A MATRICIAL CORONA THEOREM II

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ABSTRACT. We extend the “matricial corona theorem” of M. Andersson to general algebras of functions which satisfy a corona theorem.

In Trent and Zhang [7], we extended the matricial corona theorem of Fuhrmann and Vasyunin (see Nikolski [2]) to more general algebras of functions which satisfy a corona theorem. In this paper, we provide a similar extension of the matricial corona theorem of M. Andersson [1] to the same algebras of functions satisfying a corona theorem. Thus, we’re giving life to the metatheorem that “whenever a corona theorem holds for an algebra of functions, a matricial corona theorem also holds.” It should be noted that since the techniques are algebraic and general, the matricial norm estimates which follow may be very bad for any specific algebra of functions. Some examples of algebras where the theorem is applicable are discussed in [7]. We list a few more examples at the end of this paper. Also, note that a complete extension of the corona theorem to two-sided infinite matrices is not possible, in general, even for entries in $H^\infty(D)$. (See Treil [5].)

For ease of notation, we will assume that $\mathcal{A}$ denotes a multiplier algebra for a reproducing kernel Hilbert space of functions on Ω . That is, $\mathcal{H}$ is a Hilbert space of functions on Ω , such that, for each $w \in \Omega$, there exists a unique $k_w \in \mathcal{H}$, the reproducing kernel, which satisfies $f(w) = \langle f, k_w \rangle_{\mathcal{H}}$ for all $f \in \mathcal{H}$. Then $\mathcal{A} = \{g \in \mathcal{H} \mid M_g \in B(\mathcal{H}), \text{ where } (M_g f)(w) = g(w)f(w) \text{ for all } w \in \Omega\}$. We may also use $\mathcal{M}(\mathcal{H})$ to denote this multiplier algebra. If B is an algebra of bounded functions, which is not such a multiplier algebra, we only consider finite matrices, so the arguments appearing below are easily modified in this case.

For $\{f_j\}_{j=1}^\infty \subset \mathcal{A}$ and $z \in \Omega$, we let $F(z) = (f_1(z), f_2(z), \dots)$ and define $M_F(\{g_j\}_{j=1}^\infty) = \sum_{j=1}^\infty f_j g_j$, an operator acting from $\bigoplus_1^\infty \mathcal{H}$ to $\mathcal{H}$. M_F^T will de-

note the transpose of M_F acting from $\mathcal{H}$ to $\bigoplus_1^\infty \mathcal{H}$. Similarly, if $\mathcal{F}(z) = \begin{bmatrix} F_1(z) \\ \vdots \\ F_n(z) \end{bmatrix}$

for $F_j(z) = (f_{j,1}(z), f_{j,2}(z), \dots)$ with $f_{j,k} \in \mathcal{A}$ and $z \in \Omega$, we let $M_{\mathcal{F}}$ denote

$\begin{bmatrix} M_{F_1} \\ \vdots \\ M_{F_n} \end{bmatrix}$ acting from $\bigoplus_1^\infty \mathcal{H}$ to $\bigoplus_1^n \mathcal{H}$. Again, $M_{\mathcal{F}}^T = [M_{F_1}^T, \dots, M_{F_n}^T]$.

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Since $M_F^*(k_w) = (f_1 \overline{(w)} k_w, f_2 \overline{(w)} k_w, \dots)^T$, we see that

$$\sum_{j=1}^{\infty} |f_j(w)|^2 \leq \|M_F\|^2 \text{ for all } w \in \Omega.$$

We assume that $\mathcal{A}$ satisfies a “corona theorem”; that is,

Corona Theorem (CT). *Assume that $\{f_j\}_{j=1}^{\infty} \subset \mathcal{A}$ and $F = (f_1, f_2, \dots)$ satisfies*

$$(1) \quad 0 < \delta^2 \leq F(z)F(z)^*$$

and

$$(2) \quad \max\{\|M_F\|, \|M_F^T\|\} = 1.$$

Then there exists $\{g_j\}_{j=1}^{\infty} \subset \mathcal{A}$ such that if $G = (g_1, g_2, \dots)$,

$$(a) \quad FG^T = 1 \text{ in } \Omega,$$

$$(b) \quad \max\{\|M_G\|, \|M_G^T\|\} < \infty.$$

Note: (i) In some algebras such as multipliers on Dirichlet space

$$\|M_G^T\| < \infty \Rightarrow \|M_G\| < \infty,$$

so (2) above may be replaced by

$$(2') \quad \|M_F^T\| = 1.$$

(See Trent [6].)

