PATTERN FORMATION (II): THE TURING INSTABILITY

YAN GUO AND HYUNG JU HWANG

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ABSTRACT. We consider the classical Turing instability in a reaction-diffusion system as the second part of our study on pattern formation. We prove that nonlinear dynamics of a general perturbation of the Turing instability is determined by the finite number of linear growing modes over a time scale of $\ln \frac{1}{\delta}$, where δ is the strength of the initial perturbation.

1. GROWING MODES IN A REACTION-DIFFUSION SYSTEM

In this section we summarize the classical linear Turing instability criterion for a reaction-diffusion system. Consider a reaction-diffusion system of 2-species as

$$(1.1) \quad \begin{aligned} \frac{\partial U}{\partial t} &= \nabla \cdot (D_1(U, V) \nabla U) + f(U, V), \\ \frac{\partial V}{\partial t} &= \nabla \cdot (D_2(U, V) \nabla V) + g(U, V), \end{aligned}$$

where $U(\mathbf{x}, t), V(\mathbf{x}, t)$ are concentration for species, D_1, D_2 diffusion coefficients, and f, g reaction terms.

In this paper we consider a d -dimensional box $\mathbb{T}^d = (0, \pi)^d$, $d = 1, 2, 3$, with Neumann boundary conditions for U and V , i.e.,

$$(1.2) \quad \frac{\partial U}{\partial x_i} = \frac{\partial V}{\partial x_i} = 0 \quad \text{at } x_i = 0, \pi, \quad \text{for } 1 \leq i \leq d.$$

Homogeneous steady state $U = \bar{U}, V = \bar{V}$ forms a steady state provided

$$(1.3) \quad 0 = f(\bar{U}, \bar{V}) = g(\bar{U}, \bar{V}).$$

In this article, we study the nonlinear evolution of a perturbation

$$u(x, t) = U(x, t) - \bar{U}, \quad v(x, t) = V(x, t) - \bar{V}$$

around $[\bar{U}, \bar{V}]$, which satisfies the equivalent reaction-diffusion system:

$$(1.4) \quad \frac{\partial u}{\partial t} = \nabla \cdot (D_1(u + \bar{U}, v + \bar{V}) \nabla u) + f(u + \bar{U}, v + \bar{V}),$$

$$(1.5) \quad \frac{\partial v}{\partial t} = \nabla \cdot (D_2(u + \bar{U}, v + \bar{V}) \nabla v) + g(u + \bar{U}, v + \bar{V}).$$

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The corresponding linearized system then takes the form

$$(1.6) \quad u_t = \bar{D}_1 \nabla^2 u + \bar{f}_u u + \bar{f}_v v,$$

$$(1.7) \quad v_t = \bar{D}_2 \nabla^2 v + \bar{g}_u u + \bar{g}_v v,$$

where $\bar{D}_1 = D_1(\bar{U}, \bar{V}) > 0$, $\bar{D}_2 = D_2(\bar{U}, \bar{V}) > 0$, $\bar{f}_u = \frac{\partial f}{\partial u}(\bar{U}, \bar{V})$, $\bar{f}_v = \frac{\partial f}{\partial v}(\bar{U}, \bar{V})$, $\bar{g}_u = \frac{\partial g}{\partial u}(\bar{U}, \bar{V})$, and $\bar{g}_v = \frac{\partial g}{\partial v}(\bar{U}, \bar{V})$.

We use $[\cdot, \cdot]$ to denote a column vector, and let

$$\mathbf{w}(x, t) \equiv [u(x, t), v(x, t)], \quad \bar{\mathbf{W}} = [\bar{U}, \bar{V}].$$

Then the system can be written in a matrix form:

$$(1.8) \quad \begin{aligned} \frac{\partial \mathbf{w}}{\partial t} &= \nabla \cdot (D \nabla \mathbf{w}) + \mathbf{F} \\ &= (\bar{D} \nabla^2 \mathbf{w} + A \mathbf{w}) + (\nabla \cdot (D \nabla \mathbf{w}) - \bar{D} \nabla^2 \mathbf{w}) + \mathbf{F} - A \mathbf{w} \\ &\equiv \mathcal{L}(\mathbf{w}) + \mathcal{N}(\mathbf{w}), \end{aligned}$$

where

$$\begin{aligned} D &= \begin{pmatrix} D_1(\mathbf{w} + \bar{\mathbf{W}}) & 0 \\ 0 & D_2(\mathbf{w} + \bar{\mathbf{W}}) \end{pmatrix}, \quad \bar{D} = \begin{pmatrix} \bar{D}_1 & 0 \\ 0 & \bar{D}_2 \end{pmatrix}, \\ \mathbf{F} &= \begin{pmatrix} f(\mathbf{w} + \bar{\mathbf{W}}) \\ g(\mathbf{w} + \bar{\mathbf{W}}) \end{pmatrix}, \quad A = \begin{pmatrix} \bar{f}_u & \bar{f}_v \\ \bar{g}_u & \bar{g}_v \end{pmatrix}. \end{aligned}$$

Let $\mathbf{q} = (q_1, \dots, q_d) \in \Omega = (\mathbb{N} \cup \{0\})^d$ and let

$$e_{\mathbf{q}}(x) \equiv \prod_{i=1}^d \cos(q_i x_i),$$

where $\mathbf{q} \in \Omega$. Then $\{e_{\mathbf{q}}(x)\}_{\mathbf{q} \in \Omega}$ forms a basis of the space of functions in $\mathbb{T}^d$ that satisfy Neumann boundary condition (1.2).

