PERTURBATIONS AND WEYL'S THEOREM

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ABSTRACT. A Banach space operator T is completely hereditarily normaloid, $T \in \mathcal{CHN}$, if either every part, and (also) T_p^{-1} for every invertible part T_p , of T is normaloid or if for every complex number λ every part of $T - \lambda I$ is normaloid. Sufficient conditions for the perturbation $T + A$ of $T \in \mathcal{CHN}$ by an algebraic operator A to satisfy Weyl's theorem are proved. Our sufficient conditions lead us to the conclusion that the conjugate operator $(T + A)^*$ satisfies a -Weyl's theorem.

1. INTRODUCTION

A Banach space operator T , $T \in B(\mathcal{X})$, is said to be *Weyl* (resp., *Browder*) if it is Fredholm of index 0 (resp., Fredholm with finite ascent and descent). The *Weyl spectrum* $\sigma_w(T)$ (resp., the *Browder spectrum* $\sigma_b(T)$) of T is the set $\{\lambda \in \mathbf{C} : T - \lambda I \text{ is not Weyl}\}$ (resp., the set $\{\lambda \in \mathbf{C} : T - \lambda I \text{ is not Browder}\}$). Let $\sigma(T)$, $\text{iso}\sigma(T)$ and $\pi_{00}(T)$ denote, respectively, the spectrum, the isolated points of the spectrum and the isolated eigenvalues of finite multiplicity of T . In keeping with current terminology, we say that T *satisfies Weyl's theorem* (resp., *satisfies Browder's theorem*) if it satisfies the Weyl condition $\sigma(T) \setminus \sigma_w(T) = \pi_{00}(T)$ (resp., the Browder condition $\sigma_b(T) = \sigma_w(T)$). Browder and Weyl theorems for operators $T \in B(\mathcal{X})$ have recently been considered by a large number of authors; we refer the interested reader to [1, Chapter 3.8] for an excellent account (and an explanation of some of our unexplained terminology). The following implications hold: Weyl's theorem for $T \implies$ Browder's theorem for $T \iff$ Browder's theorem for T^* . Browder's theorem does not survive perturbations: thus Browder's theorem for T does not imply Browder's theorem for $T + K$ for compact or quasinilpotent (even, nilpotent) K (unless T and K commute) [8]. Recall that an operator A is *algebraic* if $p(A) = 0$ for some non-trivial polynomial $p(\cdot)$, *paranormal* if $\|Ax\|^2 \leq \|A^2x\|^2$ for every unit vector $x \in \mathcal{X}$, and A satisfies *property $H(m)$* for some integer $m \geq 1$ if the quasinilpotent part $H_0(A - \lambda I)$ of $A - \lambda I$ equals the null space of $(A - \lambda I)^m$ for all (complex numbers) $\lambda \in \mathbf{C}$. (Paranormal operators do not satisfy property $H(m)$ [2].) Perturbations of operators T satisfying property $H(m)$ by algebraic operators A commuting with T have recently been considered by Oudghiri [10]; Aiena and Guillen [2, Theorem 2.5] have extended the result from [10] to prove that if $\mathcal{X}$ is a Hilbert space, and if $T \in B(\mathcal{X})$ is a paranormal operator which commutes with the

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algebraic operator $A \in B(\mathcal{X})$, then $T + A$ satisfies Weyl's theorem. In this note, we generalize [2, Theorem 2.5] to the class $\mathcal{CHN}$ of *completely hereditarily normaloid operators* in $B(\mathcal{X})$ [3] to prove the following theorem. (Any unexplained notation in the statement of the theorem is explained in the sequel.)

Theorem 1.1. *If $T \in \mathcal{CHN}$ commutes with the algebraic operator $A \in B(\mathcal{X})$, and if either (i) $\mathcal{X}$ is separable or (ii) T_i has SVEP at points $\lambda \in \sigma_w(T_i)$ or (iii) T has SVEP at points $\lambda \in \sigma_w(T)$ or (iv) $\mathcal{X}$ is a Hilbert space or (v) $T^{-1}(0) \perp (T - \alpha I)^{-1}(0)$ for all complex numbers α , then $T + A$ satisfies Weyl's theorem and (the conjugate operator) $(T + A)^*$ satisfies *a-Weyl's theorem*.*

Our argument, which is similar in spirit to that in [2], leads to a proof of the result from [10] for operators satisfying property $H(m)$.

