ON UNBOUNDEDNESS OF MAXIMAL OPERATORS FOR DIRECTIONAL HILBERT TRANSFORMS

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ABSTRACT. We show that for any infinite set of unit vectors U in $\mathbb{R}^2$ the maximal operator defined by

$$H_U f(x) = \sup_{u \in U} \left| \text{pv} \int_{-\infty}^{\infty} \frac{f(x - tu)}{t} dt \right|, \quad x \in \mathbb{R}^2,$$

is not bounded in $L^2(\mathbb{R}^2)$.

1. INTRODUCTION

For a rapidly decreasing function f and a unit vector $u = (\cos \theta, \sin \theta)$, $\theta \in [0, 2\pi]$, we define

$$H_u f(x, y) = \text{pv} \int_{-\infty}^{\infty} \frac{f(x - t \cos \theta, y - t \sin \theta)}{t} dt,$$

which is the one-dimensional Hilbert transform along the direction u . It is well known that this operator can be extended to a bounded operator from $L^p(\mathbb{R}^2)$ to itself when $1 < p < \infty$. In this paper we study operators

$$(1) \quad H_U f(x, y) = \sup_{u \in U} |H_u f(x, y)|,$$

where U is a set of unit vectors u in $\mathbb{R}^2$. Analogous operators for the maximal functions are properly investigated. The case of lacunary U was first considered in the papers [18], [7], [15]. A final result was obtained by A. Nagel, E. M. Stein and S. Wainger in [15]. They proved the boundedness of the norms of these operators in L^p , $1 < p < \infty$, for a lacunary U . Upper bounds of such operators depending on the cardinality $\#U$ of the set U are considered in the papers [4], [5], [6], [13], [14], [19], and the definitive estimates are due to N. Katz ([13], [14]), where he obtained a logarithmic order for the norms of two different maximal operators depending on $\#U$. Various generalizations of these results were considered in series of papers ([1], [2], [3], [10], [17]).

As for the operator (1) there were no results except for the bound

$$\|H_U f(x)\|_{L^2} \lesssim \log \#U \|f\|_{L^2}.$$

This is an immediate consequence of the Menshov-Rademacher theorem (see [9] or [12]), in spite of the fact that in Katz's theorems a subtle range of ideas are used.

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It was not even known whether or not H_U is bounded in L^2 for an infinite lacunary set U . The main result of the paper is unboundedness of H_U in L^2 for any infinite U .

Theorem 1. *For any infinite set of unit vectors U the operator H_U cannot be extended to a bounded operator from $L^2(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$ for all $1 \leq p < \infty$.*

This theorem is an immediate consequence of the following estimate.

Theorem 2. *If U is a finite set and $1 \leq p < \infty$, then $\|H_U\|_{L^2 \rightarrow L^p} \geq c\sqrt{\log \#U}$, where $c > 0$ is an absolute constant.*

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2. PROOF OF THE THEOREMS

Let $f_n(x)$, $n = 1, 2, \dots, 2^m - 1$ ($f_n \not\equiv 0$), be a system of functions defined on the square

$$Q = [-\pi, \pi] \times [-\pi, \pi].$$

In some places we shall use for $f_n(x)$ *double numbering*, defined by

$$f_j^{(k)}(x) = f_n(x), \quad n = 2^k + j - 1, \quad 1 \leq j \leq 2^k, \quad k = 0, 1, \dots, m-1.$$

We shall say the sequence $f_n(x) = f_j^{(k)}(x)$ is a *tree-system*, if

$$(2) \quad \text{supp } f_{2j-1}^{(k+1)} \subset \{x \in Q : f_j^{(k)}(x) > 0\}, \quad \text{supp } f_{2j}^{(k+1)} \subset \{x \in Q : f_j^{(k)}(x) < 0\}.$$