We assume that $D_1, D_2, f, g \in C^2$ near $\bar{W}$, so that there exists, where $\eta > 0$,

$$(1.9) \quad C_\eta \equiv \max_{\|w\|_\infty \leq \eta} \left\{ \sum_{i=1}^2 \|D_i(\bar{W} + w)\|_{C^2} + \|f(\bar{W} + w)\|_{C^2} + \|g(\bar{W} + w)\|_{C^2} \right\} < \infty.$$

We look for a normal mode to the linear reaction-diffusion system (1.6) and (1.7) of the following form:

$$(1.10) \quad \mathbf{w}(x, t) = \mathbf{r}_{\mathbf{q}} \exp(\lambda_{\mathbf{q}} t) e_{\mathbf{q}}(x),$$

where $\mathbf{r}_{\mathbf{q}}$ is a vector depending on $\mathbf{q}$.

We substitute (1.10) into (1.6)-(1.7) to get

$$\lambda_{\mathbf{q}} \mathbf{r}_{\mathbf{q}} = \begin{pmatrix} \bar{f}_u - \bar{D}_1 q^2 & \bar{f}_v \\ \bar{g}_u & \bar{g}_v - \bar{D}_2 q^2 \end{pmatrix} \mathbf{r}_{\mathbf{q}} \equiv L \mathbf{r}_{\mathbf{q}},$$

where $q^2 = \sum_{i=1}^d q_i^2$. A nontrivial normal mode can be obtained by setting

$$\det \begin{pmatrix} \lambda_{\mathbf{q}} - \bar{f}_u + \bar{D}_1 q^2 & -\bar{f}_v \\ -\bar{g}_u & \lambda_{\mathbf{q}} - \bar{g}_v + \bar{D}_2 q^2 \end{pmatrix} = 0.$$

This leads to the following dispersion formula for $\lambda_{\mathbf{q}}$:

$$(1.11) \quad \lambda_{\mathbf{q}}^2 + \{-\bar{f}_u + \bar{D}_1 q^2 - \bar{g}_v + \bar{D}_2 q^2\} \lambda_{\mathbf{q}} + \{(\bar{f}_u - \bar{D}_1 q^2)(\bar{g}_v - \bar{D}_2 q^2) - \bar{f}_v \bar{g}_u\} = 0.$$

We assume first that without diffusion, the $\lambda_{\mathbf{q}}$ has a negative real part (stable):

$$(1.12) \quad \operatorname{tr} A = \bar{f}_u + \bar{g}_v < 0, \quad \det A = \bar{f}_u \bar{g}_v - \bar{f}_v \bar{g}_u > 0.$$

On the other hand, in the presence of diffusion, we assume the following diffusion-driven (linear) instability criterion by requiring there exists a q such that

$$(1.13) \quad (\bar{f}_u - \bar{D}_1 q^2) (\bar{g}_v - \bar{D}_2 q^2) - \bar{f}_v \bar{g}_u < 0,$$

which ensures that (1.11) has exactly one positive root $\lambda_{\mathbf{q}}$.

Remark 1. To guarantee existence of q satisfying (1.13), we have its discriminant to be positive:

$$(1.14) \quad (\bar{f}_u \bar{D}_2 + \bar{g}_v \bar{D}_1) > 2\sqrt{\bar{D}_1 \bar{D}_2} \det A > 0,$$

which means the range of inhibition $\sqrt{\bar{D}_2/|\bar{g}_v|}$ is larger than the range of activation $\sqrt{\bar{D}_1/|\bar{f}_u|}$. From (1.12) and (1.14), it follows that

$$(1.15) \quad \bar{f}_u \bar{g}_v < 0, \quad \text{and} \quad \bar{f}_v \bar{g}_u < 0,$$

and we have only two cases for A :

$$A = \begin{pmatrix} + & - \\ + & - \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} + & + \\ - & - \end{pmatrix},$$

where the former case is called activator-inhibitor (or predator-prey) and the latter positive feedback. It also follows from (1.12)-(1.14) that

$$\bar{D}_1 \neq \bar{D}_2.$$

For given $\mathbf{q} \in \Omega$, we denote the corresponding eigenvalues by $\lambda_{\pm}(\mathbf{q})$ and eigenvectors by $\mathbf{r}_{\pm}(\mathbf{q})$. We split it into three cases for the linear analysis:

(1) The generic case where we have two independent real eigenvectors and we denote

$$\Omega_{\text{generic}} \equiv \{\mathbf{q} \in \Omega \text{ such that } \mathbf{r}_+(\mathbf{q}) \neq \mathbf{r}_-(\mathbf{q})\}.$$

By an elementary computation of the discriminant, we have, for all q but only finitely many small q ,

$$(\bar{D}_1 - \bar{D}_2)^2 q^4 - \operatorname{tr} A (\bar{D}_1 + \bar{D}_2) q^2 + 4(\bar{f}_u \bar{D}_2 + \bar{g}_v \bar{D}_1) q^2 + (\operatorname{tr} A)^2 - 4 \det A > 0,$$

in which case we have two distinct real roots such that

$$\lambda_-(\mathbf{q}) < \lambda_+(\mathbf{q}).$$

Since $\bar{f}_v \neq 0$ in (1.15), the corresponding (linearly independent) eigenvectors $\mathbf{r}_-(\mathbf{q})$ and $\mathbf{r}_+(\mathbf{q})$ are given by

$$(1.16) \quad \mathbf{r}_{\pm}(\mathbf{q}) = \left[1, \frac{\lambda_{\pm}(\mathbf{q}) - \bar{f}_u + \bar{D}_1 q^2}{\bar{f}_v} \right].$$

It is easy to see from (1.13) that there exist only finitely many $\mathbf{q}$ such that $\lambda_+(\mathbf{q}) > 0$. We therefore denote the largest eigenvalue by $\lambda_{\max} > 0$ and define

$$\Omega_{\max} \equiv \{\mathbf{q} \in \Omega \text{ such that } \lambda_+(\mathbf{q}) = \lambda_{\max}\}.$$

We also denote $\nu > 0$ to be the gap between $\lambda_{\max}$ and the rest. Moreover, there is one q^2 (possibly two) having $\lambda_{\mathbf{q}}^+(q^2) = \lambda_{\max}$ when we regard $\lambda_{\mathbf{q}}^+$ as a function of q^2 .

