AN INFINITE-DIMENSIONAL INTEGRAL IDENTITY FOR THE SEGAL-BARGMANN TRANSFORM

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ABSTRACT. We prove an infinite-dimensional integral identity equating the integral of a function on a subspace of a linear space to the integral of its Segal-Bargmann transform over the orthogonal complement.

1. INTRODUCTION

The purpose of this paper is to prove an integral identity for functions on infinite-dimensional linear spaces which is an analog of the finite dimensional identity

$$(1.1) \quad \int_V f(x) dx = \int_{V^\perp} \hat{f}(k) dk$$

where $\hat{f}$ is the Fourier transform of a regular enough function f on $\mathbf{R}^n$, and V is a subspace of $\mathbf{R}^n$. The simplest check on this finite-dimensional identity is obtained by taking $f(x) = e^{-2\pi i ax - \pi x^2}$, where $a = (a_1, \dots, a_n) \in \mathbf{R}^n$ and $ax = \sum_{j=1}^n a_j x_j$, for which $\hat{f}(k) \stackrel{\text{def}}{=} \int f(x) e^{-2\pi i kx} dx = e^{-\pi(a+k)^2}$, and then both sides of (1.1) work out to the common value $e^{-\pi a_V^2}$, with a_V being the orthogonal projection of a on V . This finite-dimensional identity appears in Hörmander [2] (Theorem 7.1.25), where it is generalized to submanifolds of $\mathbf{R}^n$ and several consequences developed. Our objective in this paper is to prove an infinite-dimensional Gaussian version of (1.1); possibly, non-linear infinite-dimensional versions also exist.

In the infinite-dimensional setting the role of Lebesgue measure is played by Gaussian measure, which is meaningful in infinite dimensions. Let V be a closed subspace of the real, separable, Hilbert space H_0 . Let μ_V be the standard Gaussian measure on V , and $\mu_{V^\perp}$ the Gaussian measure on $V^\perp$; technically, these measures live on a certain space of distributions we denote $\mathcal{H}'$. The main result of this paper states that the identity

$$\int_{\mathcal{H}'} F d\mu_V = \int_{\mathcal{H}'} SF(ik) d\mu_{V^\perp}(k)$$

holds for any test function F on $\mathcal{H}'$. Here S is the Segal-Bargmann transform, which takes over the role of Fourier transform in infinite-dimensional analysis.

Sections 2 and 3 summarize the essential notions and results necessary to formulate and prove our main result, Theorem 4.1. Section 4 contains the formal

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statement and proof of the identity, using some auxiliary results presented in Section 3.

2. TEST FUNCTIONS OVER INFINITE-DIMENSIONAL SPACES

In this section we summarize the necessary notions concerning test functions on infinite-dimensional linear spaces. We also set up notation to be used in the rest of the paper.

2.1. Test functions and distributions. A distribution over a space X is a continuous linear functional on a space $\mathcal{E}$ of appropriately chosen ‘test functions’ over X . For analysis we would also have some measure μ on X and

$$\mathcal{E} \subset E = L^2(\mu).$$

The classical example is Schwartz space $\mathcal{S}(\mathbf{R}) \subset L^2(\mathbf{R})$. The topology on $\mathcal{E}$ is given by some family of norms. Thus, in abstract, the basic framework is a pair

$$(2.1) \quad \mathcal{H} \subset H_0,$$

where H_0 is a separable real Hilbert space with norm $|\cdot|_0$ and inner-product $\langle \cdot, \cdot \rangle$ and $\mathcal{H}$ is a nuclear space. To form $\mathcal{H}$ we take an operator A on H_0 such that there exists an orthonormal basis $\{e_k; k = 1, 2, 3, \dots\}$ for H_0 satisfying

- (1) $Ae_k = \lambda_k e_k$, for $k = 1, 2, 3, \dots$,
- (2) $1 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$,
- (3) $\sum_{k=1}^{\infty} \lambda_k^{-2} < \infty$.