2. ADDITIONAL NOTATION AND TERMINOLOGY

A part of an operator is its restriction to an invariant subspace. We say that an operator $T \in B(\mathcal{X})$ is *completely hereditarily normaloid*, $T \in \mathcal{CHN}$, if either every part of T , and (also) T_p^{-1} for every invertible part T_p of T , is normaloid or if for every $\lambda \in \mathbf{C}$ every part of $T - \lambda I$ is normaloid. The class $\mathcal{CHN}$ is large. In particular, Hilbert space operators T which are either *hyponormal* ($|T^*|^2 \leq |T|^2$) or *p-hyponormal* ($|T^*|^{2p} \leq |T|^{2p}$ for some $0 < p < 1$) or *w-hyponormal* (if T has the polar decomposition $T = U|T|$, and $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ denotes the Aluthge transform of T , then $|\tilde{T}^*| \leq |T| \leq |\tilde{T}|$) are $\mathcal{CHN}$ operators. Again, *totally *-paranormal* Hilbert space operators ($\|(T - \lambda I)^*x\|^2 \leq \|(T - \lambda I)^2x\|^2$ for every unit vector x) and *paranormal operators* $T \in B(\mathcal{X})$ ($\|Tx\|^2 \leq \|T^2x\|^2$ for all unit vectors $x \in \mathcal{X}$) are $\mathcal{CHN}$ operators. (We refer the reader to the monograph [6] for information on these classes of operators; see also [1], [3] and [7].) We shall henceforth shorten $T - \lambda I$ to $T - \lambda$.

Let $\pi_{a0}(T)$ denote the set of $\lambda \in \mathbf{C}$ such that λ is an isolated point of $\sigma_a(T)$ and $0 < \dim(T - \lambda)^{-1}(0) < \infty$, where $\sigma_a(T)$ denotes the *approximate point spectrum* of T . We say that *a-Weyl's theorem holds for T* if

$$\sigma_{aw}(T) = \sigma_a(T) \setminus \pi_{a0}(T),$$

where $\sigma_{aw}(T)$ denotes the *essential approximate point spectrum* of T (i.e., $\sigma_{aw}(T) = \bigcap \{\sigma_a(T + K) : K \in K(\mathcal{X})\}$ with $K(\mathcal{X})$ denoting the ideal of compact operators in $B(\mathcal{X})$). Recall that an operator T is *upper semi-Fredholm* (resp., *lower semi-Fredholm*) if $T(\mathcal{X})$ is closed and $\dim T^{-1}(0) < \infty$ (resp., $\dim(\mathcal{X} \setminus T(\mathcal{X})) < \infty$). If we let $\Phi_+^-(\mathcal{X})$ denote those upper semi-Fredholm operators $T \in B(\mathcal{X})$ for which the index $\dim T^{-1}(0) - \dim(\mathcal{X} \setminus T(\mathcal{X})) \leq 0$, then $\sigma_{aw}(T)$ is the complement in $\mathbf{C}$ of all those λ for which $(T - \lambda) \in \Phi_+^-(\mathcal{X})$ [1, Theorem 3.65]. We note that *a-Weyl's theorem* $\implies$ Weyl's theorem [1, Theorem 3.106].

An operator $T \in B(\mathcal{X})$ has the *single-valued extension property* at $\lambda_0 \in \mathbf{C}$, SVEP at λ_0 for short, if for every open disc $\mathcal{D}_{\lambda_0}$ centered at λ_0 the only analytic function $f : \mathcal{D}_{\lambda_0} \rightarrow \mathcal{X}$ which satisfies

$$(T - \lambda)f(\lambda) = 0 \quad \text{for all } \lambda \in \mathcal{D}_{\lambda_0}$$

is the function $f \equiv 0$. Trivially, every operator T has SVEP at points of the resolvent $\rho(T) = \mathbf{C} \setminus \sigma(T)$ and at points $\lambda \in \text{iso}\sigma(T)$. We say that *T has SVEP* if it has SVEP at every $\lambda \in \mathbf{C}$. A Banach space operator T with SVEP satisfies Browder's theorem.