(2) The defective case where we have the repeated real eigenvalues and eigenvectors:

Note that there may be possibly one q^2 (so finitely many $\mathbf{q}$) such that $\lambda_+(\mathbf{q}) = \lambda_-(\mathbf{q}) \equiv \lambda(\mathbf{q}) = \{\bar{f}_u + \bar{g}_v - (\bar{D}_1 + \bar{D}_2) q^2\}/2 < 0$ and $\mathbf{r}_+(\mathbf{q}) = \mathbf{r}_-(\mathbf{q}) \equiv \mathbf{r}(\mathbf{q})$, and we denote

$$\Omega_{\text{defective}} \equiv \{\mathbf{q} \in \Omega \text{ such that } \mathbf{r}_+(\mathbf{q}) = \mathbf{r}_-(\mathbf{q})\}.$$

In this case we find another independent vector

$$\mathbf{r}'(\mathbf{q}) = [0, \frac{1}{f_v}]$$

satisfying $(A - q^2 \bar{D} - \lambda(\mathbf{q})I) \mathbf{r}'(\mathbf{q}) = \mathbf{r}(\mathbf{q})$.

(3) The complex case where we have complex eigenvalues for q small and we denote it by $\Omega_{\text{complex}} \equiv \Omega - (\Omega_{\text{generic}} \cup \Omega_{\text{defective}})$. For $\mathbf{q} \in \Omega_{\text{complex}}$, we denote $\lambda_+(\mathbf{q}) \equiv \text{Re } \lambda(\mathbf{q}) + i \text{Im } \lambda(\mathbf{q})$ and $\mathbf{r}_+(\mathbf{q}) \equiv \text{Re } \mathbf{r}(\mathbf{q}) + i \text{Im } \mathbf{r}(\mathbf{q})$. Then we have $\lambda_-(\mathbf{q}) \equiv \text{Re } \lambda(\mathbf{q}) - i \text{Im } \lambda(\mathbf{q})$ and $\mathbf{r}_-(\mathbf{q}) \equiv \text{Re } \mathbf{r}(\mathbf{q}) - i \text{Im } \mathbf{r}(\mathbf{q})$.

Notice that $\text{Re } \mathbf{r}(\mathbf{q})$ and $\text{Im } \mathbf{r}(\mathbf{q})$ are linearly independent vectors.

Given any initial perturbation $\mathbf{w}(\mathbf{x}, 0)$, we can expand it as

$$\begin{aligned} \mathbf{w}(\mathbf{x}, 0) &= \sum_{\mathbf{q} \in \Omega} \mathbf{w}_{\mathbf{q}} e_{\mathbf{q}}(x) = \sum_{\mathbf{q} \in \Omega_{\text{generic}}} \{w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q})\} e_{\mathbf{q}}(x) \\ &+ \sum_{\mathbf{q} \in \Omega_{\text{defective}}} \{w_{\mathbf{q}} \mathbf{r}(\mathbf{q}) + w'_{\mathbf{q}} \mathbf{r}'(\mathbf{q})\} e_{\mathbf{q}}(x) \\ &+ \sum_{\mathbf{q} \in \Omega_{\text{complex}}} \{w_{\mathbf{q}}^{\text{Re}} \text{Re } \mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}^{\text{Im}} \text{Im } \mathbf{r}(\mathbf{q})\} e_{\mathbf{q}}(x), \end{aligned}$$

so that

$$\begin{aligned} (1.17) \quad \mathbf{w}_{\mathbf{q}} &= w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) \quad \text{for } \mathbf{q} \in \Omega_{\text{generic}}, \\ \mathbf{w}_{\mathbf{q}} &= w_{\mathbf{q}} \mathbf{r}(\mathbf{q}) + w'_{\mathbf{q}} \mathbf{r}'(\mathbf{q}) \quad \text{for } \mathbf{q} \in \Omega_{\text{defective}}, \\ \mathbf{w}_{\mathbf{q}} &= w_{\mathbf{q}}^{\text{Re}} \text{Re } \mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}^{\text{Im}} \text{Im } \mathbf{r}(\mathbf{q}) \quad \text{for } \mathbf{q} \in \Omega_{\text{complex}}. \end{aligned}$$

The unique solution $\mathbf{w}(x, t) = [u(x, t), v(x, t)]$ to (1.6)-(1.7) is given by

$$\begin{aligned} (1.18) \quad \mathbf{w}(x, t) &= \sum_{\mathbf{q} \in \Omega_{\text{generic}}} \{w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) \exp(\lambda_{\mathbf{q}}^- t) + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) \exp(\lambda_{\mathbf{q}}^+ t)\} e_{\mathbf{q}}(x) \\ &+ \sum_{\mathbf{q} \in \Omega_{\text{defective}}} \{(w_{\mathbf{q}} \mathbf{r}(\mathbf{q}) + w'_{\mathbf{q}} \mathbf{r}'(\mathbf{q})) + w'_{\mathbf{q}} \mathbf{r}(\mathbf{q}) t\} \exp(\lambda_{\mathbf{q}} t) e_{\mathbf{q}}(x) \\ &+ \sum_{\mathbf{q} \in \Omega_{\text{complex}}} \{w_{\mathbf{q}}^{\text{Re}} (\text{Re } \mathbf{r}(\mathbf{q}) \cos[(\text{Im } \lambda_{\mathbf{q}}) t] - \text{Im } \mathbf{r}(\mathbf{q}) \sin[(\text{Im } \lambda_{\mathbf{q}}) t]) \\ &+ w_{\mathbf{q}}^{\text{Im}} (\text{Re } \mathbf{r}(\mathbf{q}) \sin[(\text{Im } \lambda_{\mathbf{q}}) t] + \text{Im } \mathbf{r}(\mathbf{q}) \cos[(\text{Im } \lambda_{\mathbf{q}}) t])\} \exp[(\text{Re } \lambda_{\mathbf{q}}) t] e_{\mathbf{q}}(x) \\ &\equiv e^{\mathcal{L}t} \mathbf{w}(x, 0). \end{aligned}$$

