4 PLANES IN $\mathbb{R}^4$

E. BATZIES

(Communicated by Ronald A. Fintushel)

This paper is dedicated to Julia.

ABSTRACT. We establish a homeomorphism between the moduli space $A_{4,k}^{\text{ord}}(\mathbb{R})$ of ordered k -tuples $(H_1, \dots, H_k)$ of 2-dimensional linear subspaces $H_i \subset \mathbb{R}^4 \pmod{\text{GL}_4(\mathbb{R})}$ and the quotient by simultaneous conjugation of a certain open subset $(\text{GL}_2^{\text{ord}}(\mathbb{R}))^* \subset (\text{GL}_2(\mathbb{R}))^{k-3}$. For $k = 4$, this leads to an explicit computation of the moduli space $A_{4,4}(\mathbb{R})$ of central 2-arrangements in $\mathbb{R}^4 \pmod{\text{GL}_4(\mathbb{R})}$ and its subspace $A_{2,4}(\mathbb{C})$ of those classes that contain a complex hyperplane arrangement.

1. INTRODUCTION

In [4], Goresky and MacPherson propose the investigation of so-called 2-arrangements, which they define to be finite collections $\mathcal{A}$ of affine $\mathbb{R}$ -subspaces of codimension 2 in some $\mathbb{R}^{2n}$ with the additional property that all intersections $\bigcap_{H \in I} H$, $I \subset \mathcal{A}$ are even-dimensional. Clearly, complex hyperplane arrangements can be viewed as such. It is natural to ask whether for 2-arrangements there exist more homotopy types of the complement $M(\mathcal{A}) := \mathbb{R}^{2n} \setminus \bigcup_{H \in \mathcal{A}} H$ than for complex hyperplane arrangements.

In [8], Ziegler presents two central (i.e., the planes are linear) 2-arrangements of 4 planes in $\mathbb{R}^4$ which differ in cohomology algebras of their complements. One of these is a complex line arrangement in $\mathbb{C}^2$, while the other is not. A classical result of Orlik and Solomon [6] states that the cohomology algebra of the complement of a complex hyperplane arrangement is uniquely determined by the intersection lattice. Since Ziegler's arrangements obviously are lattice-equivalent, the non-complex one does not have the homotopy type of a complex hyperplane arrangement.

So far, the class of central 2-arrangements of k planes in $\mathbb{R}^4$ is not understood in general. The state of the art is Matei and Suciu's classification [5] of central 2-arrangements in $\mathbb{R}^4$ with $k \leq 6$ up to homotopy type of the complement.

In this paper a natural approach to the problem is presented and carried out in detail for the case $k = 4$: We study the moduli space $A_{4,k}(\mathbb{R})$ of central 2-arrangements of k planes in $\mathbb{R}^4$, identifying two of them if they can be transformed into one another by a general linear transformation of $\mathbb{R}^4$. By this approach, $k = 4$ is the first non-trivial case (for $k = 3$, we get a one-point-space, since any 3 trivially

Received by the editors July 27, 2001 and, in revised form, January 23, 2005.

2000 *Mathematics Subject Classification*. Primary 52C35, 32S22; Secondary 58D29.

Key words and phrases. Arrangements.

intersecting planes can be transformed into a standard configuration; see Section 2).

In computing the moduli space $A_{4,4}(\mathbb{R})$ (and the subspace $A_{2,4}(\mathbb{C}) \subset A_{4,4}(\mathbb{R})$ of those classes of 2-arrangements mod $GL_4 := GL_4(\mathbb{R})$ that contain a complex hyperplane arrangement), we not only rediscover Ziegler's result of finding exactly two homotopy types of complements for $k = 4$, but also identify $A_{4,4}(\mathbb{R})$ as an interesting topological space:

Let $\mathbb{PR}_2$ denote the two-dimensional real projective space, that is, the space of all lines $[x, y, z] := \{(\lambda x, \lambda y, \lambda z) \mid \lambda \in \mathbb{R}\}$, $(x, y, z) \neq (0, 0, 0)$, in $\mathbb{R}^3$, endowed with the quotient topology with respect to the canonical projection $\mathbb{R}^3 \twoheadrightarrow \mathbb{PR}_2$.

