thought of as a limiting case of the spaces $X_s(R^n)$ of Sweezy. These spaces form a nested family that starts at H^1 and approaches $L_0^1(R^n) = L_0^1$, the subspace of L^1 functions with vanishing integral, as $s \rightarrow \infty$; see [4] and [1].

Now, since for an L^q atom a we have

$$\int_I |a| \leq |I| \left(\frac{1}{|I|} \int_I |a(x)|^q dx \right)^{1/q} \leq 1,$$

it readily follows that $\|f\|_1 \leq \|f\|_{X_\infty}$, and X_∞ is continuously embedded in L_0^1 .

We propose to show that actually X_∞ and L_0^1 are the same space. To do this, we begin by identifying those L^∞ functions that differ by a constant, and denote the equivalence class corresponding to $\varphi \in L^\infty$ by Φ . Now let $(\mathcal{L}^\infty, \|\cdot\|_{\mathcal{L}^\infty})$ denote the Banach space L^∞ modulo constants endowed with the quotient norm, that is,

$$\|\Phi\|_{\mathcal{L}^\infty} = \inf \{ \|\varphi\|_\infty : \varphi \in \Phi \}.$$

$(\mathcal{L}^\infty, \|\cdot\|_{\mathcal{L}^\infty})$ is the dual of L_0^1 .

Moreover, since $[\varphi(x) - \varphi_I] \chi_I = [\varphi_1(x) - (\varphi_1)_I] \chi_I$ for $\varphi \sim \varphi_1$, the expression

$$\|\Phi\|_{\infty,*} = \sup_I \|[\varphi(\cdot) - \varphi_I] \chi_I\|_\infty, \quad \varphi \in \Phi,$$

also defines a norm in L^∞ modulo constants. The resulting space is denoted by $(L^{\infty,*}, \|\cdot\|_{\infty,*})$ and, as we show next, it is the dual of X_∞ .

Theorem 2. *The dual of X_∞ is $(L^{\infty,*}, \|\cdot\|_{\infty,*})$. More precisely, every Φ in $L^{\infty,*}$ induces a bounded linear functional L_Φ on X_∞ by means of the integral*

$$L_\Phi(f) = \int_{R^n} f(x) \varphi(x) dx, \quad \varphi \in \Phi,$$

and $\|L_\Phi\| \leq \|\Phi\|_{\infty,*}$.

Furthermore, given a bounded linear functional $L \in X_\infty^*$, there is a Φ in $L^{\infty,*}$ such that

$$\|\Phi\|_{\infty,*} \leq 2 \|L\| \text{ and } L(f) = L_\Phi(f), \quad f \in X_\infty.$$

Proof. Let $\Phi \in L^{\infty,*}$ and $\varphi \in \Phi$. Then the absolutely convergent integral

$$L_\Phi(f) = \int_{R^n} f(x) \varphi(x) dx, \quad f \in X_\infty,$$

is a well-defined linear functional on X_∞ . Moreover, if $f(x) = \sum_j \lambda_j a_j(x)$, $L_\Phi(f) = \sum_j \lambda_j \int_{I_j} a_j(x) \varphi(x) dx$, and

$$\begin{aligned} |L_\Phi(f)| &\leq \sum_j |\lambda_j| \left| \int_{I_j} a_j(x) \varphi(x) dx \right| \\ &\leq \sum_j |\lambda_j| \left(\int_{I_j} |a_j(x)|^{q_j} dx \right)^{1/q_j} \left(\int_{I_j} |\varphi(x) - \varphi_{I_j}|^{p_j} dx \right)^{1/p_j} \\ &\leq \sup_{I_j} \|(\varphi(\cdot) - \varphi_{I_j}) \chi_{I_j}\|_\infty \sum_j |\lambda_j| |I_j| \left(\frac{1}{|I_j|} \int_{I_j} |a_j(x)|^{q_j} dx \right)^{1/q_j}. \end{aligned}$$

Thus, taking the infimum over the atomic decompositions of f we have

$$|L_\Phi(f)| \leq \|\Phi\|_{\infty,*} \|f\|_{X_\infty},$$

and Φ induces a linear functional L_Φ on X_∞ with norm not exceeding $\|\Phi\|_{\infty,*}$.