Applying (2) several times we get

$$\begin{aligned} \text{supp } f_i^{(k+r)} \subset \{x \in Q : f_j^{(k)}(x) > 0\} &\Leftrightarrow i \in (2^r j - 2^r, 2^r j - 2^{r-1}], \\ \text{supp } f_i^{(k+r)} \subset \{x \in Q : f_j^{(k)}(x) < 0\} &\Leftrightarrow i \in (2^r j - 2^{r-1}, 2^r j], \end{aligned}$$

and then

$$(3) \quad \text{supp } f_i^{(k+r)} \cap \{x \in Q : f_j^{(k)}(x) > 0\} = \emptyset \quad \Leftrightarrow i \notin (2^r j - 2^r, 2^r j - 2^{r-1}],$$

$$(4) \quad \text{supp } f_i^{(k+r)} \cap \{x \in Q : f_j^{(k)}(x) < 0\} = \emptyset \quad \Leftrightarrow i \notin (2^r j - 2^{r-1}, 2^r j].$$

The following lemma for the Haar system has been proved by E. M. Nikishin and P. L. Ul'yanov [16], and we use the same idea to prove a general one (see also [12]).

Lemma 1. *If $f_n(x)$, $x \in Q$, $n = 1, 2, \dots, 2^m - 1$, is tree-system, then there exists a permutation σ of the numbers $\{1, 2, \dots, 2^m - 1\}$ such that*

$$\sup_{1 \leq l < 2^m} \left| \sum_{n=1}^l f_{\sigma(n)}(x) \right| \geq \frac{1}{3} \sum_{n=1}^{2^m-1} |f_n(x)|.$$

Proof. We connect with each $f_n(x)$, $n = 2^k + j - 1$, a number

$$t_n = \frac{2j-1}{2^{k+1}} \in [0, 1].$$

Note they are not equal for different n 's. Define the permutation σ so that

$$t_{\sigma(1)} < t_{\sigma(2)} < \dots < t_{\sigma(2^m-1)}.$$

We shall prove that for any $x \in Q$ there exists a number $l = l(x)$ with

$$(5) \quad f_{\sigma(n)}(x) \geq 0, \text{ if } n > l(x),$$

$$(6) \quad f_{\sigma(n)}(x) \leq 0, \text{ if } n \leq l(x).$$

Defining

$$l = l(x) = \sup\{n : 1 \leq n < 2^m, f_{\sigma(n)}(x) \leq 0\},$$

we shall have (6) immediately and if $\nu = \sigma(l(x) + 1)$, then

$$(7) \quad f_\nu(x) > 0.$$

To prove (5) it is enough to show that if (7) holds and $t_n > t_\nu$, then $f_n(x) \geq 0$. Suppose

$$\nu = 2^k + j - 1, \quad n = 2^s + i - 1.$$

According to the assumption

$$(8) \quad t_\nu = \frac{2j-1}{2^{k+1}} < \frac{2i-1}{2^{s+1}} = t_n.$$

If $s > k$, then $s = k + r$ ($r > 0$). From (8) we get

$$i > 2^r j - 2^{r-1}.$$

Therefore by (3) we obtain

$$\text{supp } f_n \cap \{f_\nu > 0\} = \emptyset.$$

Since $f_\nu(x) > 0$ we get $f_n(x) = 0$.

If $s < k$, then $k = s + r$ ($r > 0$). Applying (8) we get

$$j \leq 2^r i - 2^{r-1}.$$

Hence by (4) we have

$$\text{supp } f_\nu \cap \{f_n < 0\} = \emptyset.$$

By (7) we conclude $f_n(x) \geq 0$. So (5) and (6) are proved. Using them, we obtain

$$\max_{1 \leq l < 2^m} \left| \sum_{n=1}^l f_{\sigma(n)}(x) \right| = \max \left\{ - \sum_{n=1}^{l(x)} f_{\sigma(n)}(x), \left| \sum_{n=1}^{2^m-1} f_{\sigma(n)}(x) \right| \right\}.$$