For the Schwartz space $\mathcal{S}(\mathbf{R})$, the operator A is the harmonic oscillator Hamiltonian $-\frac{d^2}{dx^2} + \frac{x^2}{4} + \frac{1}{2}$.

Note that A^{-1} is a bounded operator with norm given by

$$(2.2) \quad \rho = \|A^{-1}\| = \frac{1}{\lambda_1}.$$

The condition that $1 < \lambda_1$ is needed when proving continuity of test functions (Theorem 2.2).

Now for each $p \geq 0$ we define the norm

$$(2.3) \quad |x|_p = |A^p x|_0 = \sqrt{\sum_{k \geq 1} \lambda_k^{2p} |\langle x, e_k \rangle|^2}$$

and let

$$(2.4) \quad H_p = \{x \in H_0; |x|_p < \infty\}.$$

This is a Hilbert space under the obvious inner-product $\langle \cdot, \cdot \rangle_p$, and

$$H_p \subset H_q \text{ for any } p \geq q$$

and the inclusion map $I_{p,p-1} : H_p \hookrightarrow H_{p-1}$ is a Hilbert-Schmidt operator. We then define $\mathcal{H}$ to be the projective limit of $\{H_p; p = 0, 1, 2, \dots\}$, and this gives us

$$\mathcal{H} = \bigcap_{p \geq 0} H_p \subset \dots \subset H_2 \subset H_1 \subset H_0.$$

Below we describe in brief how a space of test functions is constructed over the dual space $\mathcal{H}'$ using this framework.

The symmetric Fock space $\mathcal{F}_s(V)$ over a Hilbert space V is the subspace of symmetric tensors in the completion of the tensor algebra $T(V)$ under the inner-product given by

$$(2.5) \quad \langle a, b \rangle_{T(V)} = \sum_{n=0}^{\infty} n! \langle a_n, b_n \rangle_{V^{\otimes n}},$$

where $a = \{a_n\}_{n \geq 0}, b = \{b_n\}_{n \geq 0}$ are elements of $T(V)$ with a_n, b_n in the tensor power $V^{\otimes n}$. Then we have

$$(2.6) \quad \mathcal{F}_s(\mathcal{H}) \stackrel{\text{def}}{=} \cap_{p \geq 0} \mathcal{F}_s(H_p) \subset \cdots \subset \mathcal{F}_s(H_2) \subset \mathcal{F}_s(H_1) \subset \mathcal{F}_s(H_0).$$

Thus, the pair (2.1) gives rise to a corresponding pair by taking symmetric Fock spaces:

$$(2.7) \quad \mathcal{F}_s(\mathcal{H}) \subset \mathcal{F}_s(H_0).$$

The dual space $\mathcal{H}'$ of continuous real linear functionals on $\mathcal{H}$ is the union

$$\mathcal{H}' = \bigcup_{p \geq 0} H_{-p}$$

where H_{-p} is the set of real linear functionals on $\mathcal{H}$ which are continuous in the $|\cdot|_p$ norm. Note that H_{-p} is naturally isomorphic to $H'_p \simeq H_p$ and is thus a Hilbert space. We denote the norm on H_{-p} by $|\cdot|_{-p}$. Thus the inner product on H_0 extends to a bilinear pairing, also denoted by $\langle \cdot, \cdot \rangle$, between H_p and H_{-p} with

$$|\langle x, f \rangle| \leq |x|_p |f|_{-p}$$

for all $p \geq 0, x \in H_p$ and $f \in H_{-p}$. We then have a chain of inclusions

$$H_0 \simeq H_{-0} \subset H_{-1} \subset \cdots \subset \bigcup_{p \geq 0} H_{-p} = \mathcal{H}'$$

where the inner product on H_0 extends to a bilinear pairing between $\mathcal{H}$ and $\mathcal{H}'$. The dual space $\mathcal{H}'$ may be equipped with the weak or the strong or the inductive limit topologies.

Fact 2.1. The Borel sigma algebras generated by the weak, strong, and inductive topologies on $\mathcal{H}'$ are equivalent.