As for the second part of the statement, let $1 < q \leq 2$, and let $L_0^q(I)$ denote the subspace of L^q consisting of those functions in L^q that have vanishing integral and are supported in the interval I . The dual of $L_0^q(I)$ is $\mathcal{L}^p(I)$, or $L^p(I)$ mod constants, $1/p + 1/q = 1$, and as is readily seen, $\|\Phi\|_{\mathcal{L}^p} \sim \|\varphi\|_p$, where $\varphi = \psi - \psi_I$, and ψ is any function in the class Φ . Note that if $f \in L_0^q(I)$, then $f \in X_\infty$ and

$$\|f\|_{X_\infty} \leq |I|^{1/p} \|f\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Now, given $L \in X_\infty^*$, for a fixed N let $I_N = \{x : |x_i| \leq N, 1 \leq i \leq n\}$, and consider the restriction of L to $L_0^q(I_N)$. Since

$$|L(f)| \leq \|L\| \|f\|_{X_\infty} \leq \|L\| |I_N|^{1/p} \|f\|_q,$$

this restriction is continuous with respect to the norm in L^q and consequently it can be extended to a continuous linear functional on $L_0^q(I_N)$ and represented as

$$L(f) = \int_{I_N} f(x) \Phi_N(x) dx,$$

where $\Phi_N \in \mathcal{L}^p(I_N)$ and satisfies $\|\Phi_N\|_{\mathcal{L}^p(I_N)} \leq \|L\| |I_N|^{1/p}$. Since for $1 < q < \infty$ the class of simple functions with vanishing integral supported in I_N is dense in $L_0^q(I_N)$, Φ_N does not depend on the choice of q . Moreover, since Φ_1 and Φ_N induce the same linear functional on $L_0^q(I_1)$, we have

$$\varphi_N(x) = \varphi_1(x) + c_N, \quad \text{a.e. in } I_1,$$

for some constant c_N . Note that with this choice of c_N , $\varphi_N - c_N$ has vanishing integral on I_1 . This readily implies that

$$\varphi_N(x) - c_N = \varphi_M(x) - c_M, \quad \text{a.e. } |x| \leq \min(N, M).$$

Thus, if

$$\varphi(x) = \varphi_N(x) - c_N, \quad |x| \leq N,$$

$\varphi(x)$ is well defined a.e., and if $f \in L_0^q(I_N)$, we have

$$\begin{aligned} L(f) &= \int_{R^n} f(x) \varphi_N(x) dx \\ &= \int_{R^n} f(x) [\varphi_N(x) - c_N] dx \\ &= \int_{R^n} f(x) \varphi(x) dx. \end{aligned}$$

Next we consider an arbitrary interval I and estimate

$$\left(\int_I |\varphi(x) - \varphi_I|^p dx \right)^{1/p} = \sup_{\|g\|_q \leq 1} \left| \int_I [\varphi(x) - \varphi_I] g(x) dx \right|.$$

Since $[\varphi - \varphi_I] \chi_I$ has vanishing integral and $[g - g_I] \chi_I \in L_0^q(I_N) \cap X_\infty$ for some N ,

$$\begin{aligned} \left| \int_I [\varphi(x) - \varphi_I] g(x) dx \right| &= \left| \int_I \varphi(x) [g(x) - g_I] dx \right| \\ &\leq \|L\| \|[g - g_I] \chi_I\|_{X_\infty} \\ &\leq \|L\| |I|^{1/p} \|[g - g_I] \chi_I\|_q \\ &\leq 2 \|L\| |I|^{1/p} \|g\|_q. \end{aligned}$$

Thus, for all p with $2 \leq p < \infty$, we have

$$\left(\frac{1}{|I|} \int_I |\varphi(x) - \varphi_I|^p dx \right)^{1/p} \leq 2 \|L\|,$$

and consequently also

$$\sup_{p,I} \left(\frac{1}{|I|} \int_I |\varphi(x) - \varphi_I|^p dx \right)^{1/p} \leq 2 \|L\|,$$

where the sup is taken over all intervals I and $2 \leq p < \infty$. Hence, for each fixed I , by letting $p \rightarrow \infty$ it readily follows that

$$\|[\varphi(\cdot) - \varphi_I] \chi_I\|_\infty \leq 2 \|L\|.$$

In particular, $\varphi \in L^\infty$, and if $\varphi \in \Phi$, $\|\Phi\|_{\infty,*} \leq 2 \|L\|$. Summing up, we have shown that for each functional $L \in X_\infty^*$ there is $\Phi \in L^{\infty,*}$ with $\|\Phi\|_{\infty,*} \leq 2 \|L\|$, and $L(f) = L_\Phi(f)$ for $f \in X_\infty$. $\square$

We can also identify $\mathcal{L}^\infty$ and $L^{\infty,*}$.

