HAMILTON'S GRADIENT ESTIMATE FOR THE HEAT KERNEL ON COMPLETE MANIFOLDS

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ABSTRACT. In this paper we extend a gradient estimate of R. Hamilton for positive solutions to the heat equation on closed manifolds to bounded positive solutions on complete, non-compact manifolds with $Rc \geq -Kg$. We accomplish this extension via a maximum principle of L. Karp and P. Li and a Berstein-type estimate on the gradient of the solution. An application of our result, together with the bounds of P. Li and S.T. Yau, yields an estimate on the gradient of the heat kernel for complete manifolds with non-negative Ricci curvature that is sharp in the order of t for the heat kernel on $\mathbb{R}^n$.

1. INTRODUCTION

In [H], Richard Hamilton established the following estimate on the gradient of the logarithm of a positive solution to the heat equation.

Theorem (Hamilton). *Suppose $(\mathcal{M}^n, g)$ is a closed Riemannian manifold and u a positive solution to the heat equation on $\mathcal{M}^n$. If $M > 0$ and $K \geq 0$ are constants such that $Rc \geq -Kg$ and $u(x, t) \leq M$, then for all $x \in \mathcal{M}^n$ and $t > 0$, one has*

$$(1) \quad t |\nabla \log(u)|^2 \leq (1 + 2Kt) \log(M/u).$$

In this paper, we provide a proof that Hamilton's theorem also holds for complete, non-compact manifolds with Ricci curvature bounded below. Under the additional restriction of non-negative Ricci curvature, we then obtain, via the well-known bounds of Li and Yau [LY], the following estimate on the heat kernel. (Recall that on a complete, non-compact manifold, the heat kernel may be defined as the smallest positive fundamental solution to the heat equation.)

Theorem 1. *Suppose $\mathcal{M}^n$ is a complete, non-compact manifold with $Rc \geq 0$, and H its heat kernel. Then, for all $\delta > 0$, there exists a constant $C = C(n, \delta)$ such that*

$$(2) \quad |\nabla \log(H(x, y, t))|^2 \leq \frac{2}{t} \left(C + \frac{d(x, y)^2}{(4 - \delta)t} \right)$$

for all x, y in $\mathcal{M}^n$ and $t > 0$.

Theorem 1 is sharp in the order of t for the heat kernel on $\mathbb{R}^n$ and should be compared to the recent estimate of Souplet and Zhang in [SZ] which applies to the

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heat kernel on manifolds with $Rc(g) \geq -Kg$. In the special case $K = 0$, inequality (2) is comparable to their estimate at scales $d^2(x, y) \leq ct$ and offers an improvement at scales $t \ll d^2(x, y)$.

Additionally, such an estimate is required to prove that the integrand in the entropy functional $\mathcal{W}$ for the linear heat equation (cf. [N2]) is pointwise non-positive for the fundamental solution to the heat equation. The proof that the integrand in Perelman's $\mathcal{W}$ -functional is non-positive for fundamental solutions to the conjugate heat equation seems also to require a non-linear analog of this result (see [N]). Perhaps the approach detailed here could serve as a model for the proof of such an estimate.

In Section 2, we obtain a Bernstein-type estimate for bounded solutions to the heat equation which affords pointwise control on the product of the squared norm of the gradient by the elapsed time. The estimate is similar in form to those found by W.-X. Shi [S] in the Ricci Flow setting for derivatives of curvature. Such estimates have found considerable service in that setting and continue to have importance in work towards the verification of the claims of Perelman.

Note added in proof. The recent work of Ni [N2] and Zhang [Z] (see also Kuang-Zhang [KZ]) contains estimates very similar to our global estimate (Theorem 6) and its application to the heat kernel, as well as analogues for solutions coupled to a metric evolving by the Ricci flow, and their applications to Perelman's differential Harnack estimate [P].

2. A BERNSTEIN-TYPE ESTIMATE

Henceforth we shall assume that $(\mathcal{M}^n, g)$ is a smooth, complete, non-compact Riemannian manifold with Ricci curvature uniformly bounded below by $-K$, and for this section, suppose u is a smooth solution to the heat equation on some open $U \subset \mathcal{M}^n$ for $0 \leq t \leq T \leq \infty$, satisfying $|u| \leq M$. Our aim is to establish a preliminary estimate on $|\nabla u|^2$ so that the maximum principle of Ni and Tam [NT] may be applied to the quantity of interest in Hamilton's gradient estimate.