The *quasinilpotent part* $H_0(T - \lambda)$ and the *analytic core* $K(T - \lambda)$ of $(T - \lambda)$ are defined by

$$H_0(T - \lambda) = \{x \in \mathcal{X} : \lim_{n \rightarrow \infty} \|(T - \lambda)^n x\|^{\frac{1}{n}} = 0\}$$

and

$$K(T - \lambda) = \{x \in \mathcal{X} : \text{there exists a sequence } \{x_n\} \subset \mathcal{X} \text{ and } \delta > 0$$

$$\text{for which } x = x_0, (T - \lambda)x_{n+1} = x_n \text{ and } \|x_n\| \leq \delta^n \|x\| \text{ for all } n = 1, 2, \dots\}.$$

We note that $H_0(T - \lambda)$ and $K(T - \lambda)$ are (generally) non-closed hyperinvariant subspaces of $(T - \lambda)$ such that $(T - \lambda)^{-m}(0) \subseteq H_0(T - \lambda)$ for all $m = 0, 1, 2, \dots$ and $(T - \lambda)K(T - \lambda) = K(T - \lambda)$ [1]. The operator $T \in B(\mathcal{X})$ is said to be *semi-regular* if $T(\mathcal{X})$ is closed and $T^{-1}(0) \subset T^\infty(\mathcal{X}) = \bigcap_{n \in \mathbf{N}} T^n(\mathcal{X})$; T admits a *generalized Kato decomposition*, *GKD* for short, if there exists a pair of T -invariant closed subspaces (M, N) such that $\mathcal{X} = M \oplus N$, the restriction $T|_M$ is quasinilpotent and $T|_N$ is semi-regular. An operator $T \in B(\mathcal{X})$ has a *GKD* at every $\lambda \in \text{iso}\sigma(T)$, namely $\mathcal{X} = H_0(T - \lambda) \oplus K(T - \lambda)$. We say that T is of *Kato type* at a point λ if $(T - \lambda)|_M$ is nilpotent in the *GKD* for $(T - \lambda)$. If $T - \lambda$ is of Kato type, then $K(T - \lambda) = (T - \lambda)^\infty(\mathcal{X})$. Fredholm (also, semi-Fredholm) operators are of Kato type. (For more information on semi-Fredholm operators, semi-regular operators and Kato type operators, see [1, Chapters 1.5 and 3.2].)

3. RESULTS

We assume in the following that the operator $T \in \mathcal{CHN}$, and that $A \in B(\mathcal{X})$ is an *algebraic operator* (there exists a non-constant polynomial $p(\cdot)$ such that $p(A) = 0$) such that $\sigma(A) = \{\mu_1, \mu_2, \dots, \mu_n\}$ for some scalars μ_i ($1 \leq i \leq n$). We shall denote $A|_{H_0(A - \mu_i)}$ by A_i and $T|_{H_0(A - \mu_i)}$ by T_i . The commutator $TA - AT$ of T and A will be denoted by $[T, A]$: thus T and A commute if $[T, A] = 0$. Recall that if Q is a quasinilpotent operator, in particular a nilpotent operator, such that $[T, Q] = 0$, then $\sigma(T) = \sigma(T + Q)$: this well known fact will be used in the sequel without further reference.

We start with some technical lemmas, required in the proof of our main result. Recall that an operator $S \in B(\mathcal{X})$ is *algebraically-CHN* if $p(S) \in \mathcal{CHN}$ for some non-constant polynomial $p(\cdot)$.