(ii) For $\mathcal{A} = H^\infty(\Omega)$, $\Omega \subset \mathbb{C}^n$ bounded and open,

$$\|M_F\| = \|M_F^T\| = \sup_{z \in \Omega} \left(\sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{\frac{1}{2}}.$$

So (2) above is equivalent to $(2') \sup_{z \in \Omega} \left(\sum_{j=1}^{\infty} |f_j(z)|^2 \right) = 1$.

(iii) For $F = (f_1, \dots, f_j)$, a finite number of elements of $\mathcal{A}$,

(2) may be omitted from the hypotheses.

[From (i), (ii), and (iii), we see that variations of **CT** involving changing hypothesis (2) are possible. Extensions of **CT** for such versions follow from our techniques below, but we will stick to the above formulation of **CT**.]

We establish the “matricial corona theorem of M. Andersson [1]” for such algebras $\mathcal{A}$, that satisfy **CT**. Let $\mathcal{A}$ be an algebra of functions on Ω , that satisfies **CT**. Our matricial corona theorem says that if we can solve a system of linear equations with coefficients from $\mathcal{A}$ pointwise on Ω , then we can find solutions belonging to $\mathcal{A}$.

Matricial Corona Theorem (MCT). *Assume that **CT** holds for $\mathcal{A}$. Let F denote an $m \times \infty$ matrix of elements of $\mathcal{A}$ satisfying:*

$$(1) \quad \epsilon^2 P_{\text{ran } F(z)} \leq F(z)F(z)^* \text{ for all } z \in \Omega,$$

$$(1') \quad \dim(\text{ran } F(z)) \text{ is constant for } z \in \Omega,$$

$$(2) \quad \max\{\|M_F\|, \|M_F^T\|\} = 1,$$

and

$$(3) F(z)\underline{u}(z) = \underline{H}(z) \text{ in } \Omega,$$

where $\underline{H} = (H_1, \dots, H_m)^T$, $H_i \in \mathcal{A}$ and $\underline{u} : \Omega \rightarrow l_2$ is an arbitrary function. Then, there exists an $\underline{A} = (v_1, v_2, \dots)^T$ with $v_i \in \mathcal{A}$ for all i satisfying

$$(1) F(z)\underline{A}(z) = \underline{H}(z) \text{ in } \Omega$$

and

$$(2) \max\{\|M_{\underline{A}}\|, \|M_{\underline{A}}^T\|\} < \infty.$$

Note: In the case that Ω is connected and the functions in $\mathcal{A}$ are continuous on Ω , we may omit condition (1') since it follows from condition (1).

Denote the rows of F , satisfying (1) and (2) of **MCT**, by $f_1, \dots, f_m$. For $1 \leq j \leq m$, we will denote the j th term of the Koszul complex with 0-th element f_p by $Q_p^{(j)}$. This means that for each $z \in \Omega$,

$$\begin{aligned} \ker f_p(z) &\supseteq \operatorname{ran} Q_p^{(1)}(z), \\ \ker Q_p^{(1)}(z) &\supseteq \operatorname{ran} Q_p^{(2)}(z), \\ &\vdots \end{aligned}$$

and, moreover,

$$\begin{aligned} f_p(z)^* f_p(z) + Q_p^{(1)}(z) Q_p^{(1)}(z)^* &= (f_p(z) f_p(z)^*) I, \\ Q_p^{(1)}(z)^* Q_p^{(1)}(z) + Q_p^{(2)}(z) Q_p^{(2)}(z)^* &= (f_p(z) f_p(z)^*) I, \\ &\vdots \end{aligned}$$

Whenever $f_p(z) f_p(z)^* > 0$ for some $z \in \Omega$, we deduce that

$$\begin{aligned} \ker f_p(z) &= \operatorname{ran} Q_p^{(1)}(z), \\ \ker Q_p^{(1)}(z) &= \operatorname{ran} Q_p^{(2)}(z), \\ &\vdots \end{aligned}$$

In addition,

$$Q_p^{(j)}(z) Q_q^{(j+1)}(z) = -Q_q^{(j)}(z) Q_p^{(j+1)}(z).$$

See the Appendix in [6] for the existence of $\{Q_p^{(j)}(z)\}_{j=1}^\infty$ and the above properties.