If

$$- \sum_{n=1}^{l(x)} f_{\sigma(n)}(x) = \sum_{n=1}^{l(x)} |f_{\sigma(n)}(x)| < \frac{1}{3} \sum_{n=1}^{2^m-1} |f_{\sigma(n)}(x)|,$$

then we get

$$\sum_{n=l(x)+1}^{2^m-1} |f_{\sigma(n)}(x)| > 2 \sum_{n=1}^{l(x)} |f_{\sigma(n)}(x)|$$

and therefore

$$\begin{aligned} \left| \sum_{n=1}^{2^m-1} f_{\sigma(n)}(x) \right| &\geq \left| \sum_{n=l(x)+1}^{2^m-1} f_{\sigma(n)}(x) \right| - \left| \sum_{n=1}^{l(x)} f_{\sigma(n)}(x) \right| \\ &> \frac{1}{2} \sum_{n=l(x)+1}^{2^m-1} |f_{\sigma(n)}(x)| > \frac{1}{3} \sum_{n=1}^{2^m-1} |f_{\sigma(n)}(x)|. \end{aligned}$$

Thus we conclude

$$\sup_{1 \leq l < 2^m} \left| \sum_{n=1}^l f_{\sigma(n)}(x) \right| > \frac{1}{3} \sum_{n=1}^{2^m-1} |f_{\sigma(n)}(x)| = \frac{1}{3} \sum_{n=1}^{2^m-1} |f_n(x)|. \quad \square$$

Fix a Schwartz function $\phi(x)$ with

$$(9) \quad \phi(x) > 0, \quad \int_{\mathbb{R}} \phi(x) dx = 1, \quad \text{supp } \widehat{\phi} \subset [-1, 1].$$

We consider operators

$$(10) \quad \Phi_n(f) = \Phi_n(x, y, f) = n^2 \int_{\mathbb{R}} f(x-t, y-s) \phi(nt) \phi(ns) dt ds, \quad n = 1, 2, \dots$$

Applying (9), for any $f \in L^\infty(\mathbb{R}^2)$ we get

$$(11) \quad \inf_{(x,y) \in \mathbb{R}^2} f(x, y) \leq \Phi_n(x, y, f) \leq \sup_{(x,y) \in \mathbb{R}^2} f(x, y),$$

$$(12) \quad \text{supp } \widehat{\Phi_n(f)} \subset [-n, n] \times [-n, n].$$

If in addition f is compactly supported, using a standard argument, we conclude

$$(13) \quad \|\Phi_n(f) - f\|_{L^2} \rightarrow 0, \text{ as } n \rightarrow \infty.$$

If

$$(14) \quad n = 2^k + j - 1, \quad 1 \leq j \leq 2^k, \quad k = 0, 1, \dots, m-1,$$

then we denote

$$\bar{n} = 2^{k-1} + \left\lfloor \frac{j+1}{2} \right\rfloor,$$

where $[\cdot]$ means an integer part of a number. Using this notation we may write the conditions (2) by

$$(15) \quad \text{supp } f_n \subset \{(-1)^{j+1} \cdot f_{\bar{n}} > 0\}.$$

We shall consider sectors defined by

$$\{(x, y) \in \mathbb{R}^2 : x + iy = re^{i\theta}, r \geq 0, \alpha \leq \theta \leq \beta\}$$

where $0 \leq \alpha < \beta \leq 2\pi$. Some arguments in the proof of following lemma are derived from the paper [11].

Lemma 2. *Let S_n , $n = 1, 2, \dots, \nu = 2^m - 1$, be sectors on the plane. Then there exist functions $f_n \in L^2(\mathbb{R}^2)$, $n = 1, 2, \dots, \nu$, such that*

$$(16) \quad \text{supp } \widehat{f_n} \subset S_n,$$

$$(17) \quad \sum_{j=1}^{\nu} \|f_j\|_{L^2}^2 \leq c_1,$$

$$(18) \quad |\{(x, y) \in Q : \max_{1 \leq n \leq \nu} |\sum_{j=1}^n f_{\sigma(j)}(x, y)| > c_3 \sqrt{\log \nu}\}| > c_2,$$

where σ is the permutation from Lemma 1 and all the constants are absolute.