Let $\mathbb{PR}_2^*$ be the subspace of $\mathbb{PR}_2$ consisting of all lines in $\mathbb{R}^3$ that do not lie on any coordinate plane, that is,

$$\mathbb{PR}_2^* := \{[x, y, z] \in \mathbb{PR}_2 \mid xyz \neq 0\}.$$

Furthermore, let $Q \subset \mathbb{PR}_2$ be the following quadric in $\mathbb{PR}_2$:

$$Q := \{[x, y, z] \in \mathbb{PR}_2 \mid x^2 + y^2 + z^2 - 2(xy + xz + yz) = 0\}.$$

With Q^* a duplicate of $\mathbb{PR}_2^* \cap Q$ and $\pi : \mathbb{PR}_2^* \dot{\cup} Q^* \rightarrow \mathbb{PR}_2^*$ the canonical projection, we topologize $\mathbb{PR}_2^* \dot{\cup} Q^*$ by requiring Q^* to be closed in $\mathbb{PR}_2^* \dot{\cup} Q^*$ and that for all $q \in Q^*$,

$$\mathcal{B}_q := \{\pi^{-1}(U) \mid U \text{ a neighborhood of } \pi(q)\}$$

is a neighborhood basis of q . Let

$$Q_{<} := \{[x, y, z] \in \mathbb{PR}_2 \mid x^2 + y^2 + z^2 - 2(xy + xz + yz) < 0\} \cup Q^*.$$

Figure 1 illustrates the situation: Identify the four open quarters of the upper hemisphere in the figure with $\mathbb{PR}_2^*$. Then the dark area represents $Q_{<}$, its boundary Q .

FIGURE 1. The moduli space $A_{4,4}(\mathbb{R})$

Theorem 1.

$$\begin{aligned} A_{4,4}(\mathbb{R}) &\cong \mathbb{P}\mathbb{R}_2^* \dot{\cup} Q^* / \text{permutations of coordinates } x, y, z, \\ A_{2,4}(\mathbb{C}) &\cong Q_{<} / \text{permutations of coordinates } x, y, z. \end{aligned}$$

Corollary 2. *There exist exactly 2 homotopy types of complements of central 2-arrangements of 4 planes in $\mathbb{R}^4$.*

For general k we show the following: Let $A_{4,k}^{\text{ord}}(\mathbb{R})$ be the moduli space mod GL_4 of all k -tuples $(H_1, \dots, H_k)$ of planes in $\mathbb{R}^4$ such that $\{H_1, \dots, H_k\} \in A_{4,k}(\mathbb{R})$. Let

$$(\text{GL}_2^l)^* := \{(\alpha_1, \dots, \alpha_l) \in \text{GL}_2^l(\mathbb{R}) \mid \det(\alpha_i - id) \neq 0, \det(\alpha_i - \alpha_j) \neq 0 \forall i \neq j\}.$$

For $\alpha = (\alpha_1, \dots, \alpha_l)$, $\alpha' = (\alpha'_1, \dots, \alpha'_l) \in \text{GL}_2^l$ we say that α and α' are simultaneously conjugate if there exists $\delta \in \text{GL}_2$ such that $\delta\alpha_1 = \alpha'_1\delta, \dots, \delta\alpha_l = \alpha'_l\delta$.

Theorem 3.

$$A_{4,k}^{\text{ord}}(\mathbb{R}) \cong (\text{GL}_2^{k-3})^* / \text{simultaneous conjugation}.$$

2. k -TUPLES OF PLANES IN $\mathbb{R}^4$

In this section we prove Theorem 3. For calculations it is convenient to describe the planes as kernels of linear maps: Let $(\mathcal{L}_{4,2}^k)^*$ consist of all k -tuples $(\gamma_1, \dots, \gamma_k) \in \text{Hom}(\mathbb{R}^4, \mathbb{R}^2)^k$ such that $\{\ker\gamma_1, \dots, \ker\gamma_k\}$ is a central 2-arrangement in $\mathbb{R}^4$, that is:

- $\text{rk}\gamma_i = 2$ for all $i = 1, \dots, k$,
- $\ker\gamma_i \cap \ker\gamma_j = \{0\}$ for all $1 \leq i < j \leq k$.