We will need several lemmas.

Lemma 1. Let $A = (a_1, a_2, \dots)$, $a_i \in \mathcal{A}$ and assume that

$$\max\{\|M_A\|, \|M_A^T\|\} < \infty.$$

Fix $z \in \Omega$ and let

$$Q^{(j)}(z) = Q_A^{(j)}(z).$$

Then for $1 \leq j$,

$$\|M_{Q^{(j)}}\| \leq (j+1)\|M_A\| \text{ and } \|M_{Q^{(j)}}^T\| \leq (j+1)\|M_A^T\|.$$

Proof. We will show that the first inequality holds. The second follows in a similar manner. See the Appendix for notation. Let $\Pi(j)$ denote increasing j -tuples of positive integers. For $\pi = (i_1, \dots, i_j) \in \Pi(j)$, let e_π denote $e_{i_1} \wedge \dots \wedge e_{i_j}$. Now

$$\begin{aligned} Q^{(j)}(z)^* \underline{x}(z) &= \overline{A(z)} \wedge \underline{x}(z) \\ &= \left(\sum_{j=1}^{\infty} \overline{a_j(z)} e_j \right) \wedge \left(\sum_{\pi \in \Pi(j)} x_\pi(z) e_\pi \right) \\ &= \sum_{j=1}^{\infty} \sum_{\pi \in \Pi(j)} \overline{a_j(z)} x_\pi(z) e_j \wedge e_\pi \\ &= \sum_{1 \leq i_1 < i_2 < \dots < i_{j+1}} \sum_{r=1}^{j+1} (-1)^{r-1} \overline{a_r(z)} x_{i_1, \dots, \widehat{i_r}, \dots, i_{j+1}} e_{i_1, \dots, i_{j+1}}. \end{aligned}$$

Thus

$$\begin{aligned} \|M_{Q^{(j)}}^* \underline{x}\|^2 &= \sum_{\substack{\sigma \in \Pi(j+1) \\ \sigma = (i_1, \dots, i_{j+1})}} \left\| \sum_{r=1}^{j+1} (-1)^{r-1} M a_r^* x_{\sigma - \{i_r\}} \right\|^2 \\ &\leq (j+1) \sum_{i_1 < \dots < i_{j+1}} \sum_{r=1}^{j+1} \|M a_{i_r}^* x_{\sigma - \{i_r\}}\|^2 \\ &= (j+1) \sum_{i_1 < \dots < i_{j+1}} (\|M a_{i_1}^* x_{\sigma - \{i_1\}}\|^2 + \dots + \|M a_{i_{j+1}}^* x_{\sigma - \{i_{j+1}\}}\|^2) \\ &\leq (j+1) \|M_A^*\|^2 \left(\sum_{i_2 < \dots < i_{j+1}} \|x_{i_2, \dots, i_{j+1}}\|^2 + \dots + \sum_{i_1 < \dots < i_j} \|x_{i_1, \dots, i_j}\|^2 \right) \\ &\leq (j+1)^2 \|M_A\|^2 \|\underline{x}\|^2. \end{aligned}$$

So

$$\|M_{Q^{(j)}}\| \leq (j+1) \|M_A\|.$$

□

To handle the nonsurjectivity of $F(z)$, we need the following:

Definition 1. Let $B \in M_n(\mathbb{C})$. For $1 \leq k \leq n$ define

$$\det_k(B) = \sum_{\pi \in \Pi_k(n)} \det(E_\pi B E_\pi),$$

where $\Pi_k(n)$ denotes the increasing k -tuples of integers in $\{1, 2, \dots, n\}$.

Lemma 2. Let B be an $n \times n$ matrix and let $C = SBS^{-1}$ where $S, S^{-1} \in M_n(\mathbb{C})$. Then

$$\det_k(B) = \det_k(C).$$

Proof. Let $f(\lambda) = \det(\lambda I - B)$ and $g(\lambda) = \det(\lambda I - C)$. Then g and f are polynomials of degree n . The $(n-k)^{\text{th}}$ coefficient of f is $(-1)^k \det_{n-k}(B)$. But clearly $f \equiv g$, so the result follows. □

Lemma 3. Let $A \in M_n(\mathbb{C})$ with $A \geq \epsilon P$, where P is a projection of rank k . Then $\det_k(A) \geq \epsilon^{2k}$.