To do this, we employ a technique of W.-X. Shi [S] from the estimation of derivatives of curvature under the Ricci Flow (see also the treatment in the forthcoming book [CLN]), and define $F(x, t) = (4M^2 + u^2) |\nabla u|^2$ for $t > 0$. The evolution of F then possesses an advantageous $-F^2$ term, as we see below.

Lemma 2. *There exist positive constants C_1 and C_2 such that*

$$(3) \quad \frac{\partial F}{\partial t} \leq \Delta F + C_1 K F - \frac{C_2}{M^4} F^2.$$

Proof. We have

$$\left(\frac{\partial}{\partial t} - \Delta \right) |\nabla u|^2 = -2 |\nabla \nabla u|^2 - 2 Rc(\nabla u, \nabla u) \leq -2 |\nabla \nabla u|^2 + 2K |\nabla u|^2,$$

and

$$\left(\frac{\partial}{\partial t} - \Delta \right) u^2 = -2 |\nabla u|^2.$$

So

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) F &\leq (4M^2 + u^2)(2K|\nabla u|^2 - 2|\nabla\nabla u|^2) - 2|\nabla u|^4 \\ &\quad - 8u(\nabla\nabla u)(\nabla u, \nabla u) \\ &\leq -10u^2|\nabla\nabla u|^2 + 10M^2K|\nabla u|^2 - 2|\nabla u|^4 \\ &\quad - 8u(\nabla\nabla u)(\nabla u, \nabla u). \end{aligned}$$

Since

$$8|u||\nabla\nabla u||\nabla u|^2 \leq 10u^2|\nabla\nabla u|^2 + \frac{8}{5}|\nabla u|^4,$$

and $4M^2|\nabla u|^2 \leq F \leq 5M^2|\nabla u|^2$, we find

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta\right) F &\leq 10M^2K|\nabla u|^2 - \frac{2}{5}|\nabla u|^4 \\ &\leq \frac{5}{2}KF - \frac{2}{625M^4}F^2 \end{aligned}$$

as claimed. $\square$

Now, as in [LY], for any $p \in \mathcal{M}^n$ and $R > 0$ we may find a cut-off function $\eta(x) = \eta_{p,R}(x)$ equal to 1 on $B_p(R)$ and supported in $B_p(2R)$ satisfying the conditions

$$(4) \quad |\nabla \eta|^2 \leq \frac{C_3}{R^2} \eta$$

and

$$(5) \quad \Delta \eta \geq -\frac{C_3}{R^2} (1 + R\sqrt{K})$$

for some $C_3 = C_3(n) > 0$. Strictly speaking, the above estimates need only hold away from the cut-locus of p , however, for the purposes of applying the maximum principle to ηF , the well-known argument of Calabi [C] allows us to assume that they hold everywhere.

The main result of this section is the following local estimate:

Theorem 3. *Suppose u is a smooth solution to the heat equation satisfying $|u| \leq M$ on $B_p(2R) \times [0, T]$ for some $p \in \mathcal{M}^n$ and $M, R, T > 0$. Then there exists a constant $C_4 = C_4(K)$ such that*

$$(6) \quad t|\nabla u|^2 \leq C_4M^2 \left(1 + T \left(1 + \frac{1}{R^2}\right)\right)$$

holds on $B_p(R) \times [0, T]$.