We endow $A_{4,k}^{\text{ord}}(\mathbb{R})$ with the quotient topology with respect to the map

$$\begin{aligned} (\mathcal{L}_{4,2}^k)^* &\longrightarrow A_{4,k}^{\text{ord}}(\mathbb{R}) \\ (\gamma_1, \dots, \gamma_k) &\longmapsto (\ker\gamma_1, \dots, \ker\gamma_k). \end{aligned}$$

Denote by $\pi \in \text{Hom}(\mathbb{R}^4, \mathbb{R}^2)$ the orthogonal projection onto the first two coordinates, by ρ the one onto the third and fourth. For any $(\gamma_1, \gamma_2, \gamma_3) \in (\mathcal{L}_{4,2}^3)^*$ there is a linear transformation $\Phi \in \text{GL}_4$, such that $(\gamma_1 \circ \Phi^{-1}, \gamma_2 \circ \Phi^{-1}, \gamma_3 \circ \Phi^{-1}) = (-\alpha^{-1}\pi, \beta^{-1}\rho, \rho - \pi)$ for some $\alpha, \beta \in \text{GL}_2$, namely $\Phi := \begin{pmatrix} -\alpha\gamma_1 \\ \beta\gamma_2 \end{pmatrix} \in \text{GL}_4$, where $(\alpha, \beta) \in \text{GL}_2^2$ is the unique pair such that $\gamma_3 = \alpha\gamma_1 + \beta\gamma_2$. Geometrically, this means that mod GL_4 any 3 pairwise trivially intersecting planes in $\mathbb{R}^4$ are equivalent to $(\ker\pi, \ker\rho, \ker(\rho - \pi))$.

Note that for any additional plane $\ker\gamma$ the set $\{\ker\pi, \ker\rho, \ker(\rho - \pi), \ker\gamma\}$ is a central 2-arrangement if and only if $\gamma = \beta\rho - \alpha\pi$ for some $\alpha, \beta \in \text{GL}_2$ such that $\alpha - \beta \in \text{GL}_2$, or, equivalently, if and only if $\ker\gamma = \ker(\rho - \alpha'\pi)$ with $\alpha' \in \text{GL}_2^* := (\text{GL}_2^1)^*$.

Furthermore, for any two additional planes $\ker(\rho - \alpha_4\pi)$, $\ker(\rho - \alpha_5\pi)$, such that $(\pi, \rho, \rho - \pi, \rho - \alpha_4\pi)$, $(\pi, \rho, \rho - \pi, \rho - \alpha_5\pi) \in (\mathcal{L}_{4,2}^4)^*$, we have:

$$\det(\alpha_4 - \alpha_5) \neq 0 \Leftrightarrow (\pi, \rho, \rho - \pi, \rho - \alpha_4\pi, \rho - \alpha_5\pi) \in (\mathcal{L}_{4,2}^5)^*.$$

Therefore, it is natural to consider the following maps:

$$\begin{aligned} (2.1) \quad r : (\mathcal{L}_{4,2}^k)^* &\longrightarrow (\text{GL}_2^{k-3})^* \\ (\gamma_1, \dots, \gamma_k) &\longmapsto (\beta_4^{-1}\alpha_4, \dots, \beta_k^{-1}\alpha_k), \end{aligned}$$

such that $\Phi = \begin{pmatrix} -\alpha\gamma_1 \\ \beta\gamma_2 \end{pmatrix}$, $\gamma_3 = \alpha\gamma_1 + \beta\gamma_2$ and $\gamma_j \circ \Phi^{-1} = \beta_j\rho - \alpha_j\pi$ for $j = 4, \dots, k$, and

$$(2.2) \quad \begin{aligned} i : (\mathrm{GL}_2^{k-3})^* &\longrightarrow (\mathcal{L}_{4,2}^k)^* \\ (\alpha_4, \dots, \alpha_k) &\longmapsto (\pi, \rho, \rho - \pi, \rho - \alpha_4\pi, \dots, \rho - \alpha_k\pi). \end{aligned}$$