Proof. Clearly, $A \geq \epsilon P$ implies that $\det_k(A) \geq \det_k(\epsilon P)$. Now P is similar to an $n \times n$ matrix of the form $\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$. Thus $\det_k(\epsilon P) = \epsilon^{2k}$. So $\det_k(A) \geq \epsilon^{2k}$. $\square$

Assume that Ω is connected and $\mathcal{A} = \mathcal{M}(\mathcal{H})$ is composed of continuous functions on Ω . Let $F(z)$ denote an $n \times \infty$ matrix whose entries are in $\mathcal{M}(\mathcal{H})$. [Otherwise we use assumption (1') to get the conclusion of Lemma 4.]

Lemma 4. *Assume that $I \geq F(z)F(z)^* \geq \epsilon^2 P_{\text{ran } F(z)}$ for all $z \in \Omega$. Let $k = \max\{\dim(\text{ran } F(z)) : z \in \Omega\}$. Then*

$$\det_k(F(z)F(z)^*) \geq \epsilon^{2k} \text{ for all } z \in \Omega.$$

Proof. Choose $z_0 \in \Omega$ so that $\dim(\text{ran } F(z_0)) = k$. Then by Lemma 2, we have

$$\det_k(F(z_0)F(z_0)^*) \geq \epsilon^{2k}.$$

Since $z \mapsto \det_k(F(z)F(z)^*)$ is continuous on Ω and integer-valued, a connectedness argument gives us that

$$\det_k(F(z)F(z)^*) \geq \epsilon^{2k} \text{ for all } z \in \Omega.$$

$\square$

Definition 2. Assume that $\{X_j\}_{j=1}^{n+1}$ are Banach spaces and let $\{T_{jk}\}_{j,k=1}^n$ denote operators such that $T_{jk} \in B(X_{j+1}, X_j)$. Let

$$\text{“det”} \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & & & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} \end{pmatrix} := \sum_{\substack{\sigma \in P(n) \\ \sigma = \{i_1, \dots, i_n\}}} (-1)^{\text{sgn}(\sigma)} T_{1,i_1} T_{2,i_2} \cdots T_{n,i_n},$$

where the products are given in the order indicated and $P(n)$ denotes the permutations of $\{1, \dots, n\}$ and $\text{sgn}(\sigma)$ denotes the sign of the permutation σ .

Notice that the order of terms in the “product” is not allowed to change since composition is not commutative. But this “det” tells us something about independence of columns. Namely, the “det” will vanish if we have linearly dependent columns.

Basically, under the hypothesis that $F\underline{u} = \underline{H}$, Lemmas 5, 6, and 7 will help us use the fact that $\begin{bmatrix} \underline{H}^T \\ F^T \end{bmatrix}$ has the same column dependencies as F^T .

In our proof of MCT, we verify an equation involving F_1 . To get the analogous equation with F_1 replaced by F_j , we need to replace the 1's in Lemma 6 by j 's.

Lemma 5.

$$\text{“det”} \begin{pmatrix} \underline{f}_1 & \cdots & \underline{f}_p \\ Q_1^{(1)} & \cdots & Q_p^{(1)} \\ \vdots & & \vdots \\ Q_1^{(p-1)} & \cdots & Q_p^{(p-1)} \end{pmatrix} = p! \underline{f}_1 Q_2^{(1)} \cdots Q_p^{(p-1)}.$$

Proof.

$$\begin{aligned} \text{"det"} \begin{pmatrix} \underline{f}_1 & \cdots & \underline{f}_p \\ Q_1^{(1)} & \cdots & Q_p^{(1)} \\ \vdots & & \vdots \\ Q_1^{(p-1)} & \cdots & Q_p^{(p-1)} \end{pmatrix} &= \sum_{\substack{\sigma \in P(p) \\ \sigma = (i_1, \dots, i_p)}} (-1)^\sigma \underline{f}_{i_1} Q_{i_2}^{(1)} \cdots Q_{i_p}^{(p-1)} \\ &= p! \underline{f}_1 Q_2^{(1)} \cdots Q_p^{(p-1)}. \end{aligned}$$

The last line follows since for $Q_j^{(0)} \doteq \underline{f}_j$, we have

$$Q_i^{(l)} Q_j^{(l+1)} = -\delta_{ij} Q_j^{(l)} Q_i^{(l+1)}$$

for $0 \leq i, j \leq p$. □

Lemma 6.