Proposition 3. $\mathcal{L}^\infty = L^{\infty,*}$. *More precisely, the two spaces coincide and the norms are equivalent.*

Proof. Both spaces are complete with their respective norms, so it suffices to show that one norm dominates the other. Let $\varphi \in \Phi$. Since for an arbitrary constant c we have

$$\sup_I \|[\varphi(\cdot) - \varphi_I] \chi_I\|_\infty \leq 2 \|\varphi - c\|_\infty,$$

taking the infimum over c it readily follows that $\|\Phi\|_{\infty,*} \leq 2 \|\Phi\|_{\mathcal{L}^\infty}$. $\square$

We are now in a position to show that X_∞ is L_0^1 .

Lemma 4. $X_\infty = L_0^1$. *That is, the two spaces coincide and the norms are equivalent.*

Proof. We first note that for f in X_∞ , $\|f\|_{X_\infty} \leq c \|f\|_1$, which gives $\|f\|_1 \sim \|f\|_{X_\infty}$. Indeed, since by Proposition 3, $\|\Phi\|_{\mathcal{L}^\infty} \leq c \|\Phi\|_{\infty,*}$, it readily follows that

$$\begin{aligned} \|f\|_{X_\infty} &= \sup_{\|L\|_{X_\infty^*} \leq 1} |L(f)| \\ &\leq \sup_{\|\Phi\|_{\infty,*} \leq 2} |L_\Phi(f)| \\ &\leq \sup_{\|\Phi\|_{\mathcal{L}^\infty} \leq 2c} |L_\Phi(f)| \\ &\leq 2c \|f\|_1, \end{aligned}$$

and, consequently, the atomic norm and the L^1 norm are equivalent on X_∞ . In particular, since X_∞ is complete with respect to the atomic norm, it is closed in L_0^1 with the L^1 norm. But X_∞ is dense in L_0^1 : a bounded linear functional on L_0^1 that vanishes on X_∞ can be represented by integration against an L^∞ function that turns out to be constant, and so it vanishes on L_0^1 . Thus L_0^1 and X_∞ are actually the same space. $\square$

We can now prove Theorem 1. First assume that f has the atomic decomposition described there. Then $f - \lambda \chi_J \in X_\infty$, and, consequently, $f - \lambda \chi_J \in L_0^1$ and $\lambda = \int_{R^n} f(x) dx$. Moreover, since $\|f\|_1 \leq \|f - \lambda \chi_J\|_1 + |\lambda| \leq 3\|f\|_1$, we have

$$\|f\|_1 \sim |\lambda| + \|f - \lambda \chi_J\|_{X_\infty},$$

and the statement about the equivalence of the norms follows readily.

Conversely, given an integrable function f , put $\lambda = \int_{R^n} f(x) dx$, and set

$$f = [f - \lambda \chi_J] + \lambda \chi_J = f_1 + f_2,$$

say. Note that $f_1 \in L_0^1$ and f_2 are uniquely determined. Moreover, since $\|f\|_1 \sim \|f_1\|_1 + |\lambda|$, and $\|f_1\|_{X_\infty} \sim \|f_1\|_1$, the conclusion follows as above with $\sum_j \lambda_j a_j(x)$, an X_∞ atomic decomposition of f_1 in terms of q_j atoms. This completes the proof.

As a consequence of the atomic decomposition, the reader will have no difficulty in computing $d(f, H^1)$, the distance in L^1 of f to H^1 .

Theorem 5. *Let $f \in L^1$. Then $d(f, H^1) = \left| \int_{R^n} f(x) dx \right|$.*

REFERENCES

1. W. Abu-Shammala, and A. Torchinsky, *Spaces between $H^1(R^n)$ and $L^1(R^n)$* , preprint.
2. A. P. Calderón and A. Torchinsky, *Parabolic maximal functions associated with a distribution, II*, *Advances in Math.*, **24** (1977), 101–171. MR0450888 (56:9180)
3. J. García-Cuerva and J. L. Rubio de Francia, *Weighted norm inequalities and related topics*, *Notas de Matemática* **116**, North Holland, 1985. MR0807149 (87d:42023)
4. C. Sweezy, *Subspaces of $L^1(R^d)$* , *Proc. Amer. Math. Soc.* **132** (2005), 3599–3606. MR2084082 (2005h:42053)
5. A. Torchinsky, *Real-variable methods in harmonic analysis*, Dover Publications, Inc., 2004. MR2059284

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405
E-mail address: wabusham@indiana.edu

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405
E-mail address: torchins@indiana.edu