Obviously, r is a retraction with section i . The final step is to show that r induces an isomorphism

$$r_* : A_{4,k}^{\mathrm{ord}}(\mathbb{R}) \cong (\mathcal{L}_{4,2}^k)^* / \mathrm{GL}_2^k \times \mathrm{GL}_4 \longrightarrow (\mathrm{GL}_2^{k-3})^* / \mathrm{simconj}.$$

Proof. (1) r_* is well defined:

Let

$$(\gamma_1, \dots, \gamma_k) \in (\mathcal{L}_{4,2}^k)^*, \quad \gamma_3 = \alpha\gamma_1 + \beta\gamma_2,$$

$$\Phi = \begin{pmatrix} -\alpha\gamma_1 \\ \beta\gamma_2 \end{pmatrix}, \quad \gamma_j = \beta_j\rho - \alpha_j\pi, \quad j = 4, \dots, k.$$

$$(a) \text{ Let } \Psi \in \mathrm{GL}_4, \quad (\gamma'_1, \dots, \gamma'_k) = (\gamma_1 \circ \Psi^{-1}, \dots, \gamma_k \circ \Psi^{-1}).$$

It follows that

$$\gamma'_3 = \alpha\gamma'_1 + \beta\gamma'_2, \quad \Phi' = \Phi \circ \Psi^{-1},$$

$$\gamma'_j \circ \Phi'^{-1} = \gamma'_j \circ \Psi \circ \Phi^{-1} = \gamma_j \circ \Phi^{-1}$$

and therefore

$$r(\gamma'_1, \dots, \gamma'_k) = r(\gamma_1, \dots, \gamma_k).$$

$$(b) \text{ Let } (\delta_1, \dots, \delta_k) \in \mathrm{GL}_2^k, \quad (\gamma'_1, \dots, \gamma'_k) = (\delta_1\gamma_1, \dots, \delta_k\gamma_k).$$

It follows that

$$\gamma'_3 = \delta_3\gamma_3 = \delta_3(\alpha\gamma_1 + \beta\gamma_2) = \delta_3\alpha\delta_1^{-1}\gamma'_1 + \delta_3\beta\delta_2^{-1}\gamma'_2,$$

so

$$\Phi' = \begin{pmatrix} -\delta_3\alpha\delta_1^{-1}\gamma'_1 \\ \delta_3\beta\delta_2^{-1}\gamma'_2 \end{pmatrix} = \begin{pmatrix} -\delta_3\alpha\gamma_1 \\ \delta_3\beta\gamma_2 \end{pmatrix} = \begin{pmatrix} \delta_3 & 0 \\ 0 & \delta_3 \end{pmatrix} \circ \Phi.$$

This implies that

$$\begin{aligned} \gamma'_j \circ \Phi'^{-1} &= \delta_j\gamma_j \circ \Phi^{-1} \circ \begin{pmatrix} \delta_3^{-1} & 0 \\ 0 & \delta_3^{-1} \end{pmatrix} \\ &= \delta_j(\beta_j\rho - \alpha_j\pi) \circ \begin{pmatrix} \delta_3^{-1} & 0 \\ 0 & \delta_3^{-1} \end{pmatrix} \\ &= \delta_j\beta_j\delta_3^{-1}\rho - \delta_j\alpha_j\delta_3^{-1}\pi; \end{aligned}$$

thus

$$r(\gamma'_1, \dots, \gamma'_k) \sim r(\gamma_1, \dots, \gamma_k)$$

modulo simultaneous conjugation.

$$(2) \text{ The section } i \text{ induces a section } i_* : (\mathrm{GL}_2^{k-3})^* / \mathrm{simconj} \longrightarrow A_{4,k}^{\mathrm{ord}}(\mathbb{R}):$$