Although this result is known and has been used implicitly or explicitly in the literature, a complete readily accessible proof can be found in [1].

2.2. The Gaussian measure μ . In infinite dimensions the role of Lebesgue measure is played by Gaussian measure. The standard Gaussian measure μ for the pair (2.1) is a Borel measure on $\mathcal{H}'$, specified uniquely by

$$(2.8) \quad \int_{\mathcal{H}'} e^{i\hat{f}} d\mu = e^{-|f|_0^2/2} \quad \text{for all } f \in \mathcal{H},$$

where

$$\hat{f} : \mathcal{H}' \rightarrow \mathbf{R} : \phi \mapsto \phi(f).$$

There is a standard unitary isomorphism, the *Wiener-Itô isomorphism* or wave-particle duality map, which identifies the complexified Fock space $\mathcal{F}_s(H_0)_c$ with $L^2(\mathcal{H}', \mu)$. This is uniquely specified by

$$(2.9) \quad I : \mathcal{F}_s(H_0)_c \rightarrow L^2(\mathcal{H}', \mu) : \text{Exp}(x) \mapsto e^{\hat{x} - \frac{1}{2}x^2}$$

where $x \in \mathcal{H}$, $x^2 = |x|_0^2$, and

$$\text{Exp}(x) = \sum_{n \geq 0} \frac{1}{n!} x^{\otimes n}.$$

Indeed, it is readily checked that I preserves inner-products (the inner-product is as described in (2.5)). Using I , for each $\mathcal{F}_s(H_p)$ with $p \geq 0$, we have the corresponding space $[H]_p \subset L^2(\mathcal{H}', \mu)$ with the norm $\|\cdot\|_p$ induced by the norm on the space $\mathcal{F}_s(H_p)_c$. From this the chain of spaces (2.6) can be transferred into a chain of function spaces:

$$(2.10) \quad [\mathcal{H}] = \bigcap_{p \geq 0} [H]_p \subset \cdots \subset [H]_2 \subset [H]_1 \subset [H]_0 = L^2(\mathcal{H}', \mu).$$

Observe that $[\mathcal{H}]$ is a nuclear space with topology induced by the norms $\{\|\cdot\|_p; p = 0, 1, 2, \dots\}$. Thus, starting with the pair

$$\mathcal{H} \subset H_0$$

one obtains a corresponding pair

$$[\mathcal{H}] \subset L^2(\mathcal{H}', \mu).$$

Note that the measure μ uses H_0 as a real Hilbert space.

The identification of H'_0 with H_0 leads to a complete chain

$$(2.11) \quad \mathcal{H} = \bigcap_{p \geq 0} H_p \subset \cdots \subset H_1 \subset H_0 \simeq H_{-0} \subset H_{-1} \subset \cdots \subset \bigcup_{p \geq 0} H_{-p} = \mathcal{H}'.$$

In the same way we have a chain for the ‘second quantized’ spaces $\mathcal{F}_s(H_q)_c \simeq [H]_q$. The unitary isomorphism I extends to unitary isomorphisms

$$(2.12) \quad I : \mathcal{F}_s(H_{-p})_c \rightarrow [H]_{-p} \stackrel{\text{def}}{=} [H]'_p \subset [\mathcal{H}]',$$

for all $p \geq 0$. In more detail, for $a \in \mathcal{F}_s(H_{-p})_c$ the distribution $I(a)$ is specified by

$$(2.13) \quad \langle I(a), f \rangle = \langle a, I^{-1}(f) \rangle,$$

for all $f \in [\mathcal{H}]$. On the right side here we have the pairing of $\mathcal{F}_s(H_{-p})_c$ and $\mathcal{F}_s(H_p)_c$ induced by the duality pairing of H_{-p} and H_p ; in particular, the pairings above are complex bilinear (not sesquilinear).

2.3. Properties of test functions. The following theorem summarizes the properties of $[\mathcal{H}]$ we need. The results here are standard (see, for instance, Kuo’s monograph [3]), and we compile them for ease of reference.