Proof. Define F as in Lemma 2. On $\text{supp}(\eta) \times [0, T]$, we have, by Lemma 2 and equations (4) and (5),

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \Delta \right) (t\eta F) &= \eta F + t\eta \left(\frac{\partial}{\partial t} - \Delta \right) F - tF\Delta\eta - 2t \langle \nabla\eta, \nabla F \rangle \\ &= \left(\eta + 2t \frac{|\nabla\eta|^2}{\eta} - t\Delta\eta \right) F + t\eta \left(\frac{\partial}{\partial t} - \Delta \right) F \\ &\quad - 2 \left\langle \nabla(t\eta F), \frac{\nabla\eta}{\eta} \right\rangle \\ &\leq \left((1 + C_1 K t) \eta + 3t \frac{C_3}{R^2} (1 + R\sqrt{K}) \right) F - \frac{C_2}{M^4} t\eta F^2 \\ &\quad - 2 \left\langle \nabla(t\eta F), \frac{\nabla\eta}{\eta} \right\rangle. \end{aligned}$$

If ηF is not identically zero (i.e., if u is not constant on $\text{supp}(\eta)$), then $t\eta F$ attains a positive maximum at $(x_0, t_0) \in \mathcal{M}^n \times (0, T]$.

At this point,

$$\nabla(t\eta F) = 0$$

and

$$\left(\frac{\partial}{\partial t} - \Delta \right) (t\eta F) \geq 0,$$

so

$$\frac{C_2}{M^4} t_0 \eta F^2 \leq \left(1 + C_1 K T + 3T \frac{C_3}{R^2} (1 + R\sqrt{K}) \right) F.$$

Consequently, for any $(x, t) \in B_p(R) \times [0, T]$,

$$\begin{aligned} tF(x, t) &= tF(x, t)\eta(x) \\ &\leq t_0 F(x_0, t_0)\eta(x_0) \\ &\leq \frac{M^4}{C_2} \left(1 + C_1 K T + 3T \frac{C_3}{R^2} (1 + R\sqrt{K}) \right). \end{aligned}$$

But $|\nabla u|^2 \leq (1/4M^2)F$, and the claim follows. $\square$

Remark 4. If $Rc \geq 0$, the above proof shows

$$t|\nabla u|^2 \leq CM^2 \left(1 + \frac{T}{R^2} \right)$$

on $B_p(R) \times [0, T]$.

Sending $R \rightarrow \infty$ in the penultimate line of the above proof, we at once obtain

Corollary 5. *Suppose the solution u exists on all of $\mathcal{M}^n \times [0, T]$. Then there exists a constant C_5 such that*

$$(7) \quad t|\nabla u|^2 \leq C_5 M^2 (1 + KT)$$

on $\mathcal{M}^n \times [0, T]$.

3. PROOF OF THE MAIN THEOREM

Next, using the estimate of the previous section, we apply a maximum principle due originally to Karp and Li whose statement we found (in more generalized form) in a paper of Ni and Tam. The statement of their theorem in our (stationary metric) case is as follows. Here $f_+(x, t) := \max\{f(x, t), 0\}$.

Theorem (Karp-Li [KL] and Ni-Tam [NT], 1.2). *Suppose $(\mathcal{M}^n, g)$ is a complete Riemannian manifold and $f(x, t)$ a smooth function on $\mathcal{M}^n \times [0, T]$ such that $(\frac{\partial}{\partial t} - \Delta)f(x, t) \leq 0$ whenever $f(x, t) \leq 0$. Assume that*

$$(8) \quad \int_0^T \int_{\mathcal{M}^n} e^{-ar^2(x)} f_+^2(x, s) dV ds < \infty$$

for some $a > 0$, where $r(x)$ is the distance to x from some fixed $p \in \mathcal{M}^n$. If $f(x, 0) \leq 0$ for all $x \in \mathcal{M}^n$, then $f(x, t) \leq 0$ for all $(x, t) \in \mathcal{M}^n \times [0, T]$.

Hamilton's Theorem in the complete case reads as

Theorem 6. *Suppose $(\mathcal{M}^n, g)$ is a complete manifold with $Rc \geq -Kg$ for some $K \geq 0$. If $0 < u(x, t) \leq M$ is a solution to the heat equation on $\mathcal{M}^n \times [0, T]$ for $0 < T \leq \infty$, then*

$$t |\nabla \log u|^2 \leq (1 + 2Kt) \log(M/u).$$

Proof. Defining $u_\epsilon = u + \epsilon$ for $\epsilon > 0$, we obtain a solution satisfying $\epsilon < u_\epsilon \leq M + \epsilon := M_\epsilon$. Once the estimate has been proved for u_ϵ , the theorem will follow by letting $\epsilon \rightarrow 0$.