Proof. We will assume (14) everywhere below. For a given $\varepsilon > 0$ define sets $E_n = E_j^{(k)} \subset Q$ ($E_1 = E_1^{(0)} = Q$), $g_n \in L^\infty(\mathbb{R}^2)$ and $p_n, q_n \in \mathbb{Z}$ with conditions

- a) $E_n = \{(x, y) \in E_{\bar{n}} : (-1)^{j+1} \cos(p_{\bar{n}}x + q_{\bar{n}}y) > 0\}$, $n = 2, 3, \dots, \nu$,
- b) $0 \leq g_n \leq 1$, $\|g_n - \mathbb{I}_{E_n}\|_{L^2} \leq \varepsilon$, $n = 1, 2, \dots, \nu$,
- c) $\text{supp } \widehat{g_n} \subset S_n - (p_n, q_n)$, $n = 1, 2, \dots, \nu$,
- d) $\int_{E_n} |\cos(p_nx + q_ny)| dx dy > \frac{|E_n|}{3}$.

We do it by induction. Take $E_1 = E_1^{(0)} = Q$. According to (13) there exists $l > 0$ with

$$\|\Phi_l(\mathbb{I}_{E_1}) - \mathbb{I}_{E_1}\|_{L^2} < \varepsilon.$$

Define $g_1 = \Phi_l(\mathbb{I}_{E_1})$ and then applying (11) we get b) for $n = 1$. We note that if E is a measurable set then

$$\begin{aligned} (19) \quad \int_E |\cos(px + qy)| dx dy &\geq \int_E \cos^2(px + qy) dx dy \\ &= \frac{|E|}{2} + \int_E \frac{\cos(2(px + qy))}{2} dx dy \rightarrow \frac{|E|}{2} \text{ as } |p|, |q| \rightarrow \infty. \end{aligned}$$

This observation shows that for sufficiently large $p_1 = p$ and $q_1 = q$ we shall have condition d) for $n = 1$. On the other hand by (12) $\text{supp } \hat{g}_1$ is bounded. Thus for an appropriate p_1, q_1 we will have also c) (with $n = 1$). Certainly we can choose p_1 and q_1 common for both conditions c) and d). Now we suppose that the conditions a)–d) hold for any $k < n$, in particular for $\bar{n}$. We define E_n by the equality in a). Then we choose the positive integer l with

$$\|\Phi_l(\mathbb{I}_{E_n}) - \mathbb{I}_{E_n}\|_{L^2} < \varepsilon.$$

and denote $g_n = \Phi_l(\mathbb{I}_{E_n})$. Again applying (19) and using the boundedness of $\text{supp } \hat{g}_n$ we may choose integers p_n, q_n satisfying c) and d) together. Using condition a), it is easy to check that the sets E_n satisfies the conditions

$$\begin{aligned} E_j^{(k)} \cap E_{j'}^{(k)} &= \emptyset, \text{ if } j \neq j', \\ E_{2j-1}^{(k+1)} \cup E_{2j}^{(k+1)} &\subset E_j^{(k)}, \quad |E_j^{(k)} \setminus (E_{2j-1}^{(k+1)} \cup E_{2j}^{(k+1)})| = 0. \end{aligned}$$

Using this properties we conclude

$$(20) \quad \sum_{n=1}^{\nu} \mathbb{I}_{E_n}(x, y) = m, \text{ almost everywhere on } Q.$$

Now we define

$$(21) \quad f_n(x, y) = \frac{e^{i(p_n x + q_n y)} g_n(x, y)}{\sqrt{m}}.$$

Condition (16) immediately follows from c), because

$$\text{supp } \hat{f}_n = \text{supp } \hat{g}_n + (p_n, q_n).$$

On the other hand taking a small ε by b) and (20) we obtain

$$\sum_{n=1}^{\nu} \int_Q |f_n|^2 = \frac{1}{m} \sum_{n=1}^{\nu} \int_Q |g_n|^2 \leq \frac{2\nu}{m} \varepsilon^2 + \frac{2}{m} \sum_{n=1}^{\nu} \int_Q \mathbb{I}_{E_n} \leq c_1,$$

which gives (17). Now consider functions

$$(22) \quad \tilde{f}_n = \text{Re } f_n \cdot \mathbb{I}_{E_n} = \frac{\cos(p_n x + q_n y) \cdot g_n(x, y) \cdot \mathbb{I}_{E_n}(x, y)}{\sqrt{m}}.$$