Theorem 2.2. *Every function in $[\mathcal{H}]$ is μ -almost-everywhere equal to a unique continuous function on $\mathcal{H}'$. Moreover, working with these continuous versions,*

- (1) $[\mathcal{H}]$ is an algebra under pointwise operations;
- (2) pointwise addition and multiplication are continuous operations $[\mathcal{H}] \times [\mathcal{H}] \rightarrow [\mathcal{H}]$;
- (3) for any $\phi \in \mathcal{H}'$, the evaluation map

$$\delta_\phi : [\mathcal{H}] \rightarrow \mathbf{R} : F \mapsto F(\phi)$$

is continuous;

- (4) the exponentials $e^{i\hat{f} - \frac{1}{2}|f|_0^2}$, with f running over $\mathcal{H}$, span a dense subspace of $[\mathcal{H}]$.

Remark 2.3. Note that (4) immediately gives us that the span of e^{if} with f running over $\mathcal{H}$ is dense in $[\mathcal{H}]$. It also follows from (4) that the span of e^z with z running over $\mathcal{H}_c$ is dense in $[\mathcal{H}]$.

A complete characterization of the space $[\mathcal{H}]$ was obtained by Y. J. Lee (see the account in [3] on page 89).

2.4. The Segal-Bargmann transform. The Segal-Bargmann transform takes a function $F \in L^2(\mathcal{H}', \mu)$ to the function SF on the complexified space $\mathcal{H}_c$ given by

$$(2.14) \quad SF(z) = \int_{\mathcal{H}'} e^{\bar{z}-z^2/2} F d\mu, \quad z \in \mathcal{H}_c,$$

with notation as follows: if $z = a + ib$, with $a, b \in \mathcal{H}$, then

$$(2.15) \quad \tilde{z}(x) \stackrel{\text{def}}{=} zx \stackrel{\text{def}}{=} \langle x, a \rangle + i\langle x, b \rangle, \quad \text{for } x \in \mathcal{H}'$$

and $z^2 = zz$, where the product zu is specified through

$$(2.16) \quad zu \stackrel{\text{def}}{=} \langle a, s \rangle - \langle b, t \rangle + i(\langle a, t \rangle + \langle b, s \rangle)$$

if $z = a + ib$ and $u = s + it$, where $a, b, s, t \in \mathcal{H}$.

Let μ_c be the Gaussian measure $\mathcal{H}'_c$ specified by the requirement that

$$(2.17) \quad \int_{\mathcal{H}'_c} e^{ax+by} d\mu_c(x+iy) = e^{(a^2+b^2)/4}$$

for every $a, b \in \mathcal{H}$. For convenience, let us introduce the renormalized exponential function

$$(2.18) \quad c_w = e^{\bar{w}-w^2/2}$$

for all $w \in \mathcal{H}_c$. Thankfully, c_w lies in $L^2(\mathcal{H}', \mu)$, which means the integrand in (2.14) exists for all $z \in \mathcal{H}_c$. It is readily checked that for any $w \in \mathcal{H}_c$

$$(2.19) \quad [Sc_w](z) = e^{wz}, \quad \text{for all } z \in \mathcal{H}_c.$$

Thus we may take Sc_w as a function on $\mathcal{H}'_c$ given by

$$(2.20) \quad Sc_w = e^{\bar{w}},$$

where now $\bar{w}$ is a function on $\mathcal{H}'_c$ in the natural way. Then $Sc_w \in L^2(\mathcal{H}'_c, \mu_c)$ and one has

$$\langle Sc_w, Sc_u \rangle_{L^2(\mu_c)} = \langle c_w, c_u \rangle_{L^2(\mu)} = e^{w\bar{u}}.$$

This shows that S provides an isometry from the linear span of the exponentials c_w in $L^2(\mathcal{H}', \mu)$ onto the linear span of the complex exponentials $e^{\bar{w}}$ in $L^2(\mathcal{H}'_c, \mu_c)$. Passing to the closure one obtains the Segal-Bargmann unitary isomorphism

$$S : L^2(\mathcal{H}', \mu) \rightarrow Hol^2(\mathcal{H}'_c, \mu_c)$$

where $Hol^2(\mathcal{H}'_c, \mu_c)$ is the closed linear span of the complex exponential functions $e^{\bar{w}}$ in $L^2(\mathcal{H}'_c, \mu_c)$.