For any $\mathbf{u}(\cdot, t) \in [L^2(\mathbb{T}^d)]^2$, we denote $\|\mathbf{u}(\cdot, t)\| \equiv \|\mathbf{u}(\cdot, t)\|_{L^2}$. Our main result of this section is

Lemma 1. *Assume that the instability criterion (1.13) is valid. Suppose*

$$\mathbf{w}(x, t) = [u(x, t), v(x, t)] \equiv e^{\mathcal{L}t} \mathbf{w}(x, 0)$$

as in (1.18) is a solution to the linearized reaction-diffusion system (1.6)-(1.7) with initial condition $\mathbf{w}(\mathbf{x}, 0)$. Then there exists a constant $C_1 \geq 1$ depending on $\bar{U}, \bar{V}, \bar{D}_1, \bar{D}_2, A$ such that

$$\|\mathbf{w}(\cdot, t)\| \leq C_1 \exp(\lambda_{\max} t) \|\mathbf{w}(\cdot, 0)\|,$$

for all $t \geq 0$.

Proof. We first notice that from the quadratic formula for (1.11), for q large,

$$|\det[\mathbf{r}_-(\mathbf{q}), \mathbf{r}_+(\mathbf{q})]| = \frac{\lambda_{\mathbf{q}}^+ - \lambda_{\mathbf{q}}^-}{|\bar{f}_v|} \geq c \frac{|\bar{D}_1 - \bar{D}_2|}{|\bar{f}_v|} q^2.$$

Thus solving (1.17) yields, due to $\bar{D}_1 \neq \bar{D}_2$,

$$\begin{aligned} |w_{\mathbf{q}}^{\pm}| &\leq \frac{1}{|\det[\mathbf{r}_-(\mathbf{q}), \mathbf{r}_+(\mathbf{q})]|} |\mathbf{r}_{\pm}(\mathbf{q})| \times |\mathbf{w}_{\mathbf{q}}| \\ &\leq C |\mathbf{w}_{\mathbf{q}}|. \end{aligned}$$

We now treat the defective wave numbers. Since $\lambda_{\mathbf{q}} < 0$, for $q \in \Omega_{\text{defective}}$, we have

$$t \exp(\lambda_{\mathbf{q}} t) \leq C.$$

Thus we deduce the lemma on the linear growth rate by the formula (1.18). $\square$

2. MAIN RESULT

Let θ be a small fixed constant, and $\lambda_{\max}$ be the dominant eigenvalue which is the maximal growth rate. We also denote the gap between the largest growth rate $\lambda_{\max}$ and the rest by $\nu > 0$. Then for $\delta > 0$ arbitrarily small, we define the escape time T^{δ} by

$$(2.1) \quad \theta = \delta \exp(\lambda_{\max} T^{\delta}),$$

or equivalently

$$T^{\delta} = \frac{1}{\lambda_{\max}} \ln \frac{\theta}{\delta}.$$

Our main theorem is

Theorem 1. Assume that the set of $q^2 = \sum_{i=1}^d q_i^2$ satisfying instability criterion (1.13) is not empty for given parameters $\bar{U}, \bar{V}, \bar{D}_1, \bar{D}_2, A$. Let

$$\begin{aligned} \mathbf{w}_0(x) &= \sum_{\mathbf{q} \in \Omega} \{w_{\mathbf{q}}^- \mathbf{r}_-(\mathbf{q}) + w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q})\} e_{\mathbf{q}}(x) \\ &\quad + \sum_{\mathbf{q} \in \Omega_{\text{defective}}} \{w_{\mathbf{q}} \mathbf{r}(\mathbf{q}) + w'_{\mathbf{q}} \mathbf{r}'(\mathbf{q})\} e_{\mathbf{q}}(x) \\ &\quad + \sum_{\mathbf{q} \in \Omega_{\text{complex}}} \{w_{\mathbf{q}}^{\text{Re}} \text{Re } \mathbf{r}(\mathbf{q}) + w_{\mathbf{q}}^{\text{Im}} \text{Im } \mathbf{r}(\mathbf{q})\} e_{\mathbf{q}}(x) \end{aligned}$$

$\in H^2$ such that $\|\mathbf{w}_0\| = 1$. Then there exist constants $\delta_0 > 0$, $C > 0$, and $\theta > 0$, depending on $\bar{U}, \bar{V}, \bar{D}_1, \bar{D}_2, f, g$, such that for all $0 < \delta \leq \delta_0$, if the initial perturbation of the steady state $[\bar{U}, \bar{V}]$ in (1.3) is

$$\mathbf{w}^{\delta}(\mathbf{x}, 0) = \delta \mathbf{w}_0,$$

then its nonlinear evolution $\mathbf{w}^\delta(t, x)$ satisfies

$$(2.2) \quad \|\mathbf{w}^\delta(t, x) - \delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}(x)\| \leq C\{e^{-\nu t} + \delta \|w_0\|_{H^2}^2 + \delta e^{\lambda_{\max} t}\} \delta e^{\lambda_{\max} t}$$

for $0 \leq t \leq T^\delta$, and $\nu > 0$ is the gap between $\lambda_{\max}$ and the rest of $\operatorname{Re} \lambda_{\mathbf{q}}$ in (1.11).