As in [H], the function

$$P(x, t) := \varphi(t) \frac{|\nabla u_\epsilon|^2}{u_\epsilon} - u_\epsilon \log\left(\frac{M_\epsilon}{u_\epsilon}\right),$$

where $\varphi(t) := t/(1 + 2Kt)$, satisfies $(\frac{\partial}{\partial t} - \Delta)P(x, t) \leq 0$ and

$$P(x, 0) = -u_\epsilon \log(M_\epsilon/u_\epsilon) \leq 0.$$

By our assumptions on u_ϵ , we also have

$$P_+(x, t) \leq \frac{1}{\epsilon} t |\nabla u_\epsilon|^2.$$

Thus using equation (7), for any $p \in \mathcal{M}^n$, and $R > 0$, we have

$$\begin{aligned} \int_0^T \int_{B_p(R)} e^{-r^2(x)} P_+^2(x, t) dV dt &\leq \frac{1}{\epsilon^2} \int_0^T \int_{B_p(R)} e^{-r^2(x)} (t |\nabla u_\epsilon|^2)^2 dV dt \\ &\leq \frac{C_5^2 M_\epsilon^4}{\epsilon^2} (1 + KT)^2 \int_0^T \int_{\mathcal{M}^n} e^{-r^2(x)} dV dt. \end{aligned}$$

Since we assume that $Rc \geq -Kg$, it follows from the Bishop volume comparison theorem that the rightmost integral in the above inequality is finite.

Hence,

$$\int_0^T \int_{\mathcal{M}^n} e^{-r^2(x)} P_+^2(x, t) dV dt \leq \liminf_{R \rightarrow \infty} \int_0^T \int_{B_p(R)} e^{-r^2(x)} P_+^2(x, t) dV dt < \infty,$$

and we conclude that $P(x, t) \leq 0$ for all $t \leq T$. $\square$

Proof of Theorem 1. Let $H(x, y, t)$ denote the heat kernel of $(\mathcal{M}^n, g)$. For any $t > 0$ and $y \in \mathcal{M}^n$, set $u(x, s) := H(x, y, s + t/2)$. Then u is a smooth, positive solution to the heat equation on $[0, \infty)$. By Corollary 3.1 and Theorem 4.1 of [LY], for all $\delta > 0$, there is a constant $C_6 = C_6(\delta) > 0$ such that u satisfies

$$(9) \quad C_6^{-1} V \left(\sqrt{s + t/2} \right)^{-1} e^{\frac{-d^2(x, y)}{(4-\delta)(s+t/2)}} \leq u(x, s) \leq C_6 V \left(\sqrt{s + t/2} \right)^{-1}$$

for all $x, y \in \mathcal{M}^n$, and $s \geq 0$, where $V(\sqrt{s + t/2}) := \text{Vol}(B_y(\sqrt{s + t/2}))$.

Defining $M = C_6 V \left(\sqrt{t/2} \right)^{-1}$, the left inequality in (9) implies $u \leq M$ for all x and s . Moreover, since we assume $Rc \geq 0$, there exists a positive constant $C_7 = C_7(n)$ such that for all $0 \leq s \leq t/2$

$$V \left(\sqrt{t/2 + s} \right) \leq V \left(\sqrt{t} \right) \leq C_7 V \left(\sqrt{t/2} \right).$$

Thus, by the right-hand inequality in (9) and Theorem 6, we have

$$s |\nabla \log u|^2 \leq \log \left(\frac{M}{u} \right) \leq \left(\log(C_6^2 C_7) + \frac{d^2(x, y)}{(4 - \delta)(s + t/2)} \right)$$

on $\mathcal{M}^n \times [0, t/2]$.

Setting $C = \log(C_6^2(\delta)C_7(n))$ and evaluating at $s = t/2$, we conclude that

$$(t/2) |\nabla \log H|^2(x, y, t) = (t/2) |\nabla u|^2(x, t/2) \leq \left(C + \frac{d^2(x, y)}{(4 - \delta)t} \right)$$

for all $x, y \in \mathcal{M}^n$ and $t > 0$. $\square$

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