An explicit expression for $SF(z)$ is suggested by (2.14). For any $F \in [\mathcal{H}]$ and $z \in \mathcal{H}'_c$, we have

$$(2.21) \quad (SF)(z) = \langle I(\text{Exp}(z)), F \rangle$$

where the right side is the evaluation of the distribution $I(\text{Exp}(z))$ on the test function F . Indeed it may be readily checked that if $SF(z)$ is defined in this way, then $[Sc_w](z) = e^{wz}$.

In view of (2.21), it is natural to extend the Segal-Bargmann transform to distributions: for $\Phi \in [\mathcal{H}]'$, define $S\Phi$ to be the function on $\mathcal{H}_c$ given by

$$(2.22) \quad S\Phi(z) \stackrel{\text{def}}{=} \langle \Phi, I(\text{Exp}(z)) \rangle.$$

2.5. Images of test functions and distributions under S . The following result (Theorem 8.2 on page 79 in [3]) describes the image $S([\mathcal{H}]')$.

Theorem 2.4 (Potthoff–Streit). *Suppose a function G on $\mathcal{H}_c$ satisfies:*

- (1) *for any $z, w \in \mathcal{H}_c$, the function $G(\alpha z + w)$ is an entire function of $\alpha \in \mathbf{C}$,*
- (2) *there exist nonnegative constants A, p , and C such that*

$$|G(z)| \leq C e^{A|z|_p^2} \quad \text{for all } z \in \mathcal{H}_c.$$

Then there is a unique distribution $\Phi \in [\mathcal{H}]'$ for which $G = S\Phi$. Conversely, for any $\Phi \in [\mathcal{H}]'$, the function $G = S\Phi$ satisfies (1) and (2) above.

Let us note the following bound on SF , for any test function F :

Lemma 2.5. *If $F \in [\mathcal{H}]_p \subset L^2(\mu)$, with $p \in \{0, 1, 2, \dots\}$ and $z \in \mathcal{H}'_c$, then*

$$|SF(z)| \leq \|F\|_p e^{|z|_{-p}^2/2}.$$

Proof. For any $F \in [\mathcal{H}]$ and $z \in \mathcal{H}'_c$:

$$(2.23) \quad SF(z) = \langle I(\text{Exp}(z)), F \rangle.$$

Then letting $\|\cdot\|_{-p}$ denote the norm in $[H]_{-p}$, we have

$$|SF(z)| \leq \|I(\text{Exp}(z))\|_{-p} \|F\|_p = e^{|z|_{-p}^2/2} \|F\|_p,$$

where we have used the easily checked expression for the p -norm of $\text{Exp}(z)$. $\square$

3. MEASURES CORRESPONDING TO DISTRIBUTIONS

A Borel measure ν for which $[\mathcal{H}] \subset L^1(\nu)$ and the mapping $F \mapsto \int_{\mathcal{H}'} F d\nu$ is continuous on $[\mathcal{H}]$ is called a *Hida measure*. A Hida measure induces a distribution $\tilde{\nu}$ given by

$$\langle \tilde{\nu}, F \rangle = \int_{\mathcal{H}'} F d\nu \quad \text{for all } F \in [\mathcal{H}].$$

The following result characterizes Hida measures (see Theorem 15.17, page 333, in [3] for a proof of a more general form of this result):

Theorem 3.1. *A measure ν on $\mathcal{H}'$ is a Hida measure if and only if for some $p \geq 1$, H_{-p} is of full measure and*

$$\int_{H_{-p}} e^{\frac{1}{2}|x|_{-p}^2} d\nu(x) < \infty.$$

Finally we record an observation for use in the next section:

Proposition 3.2. *Suppose ν is a finite Borel measure on $\mathcal{H}'$ such that*

$$(3.1) \quad \left| \int_{\mathcal{H}'} e^{\bar{z}z - z^2/2} d\nu \right| \leq e^{A|z|_p^2} \quad \text{for every } z \in \mathcal{H}_c,$$

for some constants $p \geq 0$ and $A > 0$. Then

$$F \mapsto \int_{\mathcal{H}'} F d\nu$$

is a continuous linear functional on $[\mathcal{H}]$. In other words, there is a unique distribution $\Phi \in [\mathcal{H}]'$ for which

$$(3.2) \quad \langle \Phi, F \rangle = \int_{\mathcal{H}'} F d\nu$$

holds for every test function $F \in [\mathcal{H}]$.

Proof. Taking $G(z) = \int_{\mathcal{H}'} e^{\bar{z}-z^2/2} d\nu$ as in Theorem 2.4, the growth bound (3.1) on the S -transform of the finite measure ν implies that there is a unique distribution $\Phi \in [\mathcal{H}]'$ such that

$$\langle \Phi, e^{i\hat{x}} \rangle = \int_{\mathcal{H}'} e^{i\hat{x}} d\nu$$

for every $x \in \mathcal{H}$.

Let $F \in [\mathcal{H}]$, a test function. By Theorem 2.2 there is a sequence of functions F_n , each a linear combination of exponentials of the form $e^{i\hat{x}}$ with $x \in \mathcal{H}$, such that $F_n \rightarrow F$ in the topology of $[\mathcal{H}]$. Again, by Theorem 2.2, with the limits being in $[\mathcal{H}]$, and hence also pointwise,

$$(3.3) \quad \lim_{n \rightarrow \infty} |F_n - F|^2 = \lim_{n \rightarrow \infty} (F_n - F)(\bar{F}_n - \bar{F}) = 0$$

and

$$(3.4) \quad \lim_{m \rightarrow \infty} |F_n - F_m|^2 = |F_n - F|^2.$$

Also, $\langle \Phi, \psi \rangle = \int \psi d\nu$ for every $\psi \in L$, the linear span of the exponentials $e^{i\hat{x}}$ for $x \in \mathcal{H}$. Using these facts and further applications of continuity of multiplication of test functions, we have

$$\begin{aligned} \int |F_n - F|^2 d\nu &\leq \liminf_{m \rightarrow \infty} \int |F_n - F_m|^2 d\nu && \text{by Fatou's lemma and (3.3)} \\ &= \liminf_{m \rightarrow \infty} \langle \Phi, |F_n - F_m|^2 \rangle && \text{because } |F_n - F_m|^2 \in L \\ &= \langle \Phi, |F_n - F|^2 \rangle. \end{aligned}$$

Letting $n \rightarrow \infty$ and using continuity of $\Phi \in [\mathcal{H}]'$ we conclude that

$$F_n \rightarrow F \text{ in } L^2(\nu), \text{ as } n \rightarrow \infty.$$

Since ν is a finite measure it follows that $\int F_n d\nu \rightarrow \int F d\nu$ and, again by continuity of Φ , we also have $\langle \Phi, F_n \rangle \rightarrow \langle \Phi, F \rangle$. This proves that $\int F d\nu = \langle \Phi, F \rangle$. $\square$

4. THE IDENTITY

We continue to work with a pair

$$\mathcal{H} \subset H_0,$$

where H_0 is a real separable Hilbert space and $\mathcal{H}$ a dense subspace which is a nuclear space as described earlier. We then have the Gaussian measure μ on $\mathcal{H}'$, and the corresponding space $[H]_0 = L^2(\mu)$. There is the dense subspace $[\mathcal{H}] \subset L^2(\mu)$, whose elements are our continuous test functions, and $[\mathcal{H}]$ is closed under pointwise addition and multiplication, and contains the linear span of $\{e^{i\hat{x}} : x \in \mathcal{H}\}$ as a dense subspace.