We notice that for $0 \leq t \leq T^\delta$, $\delta e^{\lambda_{\max} t} \leq \theta$, is sufficiently small. The initial profile $\mathbf{w}_0$ is any H^2 function. In particular, as long as $w_{\mathbf{q}_0}^+ \neq 0$ for at least one $\mathbf{q}_0 \in \Omega_{\max}$, which is generic for a general H^2 perturbation, the part of its fastest growing modes satisfies

$$\|\delta e^{\lambda_{\max} t} \sum_{\mathbf{q} \in \Omega_{\max}} w_{\mathbf{q}}^+ \mathbf{r}_+(\mathbf{q}) e_{\mathbf{q}}\| \geq \delta e^{\lambda_{\max} t} |w_{\mathbf{q}_0}^+| |\mathbf{r}_+(\mathbf{q}_0)|,$$

which has the dominant leading order of $\delta e^{\lambda_{\max} t}$. Our estimate (2.2) implies that the dynamics of a general perturbation can be characterized by such linear dynamics over a long time period of $\varepsilon T^\delta \leq t \leq T^\delta$, for any fixed constant $\varepsilon > 0$. In particular, choose a fixed $\mathbf{q}_0 \in \Omega_{\max}$ and let

$$w_0(x) = \frac{\mathbf{r}_+(\mathbf{q}_0)}{|\mathbf{r}_+(\mathbf{q}_0)|} e_{\mathbf{q}_0}(x).$$

Then If $t = T^\delta$,

$$\left\| \mathbf{w}^\delta(t, \cdot) - \delta e^{\lambda_{\max} T^\delta} \frac{\mathbf{r}_+(\mathbf{q}_0)}{|\mathbf{r}_+(\mathbf{q}_0)|} e_{\mathbf{q}_0}(\cdot) \right\| \leq C\{\delta^{\nu/\lambda_{\max}} + \theta^2\},$$

hence

$$\|\mathbf{w}^\delta(t, \cdot)\| \geq \theta - C\{\delta^{\nu/\lambda_{\max}} + \theta^2\} \geq \theta/2 > 0,$$

which implies nonlinear instability as $\delta \rightarrow 0$. The instability occurs before the possible blow-up time.

Reaction-diffusion systems are often employed to study chemical and biological pattern formation and have received much attention from scientists [3], [4], [13], [12], [16], since the pioneering work of Turing [17] in 1952. This symmetry breaking instability is called diffusion-driven instability since the presence of diffusion and the difference of diffusion coefficients are essential for the instability mechanism leading to nonuniform patterns in the systems. The global aspect of the Turing instability was studied in [8], [14] which provided a better understanding of the mechanism in diffusion and cross-diffusion. After some experimental results such as in [2], [11], [15], more extensive and serious works began towards this Turing-like pattern formation across many fields of study. However, most of the works, including Turing himself, dealt with linear instability, and the nonlinear effect of the reaction-diffusion systems in large time has been in question to date from a mathematical viewpoint. In general, nonlinear instability is treated with great delicacy for nondissipative systems as in [1], [5], [9], [10]. Starting in [7], nonlinear instability was established from linear instability for dissipative systems [6], based on a bootstrap lemma. Notice in our theorem that each initial perturbation can be drastically different from another, which gives rise to the richness of the pattern; on the other hand, the finite number maximal growing modes determines the common characteristics of the pattern, over the time scale of $\ln \frac{1}{\delta}$. Our result can be interpreted as a mathematical description of early pattern formation, and it has the following several advantages over an earlier *different* result along this direction

[18]: First of all, the reaction-diffusion system considered here is *not* scaled. Second, our initial perturbation is more general; it need not be close to the space of finite number of maximal growing model. Third, a precise estimate of the time scale ($\ln \frac{1}{\delta}$) for pattern formation is given here, without an *a-priori* assumption for the smallness of the perturbation later in time as in [18]. Lastly, our proof is based on Guo-Strauss' bootstrap argument, which is much simpler. On the other hand, we believe that our method should apply to more general domains considered in [18] as well.

3. BOOTSTRAP LEMMA

We state existence of local-in-time solutions for (1.4)-(1.5).

Lemma 2 (Local existence). *For $s \geq 1$ ($d = 1$) and $s \geq 2$ ($d = 2, 3$), there exist a $T > 0$ and a constant C depending on $\bar{U}, \bar{V}, D_1, D_2, f, g$ such that*

$$\|\mathbf{w}(t)\|_{H^s} \leq C \|\mathbf{w}(0)\|_{H^s}.$$

We now derive the following energy estimates for a d -dimensional reaction-diffusion system with $d = 1, 2, 3$.

Lemma 3. *Suppose that $[u(x, t), v(x, t)]$ is a solution to the full system (1.4)-(1.5). Then for $\|w(t)\|_{H^2} \leq \eta$,*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \sum_{|\partial|=2} \int_{\mathbb{T}^d} \{|\partial u|^2 + |\partial v|^2\} d\mathbf{x} \\ & + \sum_{|\partial|=2} \int_{\mathbb{T}^d} \left\{ \frac{\bar{D}_1}{2} |\nabla \partial u|^2 + \bar{D}_2 |\nabla \partial v|^2 \right\} d\mathbf{x} + \frac{|\bar{g}_v|}{2} \sum_{|\partial|=2} \int_{\mathbb{T}^d} |\partial v|^2 \\ & \leq C_0 C_1 \|\mathbf{w}\|_{H^2} \|\nabla^3 \mathbf{w}\|^2 + C_2 \|u\|^2, \end{aligned}$$

where C_0 is the universal constant while $C_1 = C_0 C_\eta (1 + \eta)$ and

$$C_2 = \frac{\left(\frac{(\bar{f}_v + \bar{g}_u)^2}{2|\bar{g}_v|} + \bar{f}_u \right)^3}{\bar{D}_1^2}.$$