Let $(\alpha_4, \dots, \alpha_k) \in (\mathrm{GL}_2^{k-3})^*$, $\delta \in \mathrm{GL}_2$, $(\alpha'_4, \dots, \alpha'_k) = (\delta\alpha_4\delta^{-1}, \dots, \delta\alpha_k\delta^{-1})$. We calculate:

$$\begin{aligned} i(\alpha'_4, \dots, \alpha'_k) &= (\pi, \rho, \rho - \pi, \rho - \delta\alpha_4\delta^{-1}\pi, \dots, \rho - \delta\alpha_k\delta^{-1}\pi) \\ &= (\delta\pi, \delta\rho, \delta(\rho - \pi), \delta\rho - \delta\alpha_4\pi, \dots, \delta\rho - \delta\alpha_k\pi) \circ \begin{pmatrix} \delta^{-1} & 0 \\ 0 & \delta^{-1} \end{pmatrix} \\ &\sim_{\mathrm{mod}(\mathrm{GL}_2^k \times \mathrm{GL}_4)} i(\alpha_4, \dots, \alpha_k). \end{aligned}$$

$$(3) \quad i_* \circ r_* = id_{A_{4,k}^{\text{ord}}(\mathbb{R})}:$$

With notation as in 1, we obtain:

$$\begin{aligned} i \circ r(\gamma_1, \dots, \gamma_k) &= (\pi, \rho, \rho - \pi, \rho - \beta_4^{-1} \alpha_4 \pi, \dots, \rho - \beta_k^{-1} \alpha_k \pi) \\ \sim_{\text{mod GL}_2^k} (-\alpha^{-1} \pi, \beta^{-1} \rho, \rho - \pi, \beta_4 \rho - \alpha_4 \pi, \dots, \beta_k \rho - \alpha_k \pi) &= (\gamma_1, \dots, \gamma_k) \circ \Phi^{-1} \\ \sim_{\text{mod GL}_4} (\gamma_1, \dots, \gamma_k). \end{aligned} \quad \square$$

3. CENTRAL 2-ARRANGEMENTS OF 4 PLANES IN $\mathbb{R}^4$

As a first step towards Theorem 1 we prove

Lemma 4.

$$A_{4,4}^{\text{ord}}(\mathbb{R}) \cong \text{GL}_2^*/\text{conjugation} \cong \mathbb{PR}_2^* \dot{\cup} Q^*.$$

Proof. The first isomorphism is from Theorem 3. For the second, consider $d : \text{GL}_2^* \rightarrow \mathbb{PR}_2^*$, $\alpha \mapsto [\det \alpha, \det(\alpha - \text{id}), \det \text{id}]$ and the restriction of the characteristic map on $M(2 \times 2; \mathbb{R})$ to $\text{GL}_2^* : \chi : \text{GL}_2^* \rightarrow X := \{(x, y) \in \mathbb{R}^2 \mid x + y + 1 \neq 0, y \neq 0\}$, $\alpha \mapsto (-\text{trace } \alpha, \det \alpha)$. Note that there is an isomorphism $\phi : X \xrightarrow{\cong} \mathbb{PR}_2^*$ given by $(x, y) \mapsto [y, x + y + 1, 1]$ such that $d = \phi \circ \chi$.

It is known (see [3], pp. 161ff) that under a change of coordinates the fibers of χ are hyperboloids, which are singular if and only if the discriminant $\Delta = \det \alpha - (\text{trace } \alpha)^2/4$ vanishes. Furthermore, regular fibers are equivalence classes modulo conjugation, while singular fibers consist of two of them: the singular point of the cone is one and the rest makes up another. The assertion follows from the equivalence $\det \alpha - (\text{trace } \alpha)^2/4 = 0 \Leftrightarrow d(\alpha) \in Q \subset \mathbb{PR}_2^*$. $\square$

A structure of a $\mathbb{C}$ -vector space on $\mathbb{R}^4$ that extends the $\mathbb{R}$ -vector space structure is uniquely determined by an automorphism $J \in \text{GL}_4$ such that $J^2 = -\text{id}$, representing scalar multiplication with $i \in \mathbb{C}$. The standard $\mathbb{C}$ -vector space structure on $\mathbb{R}^4$ is given by the matrix $J = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$, where $I = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

We call a k -tuple $\mathcal{H} = (H_1, \dots, H_k)$ of planes complex if there exists a $\mathbb{C}$ -vector space structure on $\mathbb{R}^4$ extending the $\mathbb{R}$ -vector space structure, such that $\mathcal{H}$ is complex with respect to this $\mathbb{C}$ -vector space structure, i.e., $J(H_i) = H_i$, $i = 1, \dots, k$.