$$\begin{aligned} \text{"det"} \begin{pmatrix} H_1 & H_{i_1} & \cdots & H_{i_p} \\ \underline{f}_1 & \underline{f}_{i_1} & \cdots & \underline{f}_{i_p} \\ Q_1^{(1)} & Q_{i_1}^{(1)} & \cdots & Q_{i_p}^{(1)} \\ \vdots & & & \vdots \\ Q_1^{(p-1)} & Q_{i_1}^{(p-1)} & \cdots & Q_{i_p}^{(p-1)} \end{pmatrix} \\ = (p-1)! [H_1 \underline{f}_{i_1} Q_{i_2}^{(1)} \cdots Q_{i_p}^{(p-1)} \\ + \sum_{l=1}^p (-1)^l \underline{f}_1 H_{i_l} Q_{i_1}^{(1)} \cdots Q_{i_{l-1}}^{(l-1)} Q_{i_{l+1}}^{(l)} \cdots Q_{i_p}^{(p-1)}]. \end{aligned}$$

Proof. This follows as in the proof of Lemma 5. □

Lemma 7. Assume that $F(\cdot)$ is $n \times \infty$ with $F(z)F(z)^*$ having constant rank $p < n$. Suppose that for some function $\underline{u} : \Omega \rightarrow l^2$, $F(z)\underline{u}(z) = \underline{H}(z)$, where $\underline{H} = (H_1, \dots, H_n)$ with $H_j \in \mathcal{M}(H)$. Then for any $\pi \in \Pi_{p+1}(n)$ with $\pi = (j_1, \dots, j_{p+1})$, we have for each $z \in \Omega$,

$$\text{"det"} \begin{pmatrix} H_{i_1} & \cdots & H_{i_{p+1}} \\ \underline{f}_{i_1} & \cdots & \underline{f}_{i_{p+1}} \\ Q_{i_1}^{(1)} & \cdots & Q_{i_{p+1}}^{(1)} \\ \vdots & & \vdots \\ Q_{i_1}^{(p-1)} & \cdots & Q_{i_{p+1}}^{(p-1)} \end{pmatrix} = 0.$$

Proof. For ease of notation, let $i_1 = 1, \dots, i_{p+1} = p+1$. Fix $z \in \Omega$ and choose a $(p+1)$ -row vector of scalars, $\underline{a}$, so that

$$\underline{a} \begin{bmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_{p+1} \end{bmatrix} = 0, \text{ where } F = \begin{bmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_n \end{bmatrix}.$$

Then

$$0 = \underline{a} \begin{bmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_{p+1} \end{bmatrix} \cdot \underline{u} = \underline{a} \begin{bmatrix} H_1 \\ \vdots \\ H_{p+1} \end{bmatrix}.$$

Thus

$$\begin{bmatrix} H_1 & \cdots & H_{p+1} \\ \underline{f}_1 & \cdots & \underline{f}_{p+1} \\ Q_1^{(1)} & \cdots & Q_{p+1}^{(1)} \\ \vdots & & \vdots \\ Q_1^{(p-1)} & \cdots & Q_{p+1}^{(p-1)} \end{bmatrix} \underline{a}^T \equiv 0,$$

and the result follows since “det” is multilinear in its columns. Also, we use the fact that

$$[Q_1^{(n)}, \dots, Q_{p+1}^{(n)}] \underline{a}^T = Q_{\sum_{j=1}^{p+1} a_j \underline{f}_j}^{(k)} = Q_0^{(k)} = 0.$$

□

We are now ready to complete the proof of the matricial corona theorem.

Proof. (MCT) By Lemma 4, there is a maximum, p , $1 \leq p \leq n$, so that

$$\det_p(F(z)F(z)^*) \geq \epsilon^{2p} \text{ for all } z \in \Omega.$$

But

$$\begin{aligned} \det_p(F(z)F(z)^*) &= \sum_{\substack{\pi \in \Pi_p(n) \\ \pi = \{i_1, \dots, i_p\}}} \det(E_\pi F(z)F(z)^* E_\pi) \\ &= \sum_{\substack{\pi \in \Pi_p(n) \\ \pi = \{i_1, \dots, i_p\}}} \underline{f}_{i_1} Q_{i_2}^{(1)} \cdots Q_{i_p}^{(p-1)} Q_{i_p}^{(p-1)*} \cdots Q_{i_2}^{(1)*} \underline{f}_{i_1}^*. \end{aligned}$$