Proof. We first notice that the reaction-diffusion system preserves the evenness of the solution $\mathbf{w}(x, t)$, i.e., if $\mathbf{w}(x, t)$ is a solution, then $\mathbf{w}(-x_i, t)$ is also a solution. We can regard the Neumann problem as a special case with evenness of the periodic problem by a standard way of even extension $\mathbf{w}(x, t)$ with respect to one of the x_i . For this reason we may assume periodicity at the boundary of the extended $2\mathbf{T}^3 \equiv (-\pi, \pi)^d$. Since now there is no contribution from the boundaries, we can take the second order ∂ -derivative of (1.8) to get

$$(3.1) \quad \frac{1}{2} \frac{d}{dt} \int_{2\mathbb{T}^d} |\partial \mathbf{w}|^2 = \int_{2\mathbb{T}^d} \partial \mathbf{w}^T \partial \mathcal{L}(\mathbf{w}) + \int_{2\mathbb{T}^d} \partial \mathbf{w}^T \partial \mathcal{N}(\mathbf{w}).$$

We first treat the last nonlinear term:

$$\begin{aligned}
& - \int_{2\mathbb{T}^d} \{\nabla \partial \mathbf{w}\}^T [\partial \{D(\mathbf{w} + \bar{\mathbf{W}}) \nabla \mathbf{w}\} + \bar{D} \nabla \partial \mathbf{w}] + \{\nabla \partial \mathbf{w}\}^T \partial (\mathbf{F} - A \mathbf{w}) \\
& \leq C \|D(\mathbf{w} + \bar{\mathbf{W}}) - \bar{D}\|_\infty \|\nabla \partial \mathbf{w}\|^2 + C \|(\nabla D)(\mathbf{w} + \bar{\mathbf{W}})\|_\infty \|\nabla \mathbf{w}\|_\infty \|\partial \mathbf{w}\| \|\nabla \partial \mathbf{w}\| \\
& \quad + C \|(\partial D)(\mathbf{w} + \bar{\mathbf{W}})\|_\infty \|\nabla \mathbf{w}\|_{L^4}^2 \|\nabla \mathbf{w}\|_\infty \|\nabla \partial \mathbf{w}\| \\
& \quad + C \|(\partial \mathbf{F})(\mathbf{w} + \bar{\mathbf{W}})\|_\infty \|\nabla \mathbf{w}\|_\infty \|\nabla \mathbf{w}\| \|\nabla \partial \mathbf{w}\| \\
& \quad + C \|\nabla \mathbf{F}(\mathbf{w} + \bar{\mathbf{W}}) - A\|_\infty \|\partial \mathbf{w}\| \|\nabla \partial \mathbf{w}\|.
\end{aligned}$$

We apply the following Sobolev imbedding to control the L^∞ norms:

$$(3.2) \quad \|g\|_{L^\infty(2\mathbb{T}^d)} \leq C_0 \|g\|_{H^2(2\mathbb{T}^d)},$$

for $d \leq 3$. Moreover, from the periodic boundary conditions,

$$\int_{2\mathbb{T}^d} \nabla u = \int_{2\mathbb{T}^d} \nabla v = 0,$$

we also use the Poincare inequality

$$(3.3) \quad \|g\| \leq \|g\|_{L^4(2\mathbb{T}^d)} \leq C_0 \|\nabla g\| \quad \text{if } d \leq 3$$

to further get

$$\|\nabla \mathbf{w}\|_\infty \leq C_0 \|\nabla \mathbf{w}\|_{H^2} \leq C_0 \sum_{|\partial|=2} \|\partial \nabla \mathbf{w}\|,$$

where C_0 is a universal constant. From (1.9) and the assumption $\|\mathbf{w}\|_{H^2} \leq \eta$, the last nonlinear term in (3.1) is bounded by

$$C_0 C_\eta (1 + \eta) \|\mathbf{w}\|_{H^2} \|\nabla \partial \mathbf{w}\|^2.$$

We now estimate the second linear term in (3.1):

$$\begin{aligned}
& - \int_{2\mathbb{T}^d} \{\bar{D}_1 |\nabla \partial u|^2 + \bar{D}_2 |\nabla \partial v|^2\} + \bar{g}_v \int_{2\mathbb{T}^d} |\partial v|^2 \\
& + (\bar{f}_v + \bar{g}_u) \int_{2\mathbb{T}^d} \partial u \partial v + \bar{f}_u \int_{2\mathbb{T}^d} |\partial u|^2.
\end{aligned}$$

The last two terms are bounded by

$$\begin{aligned}
& (\bar{f}_v + \bar{g}_u) \int_{2\mathbb{T}^d} \partial u \partial v + \bar{f}_u \int_{2\mathbb{T}^d} |\partial u|^2 \\
& \leq \frac{|\bar{g}_v|}{2} \int_{2\mathbb{T}^d} |\partial v|^2 + \left\{ \frac{(\bar{f}_v + \bar{g}_u)^2}{2|\bar{g}_v|} + \bar{f}_u \right\} \int_{2\mathbb{T}^d} |\partial u|^2.
\end{aligned}$$

Thus we can bound the linear term in (3.1) by ($\bar{g}_v < 0$):

$$\begin{aligned}
& - \int_{2\mathbb{T}^d} \{\bar{D}_1 |\nabla \partial u|^2 + \bar{D}_2 |\nabla \partial v|^2\} - \frac{|\bar{g}_v|}{2} \int_{2\mathbb{T}^d} |\partial v|^2 \\
& + \left\{ \frac{(\bar{f}_v + \bar{g}_u)^2}{2|\bar{g}_v|} + \bar{f}_u \right\} \int_{2\mathbb{T}^d} |\partial u|^2.
\end{aligned}$$

By the interpolation between $\|\nabla \partial u\|$ and $\|u\|$, the last term above is bounded by

$$\left\{ \frac{(\bar{f}_v + \bar{g}_u)^2}{2|\bar{g}_v|} + \bar{f}_u \right\} \left\{ a \int_{2\mathbb{T}^d} \|\nabla \partial u\|^2 + \frac{1}{4a^2} \int_{2\mathbb{T}^d} \|u\|^2 \right\}$$

for any $a > 0$. We can choose a such that

$$\left\{ \frac{(\bar{f}_v + \bar{g}_u)^2}{2|\bar{g}_v|} + \bar{f}_u \right\} a = \frac{1}{2} \bar{D}_1.$$

Collecting terms, we conclude the proof. $\square$

We are now ready to establish the bootstrap lemma, which controls the H^2 growth of $\mathbf{w}(x, t)$ in term of its L^2 growth.