Note that an equivalence class $[(H_1, \dots, H_k)] \in A_{4,k}^{\text{ord}}(\mathbb{R})$ contains a complex element if and only if it contains an element that is complex with respect to any fixed complex structure. This is because for any $J_i \in \text{GL}_4$ such that $J_i^2 = -\text{id}$, $i = 1, 2$, there exists $\phi \in \text{GL}_4$ such that $J_2 = \phi J_1 \phi^{-1}$, so that if $\mathcal{H}$ is complex with respect to J_1 , so is $\phi(\mathcal{H})$ with respect to J_2 .

We define $A_{2,k}^{\text{ord}}(\mathbb{C})$ to be the set of equivalence classes in $A_{4,k}^{\text{ord}}(\mathbb{R})$ that contain a complex element.

Lemma 5. *The isomorphism from Lemma 4 induces an isomorphism*

$$A_{2,4}^{\text{ord}}(\mathbb{C}) \cong Q_{<} \subset \mathbb{PR}_2^* \dot{\cup} Q^*.$$

Proof. Let $\mathcal{H} = (H_1, \dots, H_4)$ be complex with respect to the standard $\mathbb{C}$ -vector space structure. It follows that there are $\gamma_1, \dots, \gamma_4 \in \text{Hom}(\mathbb{C}^2, \mathbb{C})$ of complex rank 1 such that $H_i = \ker \gamma_i$, $i = 1, \dots, 4$ and that $r(\mathcal{H})$ represents a complex number. The assertion follows from the fact (see for example [3], p. 234) that $\alpha \in \text{GL}_2$ is equivalent modulo conjugation to an element representing a complex number if and only if it is either a multiple of the identity or its discriminant Δ satisfies $\Delta > 0$. $\square$

We are now ready to prove Theorem 1.

Proof. Since $A_{4,4}(\mathbb{R}) = A_{4,4}(\mathbb{R})^{\text{ord}}/S_4$, where S_4 acts on $A_{4,4}(\mathbb{R})^{\text{ord}}$ by permutation of the 4 planes, we need to find the induced operation of S_4 on $\text{GL}_2^*/\text{conj}$ under the isomorphism $r_* : A_{4,4}(\mathbb{R})^{\text{ord}} \xrightarrow{\cong} \text{GL}_2^*/\text{conj}$. For this, let $[\alpha] \in \text{GL}_2^*/\text{conj}$, $\mathcal{H} = (\ker\pi, \ker\rho, \ker(\rho - \pi), \ker(\rho - \alpha\pi))$. We calculate $r_*([\sigma(\mathcal{H})])$ for $\sigma \in \{(1, 2), (2, 3), (3, 4)\}$.

- (1) $\sigma = (1, 2)$:
 $\sigma(\mathcal{H}) = (\ker\gamma_1, \dots, \ker\gamma_4)$ with

$$\gamma_1 = \rho, \gamma_2 = \pi, \gamma_3 = \rho - \pi, \gamma_4 = \rho - \alpha\pi.$$

Definition 2.1 of $r : (\mathcal{L}_{4,2}^4)^* \rightarrow \text{GL}_2^*$ leads to:

$$\gamma_3 = \text{id}\gamma_1 + (-\text{id})\gamma_2, \Phi^{-1} = \Phi = \begin{pmatrix} -\rho \\ -\pi \end{pmatrix}, \gamma_4 \circ \Phi^{-1} = \alpha\rho - \pi;$$

therefore

$$r((\gamma_1, \dots, \gamma_4)) = \alpha^{-1}.$$

- (2) $\sigma = (2, 3)$:
 $\sigma(\mathcal{H}) = (\ker\gamma_1, \dots, \ker\gamma_4)$ with

$$\gamma_1 = \pi, \gamma_2 = \rho - \pi, \gamma_3 = \rho, \gamma_4 = \rho - \alpha\pi.$$