By Lemma 1,

$$\|M_{\underline{f}_{i_1}} M_{Q_{i_2}^{(1)}} \cdots M_{Q_{i_p}^{(p-1)}}\| \leq \|M_{\underline{f}_{i_1}}\| \|M_{Q_{i_2}^{(1)}}\| \cdots \|M_{Q_{i_p}^{(p-1)}}\| < \infty$$

and, similarly,

$$\|(M_{\underline{f}_{i_1}} M_{Q_{i_2}^{(1)}} \cdots M_{Q_{i_p}^{(p-1)}})^T\| < \infty.$$

Thus by the corona theorem in $\mathcal{M}(\mathcal{H})$, for each $\pi \in \Pi_p(n)$ there is a $\underline{v}_\pi$ with entries in $\mathcal{M}(\mathcal{H})$ so that

$$p! \sum_{\substack{\pi \in \Pi_p(n) \\ \pi = \{i_1, \dots, i_p\}}} \underline{f}_{i_1} Q_{i_2}^{(1)} \cdots Q_{i_p}^{(p-1)} \underline{v}_\pi \equiv 1$$

and

$$\max_{\pi \in \Pi_p(n)} \{\|M_{\underline{v}_\pi}\|, \|M_{\underline{v}_\pi}^T\|\} < \infty.$$

Computing as in the Appendix of Trent and Zhang [7], we may write this as

$$\begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_n \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ \vdots \\ Q_n^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(p-1)} \\ \vdots \\ Q_n^{(p-1)} \end{pmatrix} \cdot \underline{v} \equiv 1,$$

where $\underline{v}$ is a vector with $\binom{n}{p}$ entries $\underline{v}_\pi$ for $\pi \in \Pi_p(n)$. Then we have

$$\max\{\|M_{\underline{v}}\|, \|M_{\underline{v}}^T\|\} < \infty.$$

We claim that the vector

$$\underline{A} = p \begin{pmatrix} H_1 I \\ \vdots \\ H_n I \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ \vdots \\ Q_n^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(p-1)} \\ \vdots \\ Q_n^{(p-1)} \end{pmatrix} \cdot \underline{v}$$

gives us a solution to our problem. We need only show that

$$F \underline{A} = \underline{H},$$

since a finite number of applications of Lemma 1 gives us

$$\max\{\|M_{\underline{A}}\|, \|M_{\underline{A}}^T\|\} < \infty.$$

It suffices to show that

$$\underline{f}_1 \underline{A} = H_1.$$

The other rows of F follow similarly:

$$\underline{f}_1 \underline{A} = p \begin{pmatrix} H_1 \underline{f}_1 \\ \vdots \\ H_n \underline{f}_1 \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ \vdots \\ Q_n^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(p-1)} \\ \vdots \\ Q_n^{(p-1)} \end{pmatrix} \cdot \underline{v}.$$

Let

$$\begin{aligned} B &= p \begin{pmatrix} H_1 \underline{f}_1 \\ \vdots \\ H_n \underline{f}_1 \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ \vdots \\ Q_n^{(1)} \end{pmatrix} \wedge \cdots \wedge \begin{pmatrix} Q_1^{(p-1)} \\ \vdots \\ Q_n^{(p-1)} \end{pmatrix} \\ &= p! \begin{pmatrix} H_1 \underline{f}_1 \\ \vdots \\ H_n \underline{f}_1 \end{pmatrix} \wedge \sum_{\substack{\sigma \in \Pi_{p-1}(n) \\ \sigma = (j_2, \dots, j_p)}} Q_{j_2}^{(1)} \cdots Q_{j_p}^{(p-1)} \underline{e}_\sigma \\ &= p! \sum_{j=1}^n \sum_{\substack{\sigma \in \Pi_{p-1}(n) \\ \sigma = (j_2, \dots, j_p) \\ -1 \notin \sigma}} H_j \underline{f}_1 Q_{j_2}^{(1)} \cdots Q_{j_p}^{(p-1)} \underline{e}_j \wedge \underline{e}_\sigma \\ &= p! H_1 \sum_{\substack{\sigma \in \Pi_{p-1}(n) \\ \sigma = (j_2, \dots, j_p) \\ 1 < j_2 < \cdots < j_p}} \underline{f}_1 Q_{j_2}^{(1)} \cdots Q_{j_p}^{(p-1)} \underline{e}_1 \wedge \underline{e}_\sigma \\ &\quad + p! \sum_{j=2}^n \sum_{\substack{\sigma \in \Pi_{p-1}(n) \\ \sigma = (j_2, \dots, j_p) \\ 1, j \notin \sigma}} H_j \underline{f}_1 Q_{j_2}^{(1)} \cdots Q_{j_p}^{(p-1)} \underline{e}_j \wedge \underline{e}_\sigma \\ &= p! \sum_{\substack{\pi \in \Pi_p(n) \\ 1 \in \pi \\ \pi = (1, i_2, \dots, i_p)}} \underline{f}_1 Q_{i_1}^{(1)} \cdots Q_{i_p}^{(p-1)} \underline{e}_\pi \\ &\quad + p! \sum_{\substack{\pi \in \Pi_p(n) \\ 1 \notin \pi \\ \pi = (i_1, \dots, i_p)}} \sum_{l=1}^p (-1)^{l-1} H_{i_l} \underline{f}_1 Q_{i_1}^{(1)} \cdots Q_{i_{l-1}}^{(l-1)} Q_{i_{l+1}}^{(l)} \cdots Q_{i_p}^{(p-1)} \underline{e}_\pi. \end{aligned}$$