Applying b) and (21) we get

$$(23) \quad \|\tilde{f}_n - \text{Re } f_n\|_{L^2}^2 = \int_{\mathbb{R}^2 \setminus E_n} |\text{Re } f_n|^2 \leq \int_{\mathbb{R}^2 \setminus E_n} |f_n|^2 = \frac{1}{m} \int_{\mathbb{R}^2 \setminus E_n} |g_n|^2 \leq \frac{\varepsilon^2}{m}.$$

On the other hand we have

$$\begin{aligned}
 (24) \quad \max_{1 \leq n \leq N} \left| \sum_{j=1}^n f_{\sigma(j)} \right| &\geq \max_{1 \leq n \leq N} \left| \sum_{j=1}^n \operatorname{Re} f_{\sigma(j)} \right| \\
 &\geq \max_{1 \leq n \leq N} \left| \sum_{j=1}^n \tilde{f}_{\sigma(j)} \right| - \sum_{j=1}^{\nu} |\tilde{f}_j - \operatorname{Re} f_j|.
 \end{aligned}$$

From (23) we obtain

$$\left\| \sum_{j=1}^{\nu} |\tilde{f}_j - \operatorname{Re} f_j| \right\|_{L^2} \leq \frac{\nu \varepsilon}{\sqrt{m}}.$$

Therefore taking a small $\varepsilon > 0$ we can say that

$$(25) \quad \left| \left\{ (x, y) \in Q : \sum_{j=1}^{\nu} |\tilde{f}_j - \operatorname{Re} f_j| > 1 \right\} \right| \leq \delta,$$

for any given $\delta > 0$. From (24) and (25) we conclude that to prove (18) and also the lemma, it is enough to prove

$$(26) \quad |\{(x, y) \in Q : \max_{1 \leq n \leq \nu} \left| \sum_{j=1}^n \tilde{f}_{\sigma(j)}(x, y) \right| > c_3 \sqrt{\log \nu}\}| > c_2.$$

Let us show that $\tilde{f}_n$ is a tree-system, i.e. it satisfies (15). Since $g_n > 0$ from (22) we get that $\tilde{f}_n(x, y)$ and $\cos(p_n x + q_n y)$ have same sign in the set E_n . Therefore by a) we obtain

$$\begin{aligned}
 \operatorname{supp} \tilde{f}_n \subset E_n &= \{(x, y) \in E_n : (-1)^{j+1} \cos(p_n x + q_n y) > 0\} \\
 &= \{(x, y) \in E_n : (-1)^{j+1} \tilde{f}_n(x, y) > 0\} = \{(x, y) \in E_n : (-1)^{j+1} \tilde{f}_n(x, y) > 0\} \\
 &= \{(x, y) \in Q : (-1)^{j+1} \tilde{f}_n(x, y) > 0\}.
 \end{aligned}$$

Hence $\tilde{f}_n$ is a tree-system. So according to Lemma 1 we have

$$(27) \quad \max_{1 \leq n \leq N} \left| \sum_{j=1}^n \tilde{f}_{\sigma(j)}(x, y) \right| \geq \frac{1}{3} \sum_{j=1}^{\nu} |\tilde{f}_j(x, y)|.$$

From (21) and the conditions b) and d) we get

$$\begin{aligned}
 \int_Q |\tilde{f}_n| &= \frac{1}{\sqrt{m}} \int_{E_n} |g_n \cos(p_n x + q_n y)| dx dy \geq \frac{1}{\sqrt{m}} \int_{E_n} |\cos(p_n x + q_n y)| dx dy \\
 &\quad - \frac{1}{\sqrt{m}} \int_{E_n} |(1 - g_n) \cos(p_n x + q_n y)| dx dy \geq \frac{|E_n|}{3\sqrt{m}} - \frac{\varepsilon}{\sqrt{m}}.
 \end{aligned}$$

If we take $\varepsilon > 0$ to be small, then from (20) we obtain

$$\int_Q \sum_{j=1}^{\nu} |\tilde{f}_j| \geq \frac{1}{3\sqrt{m}} \sum_{n=1}^{\nu} |E_n| - \frac{\nu \varepsilon}{\sqrt{m}} \gtrsim \sqrt{m}.$$

Combining this and (27) we get

$$(28) \quad \int_Q \max_{1 \leq n \leq N} \left| \sum_{j=1}^n \tilde{f}_{\sigma(j)}(x, y) \right| \gtrsim \sqrt{m}.$$