We have

$$\gamma_3 = \text{id}\gamma_1 + \text{id}\gamma_2, \Phi^{-1} = \Phi = \begin{pmatrix} -\pi \\ \rho - \pi \end{pmatrix}, \gamma_4 \circ \Phi^{-1} = \rho - (\text{id} - \alpha)\pi;$$

therefore

$$r((\gamma_1, \dots, \gamma_4)) = \text{id} - \alpha.$$

- (3) $\sigma = (3, 4)$:
 $\sigma(\mathcal{H}) = (\ker\gamma_1, \dots, \ker\gamma_4)$ with

$$\gamma_1 = \pi, \gamma_2 = \rho, \gamma_3 = \rho - \alpha\pi, \gamma_4 = \rho - \pi.$$

We have

$$\gamma_3 = -\alpha\gamma_1 + \text{id}\gamma_2, \Phi = \begin{pmatrix} \alpha\pi \\ \rho \end{pmatrix}, \Phi^{-1} = \begin{pmatrix} \alpha^{-1}\pi \\ \rho \end{pmatrix}, \gamma_4 \circ \Phi^{-1} = \rho - \alpha^{-1}\pi;$$

therefore

$$r((\gamma_1, \dots, \gamma_4)) = \alpha^{-1}.$$

The assertion now follows from (for the definition of the map $d : \text{GL}_2^* \rightarrow \mathbb{PR}_2^*$, see proof of Lemma 4):

- (1)

$$\begin{aligned} d(\alpha^{-1}) &= [\det \alpha^{-1}, \det(\alpha^{-1} - \text{id}), \det \text{id}] \\ &= [\det \text{id}, \det(\text{id} - \alpha), \det \alpha] \\ &= [\det \text{id}, \det(\alpha - \text{id}), \det \alpha] \\ &= \sigma_{x,z}(d(\alpha)), \end{aligned}$$

where $\sigma_{x,z}$ interchanges the x and z coordinates in $\mathbb{PR}_2^*$, and

(2)

$$\begin{aligned}
d(\text{id} - \alpha) &= [\det(\text{id} - \alpha), \det((\text{id} - \alpha) - \text{id}), \det \text{id}] \\
&= [\det(\alpha - \text{id}), \det \alpha, \det \text{id}] \\
&= \sigma_{x,y}(d(\alpha)),
\end{aligned}$$

where $\sigma_{x,y}$ interchanges the x and y coordinates in $\mathbb{PR}_2^*$. $\square$

Finally, we prove Corollary 2.

Proof. There are 4 connected components of $\mathbb{PR}_2^* \dot{\cup} Q^*$, namely,

$$A_{\epsilon_1, \epsilon_2} := \{(x, y, z) \in \mathbb{PR}_2^* \dot{\cup} Q^* \mid xz \epsilon_1 0, yz \epsilon_2 0\}, (\epsilon_1, \epsilon_2) \in \{<, >\}^2.$$

Since $A_{>, >}$ is invariant under permutations of the coordinates x, y, z while the other three are identified by them, it follows that $A_{4,4}(\mathbb{R})$ consists of two connected components. A result by Randell [7] asserts that the homotopy types of complements of arrangements are constant on connected components. On the other hand, Ziegler presents in [8] two such arrangements with differing cohomology algebras, and consequently, differing homotopy types of their complements. This proves the assertion. $\square$

4. ACKNOWLEDGMENT

I am indebted to Professor Friedrich Wilhelm Knöller for introducing me to the subject and for numerous helpful discussions.

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FACHBEREICH MATHEMATIK UND INFORMATIK, UNIVERSITÄT MARBURG, 35032 MARBURG, GERMANY

E-mail address: batzies@web.de