But by Lemma 6, for $\pi \in \Pi_p(n)$, $1 \notin \pi$, $\pi = (i_1, \dots, i_p)$, we have

$$H_1 \underline{f}_{i_1} Q_{i_2}^{(1)} \dots Q_{i_p}^{(p-1)} = \sum_{l=1}^p (-1)^{l-1} H_{i_l} \underline{f}_{i_1} Q_{i_1}^{(1)} \dots Q_{i_{l-1}}^{(l-1)} Q_{i_{l+1}}^{(l)} \dots Q_{i_p}^{(p-1)}.$$

So B is equal to

$$\begin{aligned} & p! \sum_{\substack{\pi \in \Pi_p(n) \\ \pi = (i_1, \dots, i_p)}} H_1 \underline{f}_{i_1} Q_{i_2}^{(1)} \dots Q_{i_p}^{(p-1)} e_\pi \\ &= H_1 \left(p! \sum_{\substack{\pi \in \Pi_p(n) \\ \pi = (i_1, \dots, i_p)}} \underline{f}_{i_1} Q_{i_2}^{(1)} \dots Q_{i_p}^{(p-1)} e_\pi \right). \end{aligned}$$

Thus

$$\begin{aligned} \underline{f}_1 A &= B \cdot \underline{v} \\ &= H_1 \begin{pmatrix} \underline{f}_1 \\ \vdots \\ \underline{f}_n \end{pmatrix} \wedge \begin{pmatrix} Q_1^{(1)} \\ \vdots \\ Q_n^{(1)} \end{pmatrix} \wedge \dots \wedge \begin{pmatrix} Q_1^{(p-1)} \\ \vdots \\ Q_n^{(p-1)} \end{pmatrix} \cdot \underline{v} \\ &= H_1. \end{aligned}$$

This completes the proof. $\square$

Some applications:

(1) In Trent and Zhang [7], the following applications and their references are considered:

- (a) $\mathcal{A} = H^\infty(\mathcal{D})$, $\mathcal{D} \subset \mathbb{C}$ is finitely connected,
- (b) $\mathcal{A} = H^\infty(D) \cap \mathcal{D}^2(D)$, where $\mathcal{D}^2(D)$ denotes Dirichlet space,
- (c) $\mathcal{A} = \mathcal{M}(\mathcal{D}^2(D))$,
- (d) $\mathcal{H}^p(D^n)$ corona versions for $1 < p < \infty$.

(2) Let $S \subset \partial D$ and let A_S denote $H^\infty(D) \cap C(D \cup S)$. By results of Stray and Rosay, the corona theorem holds (for a finite number of functions) in A_S (see Sasane [3]). Thus a matricial version follows. For the particular case when $S = \{1\}$, we may identify $A_{\{1\}}$ with $H^\infty(\Pi_+)$, where Π_+ denotes the right half-plane, $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$. This algebra arises naturally in control theory. The algebra $H^\infty(\Pi_+)$ satisfies an infinite corona theorem, so our results give a matricial version.

For the algebra A_S , these matricial results and more general ones (in the case that $\epsilon^2 \leq F(z)F(z)^*$ for all $z \in D$ and S is open) have independently been discovered in Sasane [4].

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