Lemma 3.1. *Operators T and their translates $T + \mu$ for a fixed scalar μ are algebraically $\mathcal{CHN}$.*

Proof. Simply define the polynomials $p(\cdot)$ and $p_1(\cdot)$ by $p(z) = z$ and $p_1(z) = p(z) - \mu$. $\square$

Recall from [3, Lemma 2.18] that an operator $S \in B(\mathcal{X})$ satisfies *a-Browder's theorem* if and only if S has SVEP at points $\lambda \notin \sigma_{aw}(S)$, and that *a-Browder's theorem* implies Browder's theorem. (S is said to satisfy *a-Browder's theorem* if $\sigma_{aw}(S) = \{\lambda \in \mathbf{C} : S - \lambda \notin \Phi_+^-(\mathcal{X}) \text{ or } \text{ascent}(S - \lambda) \text{ is infinite}\}$.)

Lemma 3.2. *Algebraically $\mathcal{CHN}$ operators satisfy Browder's theorem.*

Proof. If $p(S) \in \mathcal{CHN}$ for some non-constant polynomial $p(\cdot)$, then $p(S)$ has SVEP at points λ such that $p(S) - \lambda$ is semi-Fredholm [3, Corollary 2.10]. Hence S has SVEP at points $\mu \in \sigma(S)$ such that $p(\mu) = \lambda$ [1, Theorem 2.39]. Since the semi-Fredholm spectrum of an operator satisfies the spectral mapping theorem, S has

SVEP at points μ where $S - \mu$ is semi-Fredholm. In particular, S has SVEP at points $\mu \notin \sigma_{aw}(S) \implies S$ satisfies a -Browder's theorem $\implies S$ satisfies Browder's theorem. $\square$

The following lemma is taken from [3, Proposition 2.1].

Lemma 3.3. *Points $\lambda \in \text{iso}\sigma(T)$ are simple poles of the resolvent of T . In particular, $H_0(T - \lambda) = (T - \lambda)^{-1}(0)$ at points $\lambda \in \text{iso}\sigma(T)$.*

Lemma 3.4. *If $[T, N] = 0$ for some nilpotent operator $N \in B(\mathcal{X})$, then*

$$H_0(T + N - \lambda) = (T + N - \lambda)^{-m}(0),$$

for some integer $m \geq 1$, at points $\lambda \in \text{iso}\sigma(T)$.

Proof. We may assume that $N^m = 0$. Choose an integer $n > m$. Then, for every $x \in \mathcal{X}$ and complex number λ ,

$$\begin{aligned} \|(T - \lambda)^n x\|^{\frac{1}{n}} &= \|((T + N - \lambda) - N)^n x\|^{\frac{1}{n}} \\ &= \left\| \sum_{j=0}^{m-1} (-1)^j \binom{n}{j} N^j (T + N - \lambda)^{n-j} x \right\|^{\frac{1}{n}} \\ &\leq \sum_{j=0}^{m-1} \left[\binom{n}{j} \|N\|^j \right]^{\frac{1}{n}} \|(T + N - \lambda)^{n-j} x\|^{\frac{1}{n}}, \end{aligned}$$

which implies that

$$H_0(T - \lambda) \subseteq H_0(T + N - \lambda).$$

By symmetry,

$$H_0(T + N - \lambda) \subseteq H_0(T + N - \lambda - N) = H_0(T - \lambda).$$

Hence

$$H_0(T - \lambda) = H_0(T + N - \lambda).$$

Choose $\lambda \in \text{iso}\sigma(T)$. Then $H_0(T - \lambda) = (T - \lambda)^{-1}(0)$. Since $x \in (T - \lambda)^{-1}(0) \implies (T + N - \lambda)^m x = N^m x = 0$, it follows that

$$H_0(T + N - \lambda) = (T - \lambda)^{-1}(0) \subseteq (T + N - \lambda)^{-m}(0).$$

Finally, since $(T + N - \lambda)^{-m}(0) \subseteq H_0(T + N - \lambda)$ for all integers $m \geq 1$, the conclusion follows. $\square$