On the other hand by (20), (21) and b) for any $(x, y) \in Q$ we have

$$(29) \quad \max_{1 \leq n \leq N} \left| \sum_{j=1}^n \tilde{f}_{\sigma(j)}(x, y) \right| \leq \sum_{j=1}^{\nu} |\tilde{f}_j(x, y)| \leq \frac{1}{\sqrt{m}} \sum_{j=1}^{\nu} \mathbb{I}_{E_n}(x, y) \leq \sqrt{m}.$$

From (28) and (29) follows (26). $\square$

Proof of Theorem 2. For any region $S \subset \mathbb{R}^2$ we denote

$$T_S f(x, y) = \int_S e^{i(\xi x + \eta y)} \widehat{f}(\xi, \eta) d\xi d\eta.$$

Since the multiplier for the Hilbert transform is $i \cdot \text{sign } x$, for any direction $u = (\cos \theta, \sin \theta)$ we have

$$\widehat{H_u} f(x, y) = i \cdot \text{sign}(x \cos \theta + y \sin \theta) \widehat{f}(x, y).$$

Thus we conclude

$$(30) \quad H_u f = i(2 \cdot T_{\Gamma_u} f - f)$$

where

$$\Gamma_u = \{(x, y) \in \mathbb{R}^2 : x \cos \theta + y \sin \theta \geq 0\}.$$

Denote

$$T_U f = \sup_{u \in U} |T_{\Gamma_u} f|.$$

Let $U = \{u_k = (\cos \theta_k, \sin \theta_k) : k = 1, 2, \dots, N\}$ be the set of directions from Theorem 2. Without loss of generality we can assume $\theta_k \in (0, \pi/2)$, $\theta_1 < \theta_2 < \dots < \theta_N$ and $N = 2^m$. According to (30), to prove Theorem 2 it is enough to prove that

$$(31) \quad \|T_U f\|_{L^1} \gtrsim \sqrt{\log N} \|f\|_{L^2}$$

for some function $f \in L^2(\mathbb{R}^2)$. We denote by S_k , $k = 1, 2, \dots, \nu = 2^m - 1$, the sectors obtained by the vectors $(u_k)^\perp = (\cos \theta_k, -\sin \theta_k)$ and $(u_{k+1})^\perp = (\cos \theta_{k+1}, -\sin \theta_{k+1})$, i.e.

$$(32) \quad S_k = \{(x, y) \in \mathbb{R}^2 : x \geq 0, x \cos \theta_k + y \sin \theta_k \geq 0, x \cos \theta_{k+1} + y \sin \theta_{k+1} \leq 0\}.$$

Hence if we suppose

$$\text{supp } \widehat{f} \subset \bigcup_{k=1}^{\nu} S_k,$$

then we can write

$$(33) \quad T_U f(x, y) = \sup_{1 \leq l \leq \nu} \left| \sum_{k=1}^l T_{S_k} f(x, y) \right|.$$

We define functions f_n satisfying the conditions of Lemma 2 corresponding to the sectors $S'_n = S_{\sigma^{-1}(n)}$ in (32). Denote

$$f = \sum_{k=1}^{\nu} f_k.$$

Since S_n are mutually disjoint the functions f_n are orthogonal. Thus by (17) we get $\|f\|_{L^2} \leq c_1$. From (16) we have $\text{supp } \widehat{f}_n \subset S_{\sigma^{-1}(n)}$, $n = 1, 2, \dots, \nu$, i.e. $\text{supp } \widehat{f}_{\sigma(n)} \subset S_n$ and therefore

$$T_{S_n} f(x) = f_n(x).$$

According to (33) we obtain

$$T_U f(x, y) = \max_{1 \leq l \leq N} \left| \sum_{j=1}^l f_{\sigma(j)}(x, y) \right|.$$

Using (17) and (18), we get

$$|\{(x, y) \in Q : T_U f(x, y) > c_3 \sqrt{\log \nu}\}| > c_2,$$

and therefore $\|T_U f\|_{L^p} \gtrsim \sqrt{\log N} \|f\|_{L^2}$. $\square$

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