Lemma 4. *Suppose that $\mathbf{w}(x, t)$ is a solution to the full system (1.4)-(1.5) such that*

$$\|\mathbf{w}(\cdot, t)\|_{H^2} \leq \min \left\{ \eta, \frac{\bar{D}_1}{2C_0C_1}, \frac{\bar{D}_2}{C_0C_1} \right\}$$

and

$$(3.4) \quad \|\mathbf{w}(\cdot, t)\| \leq 2C_1 e^{\lambda_{\max} t} \|\mathbf{w}(\cdot, 0)\|.$$

Then We Have

$$\|\mathbf{w}(t)\|_{H^2}^2 \leq C_3 \{ \|\mathbf{w}(0)\|_{H^2}^2 + e^{2\lambda_{\max} t} \|\mathbf{w}(\cdot, 0)\|^2 \}$$

where $C_3 = \frac{3}{2} \max \{ \frac{4C_1^2 C_2}{\lambda_{\max}}, 4C_1^2, 1 \} \geq 1$.

Proof. We first consider the second-order derivatives of $\mathbf{w}(x, t)$. From the previous lemma and our assumption for $\|\mathbf{w}\|_{H^2}$, we deduce that

$$\frac{1}{2} \frac{d}{dt} \sum_{|\alpha|=2} \int_{\mathbb{T}^d} \{ |\partial u|^2 + |\partial v|^2 \} d\mathbf{x} \leq C_2 \|u\|^2.$$

Therefore, by (3.4) and an integration from 0 to t , we have

$$\begin{aligned} & \sum_{|\partial|=2} \int_{\mathbb{T}^d} \{ |\partial u(t)|^2 + |\partial v(t)|^2 \} \\ & \leq \sum_{|\partial|=2} \int_{\mathbb{T}^d} \{ |\partial u(0)|^2 + |\partial v(0)|^2 \} + \frac{4C_2 C_1^2}{\lambda_{\max}} e^{2\lambda_{\max} t} \|\mathbf{w}(\cdot, 0)\|^2. \end{aligned}$$

For the 0-th and the first derivatives, we use the assumption (3.4) and interpolation to get

$$\begin{aligned} \|\mathbf{w}(t)\|_{L^2}^2 & \leq 4C_1^2 e^{2\lambda_{\max} t} \|\mathbf{w}(\cdot, 0)\|^2, \\ \|\partial^1 \mathbf{w}(t)\|_{L^2}^2 & \leq \frac{1}{2} \|\mathbf{w}(t)\|_{L^2}^2 + \frac{1}{2} \|\partial^2 \mathbf{w}(t)\|_{L^2}^2, \end{aligned}$$

where $|\partial^1| = 1$ and $|\partial^2| = 2$. Thus the lemma follows. $\square$

4. NONLINEAR INSTABILITY AND PATTERN FORMATION

We now prove Theorem 1:

Proof. Let $\mathbf{w}^\delta(x, t)$ be the family of solutions to the reaction-diffusion system (1.4)-(1.5) with initial data $\mathbf{w}^\delta(x, 0) = \delta \mathbf{w}_0$. Define T^* by

$$T^* = \sup \left\{ t \mid \|\mathbf{w}^\delta(t) - \delta e^{\mathcal{L}t} \mathbf{w}_0\| \leq \frac{C_1}{2} \delta \exp(\lambda_{\max} t) \right\}.$$

Note that T^* is well defined. We also define

$$T^{**} = \sup \left\{ t \mid \|\mathbf{w}(t)\|_{H^2} \leq \min \left\{ \eta, \frac{\bar{D}_1}{2C_0C_1}, \frac{\bar{D}_2}{C_0C_1} \right\} \right\}.$$

We now derive estimates for the H^2 norm of

$$\mathbf{w}^\delta(x, t) \text{ for } 0 \leq t \leq \min\{T^\delta, T^*, T^{**}\}.$$

First of all, by the definition of T , for $t \leq T$ and Lemma 1

$$\|\mathbf{w}^\delta(t)\| \leq \frac{3C_1}{2}\delta \exp(\lambda_{\max}t).$$

Moreover, using Lemma 4 and applying a bootstrap argument yields

$$(4.1) \quad \|\mathbf{w}^\delta(t)\|_{H^2} \leq \sqrt{C_3}\{\delta\|\mathbf{w}_0\|_{H^2} + \delta e^{\lambda_{\max}t}\}.$$

We now estimate the L^2 norm of $\mathbf{w}^\delta(x, t)$ for $0 \leq t \leq \min\{T^\delta, T^*, T^{**}\}$. We apply Duhamel's principle to obtain

$$\mathbf{w}^\delta(t) = \delta e^{\mathcal{L}t} \mathbf{w}_0 - \int_0^t e^{\mathcal{L}(t-\tau)} \mathcal{N}(\mathbf{w}^\delta(\tau)) d\tau.$$