Lemma 3.5. *$A_i - \mu_i$ is nilpotent for all $1 \leq i \leq n$.*

Proof. Evidently, $\sigma(A_i) = \{\mu_i\}$ and $p(A_i) = 0$ (for some non-constant polynomial $p(\cdot)$). Since $\sigma(p(A_i)) = p(\sigma(A_i))$, $p(\mu_i) = 0$, and so

$$0 = p(A_i) = p(A_i) - p(\mu_i) = (A_i - \mu_i)^{m_i} g(A_i)$$

for some integer $m_i \geq 1$ and invertible $g(A_i)$. Hence $A_i - \mu_i = A|_{H_0(A - \mu_i)} - \mu_i I|_{H_0(A - \mu_i)}$ is nilpotent. $\square$

Combining Lemmas 3.4 and 3.5 we are able to relate the quasinilpotent part of $T + A - \lambda$ to its kernel at points $\lambda \in \text{iso}\sigma(T + A)$ as follows.

Lemma 3.6. *If $[T, A] = 0$, then $H_0(T + A - \lambda) = (T + A - \lambda)^{-m}(0)$, for some integer $m \geq 1$, for every $\lambda \in \text{iso}\sigma(T + A)$.*

Proof. Since the subspace $H_0(A - \mu_i)$ coincides with the range of the spectral projection of A associated with μ_i [1, Theorem 3.74], the hypothesis $[T, A] = 0$ implies that $[T_i, A_i] = 0$. By Lemma 3.5, $(A_i - \mu_i)^{m_i} = 0$. Evidently, either $\lambda - \mu_i \notin \sigma(T_i)$ or $\lambda - \mu_i \in \text{iso}\sigma(T_i)$; $1 \leq i \leq n$. If $\lambda - \mu_i \notin \sigma(T_i)$, then $T_i - (\lambda - \mu_i)$ is invertible, which implies that $\{T_i - (\lambda - \mu_i)\} + \{A_i - \mu_i\}$ is invertible, and hence that

$$H_0(T_i + A_i - \lambda) = H_0((T_i + A_i - \mu_i) - (\lambda - \mu_i)) = \{0\} = (T_i + A_i - \lambda)^{-m_i}(0).$$

If, instead, $\lambda - \mu_i \in \text{iso}\sigma(T_i)$, then, by Lemma 3.4,

$$\begin{aligned} H_0(T_i + A_i - \lambda) &= H_0((T_i + A_i - \mu_i) - (\lambda - \mu_i)) \\ &= ((T_i + A_i - \mu_i) - (\lambda - \mu_i))^{-m_i}(0) \\ &= (T_i + A_i - \lambda)^{-m_i}(0). \end{aligned}$$

Let $m = \max\{m_1, m_2, \dots, m_n\}$. Then

$$\begin{aligned} H_0(T + A - \lambda) &= \bigoplus_{i=1}^n H_0(T_i + A_i - \lambda) = \bigoplus_{i=1}^n (T_i + A_i - \lambda)^{-m_i}(0) \\ &= (T + A - \lambda)^{-m}(0). \end{aligned} \quad \square$$

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It is evident from the decomposition $\mathcal{X} = \bigoplus_{i=1}^n H_0(A - \mu_i)$ that if

$$\lambda \in \text{iso}\sigma(T + A),$$

then, for some $1 \leq j \leq n$,

$$\lambda \in \text{iso}\sigma(T_j + A_j) = \text{iso}\sigma(T_j + \mu_j + (A_j - \mu_j)) = \text{iso}\sigma(T_j + \mu_j),$$

so that $\lambda - \mu_j \in \text{iso}\sigma(T_j)$. It thus follows from the proof of Lemma 3.6 that $H_0(T + A - \lambda) = (T_j + A_j - \lambda)^{-m_j}(0)$. Consequently

$$H_0(T + A - \lambda) \subseteq \bigoplus_{i=1}^n (T_i + A_i - \lambda)^{-m_i}(0) = (T + A - \lambda)^{-m}(0),$$

and hence

$$H_0(T + A - \lambda) = (T + A - \lambda)^{-m}(0).$$

This, since $\lambda \in \text{iso}\sigma(T + A)$ implies that $\mathcal{X} = (T + A - \lambda)^{-m}(0) \oplus K(T + A - \lambda)$, whence $K(T + A - \lambda) = (T + A - \lambda)^m \mathcal{X}$ and λ is a pole of the resolvent of $T + A$ [9, Propositions 38.4 and 50.2]. In particular:

Corollary 3.7. *Operators $T + A$, $[T, A] = 0$, are of Kato type at points*

$$\lambda \in \text{iso}\sigma(T + A).$$

Proof. $T + A - \lambda|_{H_0(T+A-\lambda)}$ is nilpotent and $T + A - \lambda|_{K(T+A-\lambda)}$ is invertible. $\square$

Lemma 3.8. *If $[T, A] = 0$, then $T_i + A_i$ satisfies Browder's theorem for all $1 \leq i \leq n$.*

Proof. Recall from [8, Theorem 11] that if Browder's theorem holds for an operator S , then Browder's theorem holds for $S + Q$ for every quasi-nilpotent operator Q such that $[S, Q] = 0$. Since $T_i + \mu_i$ satisfies Browder's theorem (by Lemmas 3.1 and 3.2), and since $A_i - \mu_i$ is nilpotent (see Lemma 3.5), the proof follows. $\square$

The direct sum of a pair of Banach space operators, each of which satisfies Browder's theorem, may fail to satisfy Browder's theorem. (Consider, for example, the direct sum $U \oplus U^*$, where U is the forward unilateral shift on the (sequence) space ℓ_p .) Thus our conclusion that each $T_i + A_i$ satisfies Browder's theorem does not guarantee that $T + A = \bigoplus_{i=1}^n T_i + A_i$ satisfies Browder's theorem. We require something more. The following lemma considers a few such conditions, but before the lemma some terminology.

A subspace M of $\mathcal{X}$ is said to be orthogonal to a subspace N of $\mathcal{X}$, $M \perp N$, in the sense of G. Birkhoff, if $\|a\| \leq \|a + b\|$ for every $a \in M$ and $b \in N$ [5, p 93]. This asymmetric definition of orthogonality coincides with the usual concept of orthogonality in the case in which $\mathcal{X}$ is a Hilbert space. We say that the subspaces M and N are mutually orthogonal, denoted $M \perp_m N$, if $M \perp N$ and $N \perp M$.

Lemma 3.9. *If $[T, A] = 0$, then $T_i + A_i$ has SVEP for all $1 \leq i \leq n$ whenever one of the following conditions is satisfied:*

- (i) $\mathcal{X}$ is separable.
- (ii) T_i has SVEP at points $\lambda \in \sigma_w(T_i)$.
- (iii) T has SVEP at points $\lambda \in \sigma_w(T)$.
- (iv) $\mathcal{X}$ is a Hilbert space.
- (v) $T^{-1}(0) \perp (T - \alpha)^{-1}(0)$ for all $(0 \neq) \alpha \in \mathbf{C}$.

Proof. Recall from [3, Proposition 2.5] that if $S \in \mathcal{CHN}$, and α and β , $|\alpha| < |\beta|$, are eigenvalues of S with corresponding eigenspaces N and M , then $M \perp N$ in the case in which $\alpha = 0$ and $M \perp_m N$ in the case in which $\alpha \neq 0$.

(i) We prove that T_i has a countable number of eigenvalues, and hence has SVEP. Assume to the contrary that T_i has an uncountable number of eigenvalues. Then (by the orthogonality of the eigenspaces corresponding to distinct non-zero eigenvalues of T_i) there exists an uncountable number of unit vectors a_s and a_t such that $1 \leq \|a_s - a_t\|$, and hence that the subspace $H_0(A_i - \mu_i)$ is not separable. Since this contradicts the separability of $\mathcal{X}$, T_i has SVEP. Hence $T_i + \mu_i$, $1 \leq i \leq n$, has SVEP. Since $A_i - \mu_i$ is nilpotent (Lemma 3.5), $T_i + A_i = (T_i + \mu_i) + (A_i - \mu_i)$ has SVEP.