Recall that for $f \in \mathcal{H}$ we have the evaluation function $\hat{f} : \mathcal{H}' \rightarrow \mathbf{R} : \phi \mapsto \langle \phi, f \rangle$. If V is a closed subspace of H_0 , then there is a unique probability measure μ_V on $\mathcal{H}'$ specified through

$$(4.1) \quad \int_{\mathcal{H}'} e^{i\hat{f}} d\mu_V = e^{-\frac{1}{2}\|f_V\|_0^2}$$

for every $f \in \mathcal{H}$, where f_V denotes the orthogonal projection of f onto a closed subspace V of H_0 . Note that if $f \in V^\perp$, then the above Fourier transform implies that $\hat{f}$ is zero μ_V -a.e. Thus, in this sense, μ_V is a Gaussian measure concentrated on the subspace V . Observe that the characteristic function of μ_V immediately implies μ_V is a Hida measure by Proposition 3.2.

Our main result is:

Theorem 4.1. *Let V be a closed subspace of the real, separable, Hilbert space H_0 . Let μ_V be the Gaussian measure for V , and $\mu_{V^\perp}$ the Gaussian measure for $V^\perp$, constructed as before. Then, for any test function $F \in [\mathcal{H}]$ on $\mathcal{H}'$, we have*

$$(4.2) \quad \int_{\mathcal{H}'} F d\mu_V = \int_{\mathcal{H}'} SF(ik) d\mu_{V^\perp}(k).$$

Proof. Consider first the function c_z for F , where $c_z = e^{\tilde{z} - \frac{1}{2}z^2}$, where $z \in \mathcal{H}_c$. Then the left side of (4.2) is

$$\int_{\mathcal{H}'} c_z d\mu_V = e^{\frac{1}{2}z_V^2} e^{-\frac{1}{2}z^2} = e^{-\frac{1}{2}z_{V^\perp}^2}.$$

Here, for $z = a + ib \in \mathcal{H}_c$ and K a closed subspace of H_0 , we denote by z_K the element $a_K + ib_K$, where a_K and b_K are the respective orthogonal projections onto K . The right side of (4.2) is

$$\begin{aligned} \int_{\mathcal{H}'} Sc_z(i\phi) d\mu_{V^\perp}(\phi) &= \int_{\mathcal{H}'} e^{i\tilde{z}} d\mu_{V^\perp} \\ &= e^{-\frac{1}{2}z_{V^\perp}^2}. \end{aligned}$$

Thus our identity (4.2) holds for the functions c_z , and hence for F in the linear span L of these functions. Now take F to be any test function in $[\mathcal{H}]$. Then, by Theorem 2.2, there is a sequence of functions $F_n \in L$ which converges in the topology of $[\mathcal{H}]$ to F and hence also pointwise. Moreover, for every $p \geq 0$

$$\lim_{n \rightarrow \infty} F_n = F \quad \text{in } \|\cdot\|_p\text{-norm.}$$

From the identity

$$(4.3) \quad \int_{\mathcal{H}'} F_n d\mu_V = \int_{\mathcal{H}'} SF_n(ik) d\mu_{V^\perp}(k)$$

Proposition 3.2 implies that as $n \rightarrow \infty$ the left side converges to $\int_{\mathcal{H}'} F d\mu_V$. On the other hand, by the bound in Lemma 2.5, the integrands on the right are bounded by $\|F_n\|_t e^{|k|^2_{-t}/2}$, for every $t \geq 0$. As noted before $\mu_{V^\perp}$ is a Hida measure, and so by Theorem 3.1 there is a $p \geq 1$ for which

$$(4.4) \quad \int_{\mathcal{H}'} e^{|k|^2_{-p}/2} d\mu_{V^\perp}(k) < \infty.$$

Combining these facts we see that the dominated convergence applies to the right side of (4.3) and the latter converges to $\int_{\mathcal{H}'} SF(ik) d\mu_{V^\perp}(k)$. This completes the proof of the identity. $\square$

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