Using Lemma 1, (3.2), (3.3), and Lemma 4 yields, for $0 \leq t \leq \min\{T^\delta, T^*, T^{**}\}$,

$$\begin{aligned} & \|\mathbf{w}^\delta(t) - \delta e^{\mathcal{L}t} \mathbf{w}_0\| \\ & \leq C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} \|\{\nabla \cdot (D \nabla \mathbf{w}^\delta) - \bar{D} \nabla^2 \mathbf{w}^\delta\} + \mathbf{F} - A \mathbf{w}^\delta\| d\tau \\ & \leq C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} \|D\|_{C^1} \|\mathbf{w}^\delta(\tau)\|_\infty \|\mathbf{w}^\delta(\tau)\|_{H^2} d\tau \\ & \quad + C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} \|D\|_{C^1} \|\nabla \mathbf{w}^\delta(\tau)\|_{L^4} \|\nabla \mathbf{w}^\delta(\tau)\|_{L^4} d\tau \\ & \quad + C_1 \int_0^t e^{\lambda_{\max}(t-\tau)} \|F\|_{C^2} \|\mathbf{w}^\delta(\tau)\|_\infty \|\mathbf{w}^\delta(\tau)\| d\tau \\ & \leq C_1 C_0^2 C_\eta \int_0^t e^{\lambda_{\max}(t-\tau)} \|\mathbf{w}^\delta(\tau)\|_{H^2}^2 d\tau, \end{aligned}$$

from assumption (1.9), where the C^1 norm is taken for $\|w\|_{H^2} \leq \eta$. We note that for $\tau \leq \min\{T^\delta, T^*, T^{**}\}$,

$$\begin{aligned} \|\mathbf{w}^\delta(\tau)\|_{H^2} & \leq \sqrt{C_3}\{\delta\|\mathbf{w}_0\|_{H^2} + \delta e^{\lambda_{\max}t}\} \\ & \leq 2\sqrt{C_3}\theta, \end{aligned}$$

for small δ such that $\delta\|\mathbf{w}_0\|_{H^2} \leq \theta$ with $\theta < 1$ small. We recall T^δ in (2.1) where θ is chosen such that

$$(4.2) \quad C_0^2 C_3 C_\eta \theta < \frac{\lambda_{\max}}{4},$$

$$(4.3) \quad 2\sqrt{C_3}\theta < \min \left\{ \eta, \frac{\bar{D}_1}{2C_0C_1}, \frac{\bar{D}_2}{C_0C_1} \right\}.$$

By our choice of $t \leq \min\{T^\delta, T^*, T^{**}\}$, it is further bounded by

$$\begin{aligned}
 (4.4) \quad & \|\mathbf{w}^\delta(t) - \delta e^{\mathcal{L}t} \mathbf{w}_0\| \\
 & \leq C_1 C_0^2 C_\eta C_3 \int_0^t e^{\lambda_{\max}(t-\tau)} \{\delta^2 \|\mathbf{w}_0\|_{H^2}^2 + \delta^2 e^{2\lambda_{\max}\tau}\} d\tau \\
 & \leq C_1 C_0^2 C_\eta C_3 \left\{ \frac{\|\mathbf{w}_0\|_{H^2}^2 \delta}{\lambda_{\max}} + \frac{1}{\lambda_{\max}} \delta e^{\lambda_{\max}t} \right\} \delta e^{\lambda_{\max}t}.
 \end{aligned}$$

We now prove by contradiction that for δ sufficiently small,

$$T^\delta = \min\{T^\delta, T^*, T^{**}\},$$

and therefore our theorem follows by further separating $\mathbf{q} \in \Omega_{\max}$ and move $\mathbf{q} \notin \Omega_{\max}$ in (1.18) to the right hand side.

If T^{**} is the smallest, we can let $t = T^{**} \leq T^\delta$ in (4.1) to obtain

$$\begin{aligned}
 \|\mathbf{w}^\delta(T^{**})\|_{H^2} & \leq \sqrt{C_3} \{\delta \|\mathbf{w}_0\|_{H^2} + \delta e^{\lambda_{\max}T^\delta}\} \\
 & = \sqrt{C_3} \{\delta \|\mathbf{w}_0\|_{H^2} + \theta\} \leq 2\sqrt{C_3}\theta,
 \end{aligned}$$

for small δ such that $\delta \|\mathbf{w}_0\|_{H^2} \leq \theta$. By the choice of θ in (4.3), we have

$$\|\mathbf{w}(T^{**})\|_{H^2} < \min \left\{ \eta, \frac{\bar{D}_1}{2C_0C_1}, \frac{\bar{D}_2}{C_0C_1} \right\}.$$

This is a contradiction to the definition of T^{**} .

On the other hand, if T^* is the smallest, we let $t = T^*$ in (4.4) to get

$$\begin{aligned}
 & \|\mathbf{w}^\delta(T^*) - \delta e^{\mathcal{L}t} \mathbf{w}_0\| \\
 & \leq C_1 C_0^2 C_3 C_\eta \left\{ \frac{\|\mathbf{w}_0\|_{H^2}^2 \delta}{\lambda_{\max}} + \frac{1}{\lambda_{\max}} \delta e^{\lambda_{\max}T^\delta} \right\} \delta e^{\lambda_{\max}T^*} \\
 & \leq C_1 C_0^2 C_3 C_\eta \left\{ \frac{\|\mathbf{w}_0\|_{H^2}^2 \delta}{\lambda_{\max}} + \frac{\theta}{\lambda_{\max}} \right\} \delta e^{\lambda_{\max}T^*} \\
 & < \frac{C_1}{2} \delta e^{\lambda_{\max}T^*},
 \end{aligned}$$

for $C_0^2 C_3 C_\eta \frac{\|\mathbf{w}_0\|_{H^2}^2 \delta}{\lambda_{\max}} < 1/4$ for δ small, by our choice of θ in (4.2). This again contradicts the definition of T , and our theorem follows. $\square$

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DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RHODE ISLAND 02912
E-mail address: guoy@dam.brown.edu

SCHOOL OF MATHEMATICS, TRINITY COLLEGE DUBLIN, DUBLIN 2, IRELAND & DEPARTMENT
 OF MATHEMATICS, POSTECH, POHANG 790-784, KOREA
E-mail address: hjhwang@postech.edu