(ii) As we saw in the proof of Lemma 3.2, T has SVEP at points $\lambda \notin \sigma_w(T)$. Hence, if T has SVEP at points $\lambda \in \sigma_w(T)$, then T has SVEP. Recall that an upper triangular matrix $S = \bigoplus_{i=1}^n S_i$ (with operator entries) has SVEP if and only if the elements along the main diagonal of the matrix have SVEP. (This is proved for 2×2 upper triangular matrices in [1, Theorem 2.9]; the general case follows from a finite induction argument.) Thus, if $T = \bigoplus_{i=1}^n T_i$ has SVEP, then T_i has SVEP for all $1 \leq i \leq n$. As in the proof of part (i), this implies that $T_i + A_i$ has SVEP.

(iii) Evident (from above).

(iv) The orthogonality property (stated above) implies in the case in which $\mathcal{X}$ is a Hilbert space that $(T - \alpha)^{-1}(0) \perp_m (T - \beta)^{-1}(0)$ for all complex numbers α and β . If T does not have SVEP at α , then there exists an open disc $\mathcal{U}$ centered at α and a non-trivial analytic function $f : \mathcal{U} \rightarrow \mathcal{X}$ such that $f(\lambda) \in (T - \lambda)^{-1}(0)$ for all $\lambda \in \mathcal{U}$. Let $\beta \in \mathcal{U}$ be such that $f(\beta) \neq 0$. Then $f(\beta) \in (T - \beta)^{-1}(0)$, and we may assume that $\|f(\beta)\| = 1$. If $\lambda \in \mathcal{U} \setminus \{\beta\}$, then the orthogonality of $(T - \beta)^{-1}(0)$ and $(T - \lambda)^{-1}(0)$ yields $1 = \|f(\beta)\| \leq \|f(\beta) - f(\lambda)\|$, from which it follows that f is not continuous at β , a contradiction. Hence T has SVEP. Now argue as above to prove that $T_i + A_i$ has SVEP.

(v) The proof in this case is similar to that for case (iv), for the hypothesis $T^{-1}(0) \perp (T - \alpha)^{-1}(0)$ implies that $(T - \alpha)^{-1}(0) \perp_m (T - \beta)^{-1}(0)$ for all complex numbers α and β . $\square$

The proof of our main result, Theorem 1.1, is now a mere formality.

Theorem 3.10. *If $[T, A] = 0$, and if one of the conditions of Lemma 3.9 is satisfied, then $T + A$ satisfies Weyl's theorem and (the conjugate operator) $(T + A)^*$ satisfies a -Weyl's theorem.*

Proof. By Lemma 3.9, every element along the main diagonal in the upper triangular matrix $\bigoplus_{i=1}^n T_i + A_i$ has SVEP; hence $T + A = \bigoplus_{i=1}^n T_i + A_i$ has SVEP (and therefore satisfies Browder's theorem). To complete the proof of the theorem, we recall from [4, Theorems 3.3 and 3.6(ii)] that a Banach space operator with SVEP satisfies Weyl's theorem, and its conjugate operator satisfies a -Weyl's theorem, whenever the operator is of Kato type at the isolated points of its spectrum. Since our operator $T + A$ is of Kato type at points $\lambda \in \text{iso}\sigma(T + A)$, see Corollary 3.7, the theorem is proved. $\square$

The argument above applies to $H(m)$ operators. Evidently, operators $T \in H(m)$ have SVEP (so that $T + A$ has SVEP whenever $[T, A] = 0$), and are Kato type at points $\lambda \in \text{iso}\sigma(T)$. Arguing as in Lemmas 3.4, 3.5 and 3.6, it follows that if $[T, A] = 0$, then $T + A$ is of Kato type at points $\lambda \in \text{iso}\sigma(T + A)$. Hence, if $[T, A] = 0$, then $T + A$ satisfies Weyl's theorem [10] and $(T + A)^*$ satisfies a -Weyl's theorem.

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