A SMOOTH INVARIANT NOT A SMOOTH FUNCTION OF THE INVARIANT POLYNOMIALS

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ABSTRACT. An explicit example is given of a smooth function invariant under a linear group action that is not a smooth function of the invariant polynomials.

A deep theorem of SCHWARZ [1] asserts that a smooth function invariant under the linear action of a compact Lie group is a smooth function of the invariant polynomials. A simple example is given to show that this does not extend to non-compact groups, at least if the action is not proper.

Consider the motion of a free particle in the plane $\mathbb{R}^2$, with Cartesian coordinates x_1, x_2 . This is modelled by the Hamiltonian system with the phase space $P = T^*\mathbb{R}^2$ and the Hamiltonian $h = \frac{1}{2}(v_1^2 + v_2^2)$. The Hamiltonian flow is the linear action $(t, (x_1, x_2, v_1, v_2)) \rightarrow (x_1 + tv_1, x_2 + tv_2, v_1, v_2)$, which is two copies of the representation on $\mathbb{R}^2$ given by the matrix $t \rightarrow \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. Observe that not only are v_1, v_2 and $j := x_1v_2 - x_2v_1$ constants of motion, but are a basis for the invariant polynomials by the following lemma.

Lemma. *Suppose that f is a polynomial on P that is invariant under the Hamiltonian flow. Then f is a polynomial in the invariant polynomials v_1, v_2, j .*

Proof. Let f be invariant under the flow. Then expanding by Taylor's theorem and denoting by ∂_k differentiation with respect to x_k

$$\begin{aligned} f(x_1, x_2, v_1, v_2) &= f(x_1 + tv_1, x_2 + tv_2, v_1, v_2) \\ &= f(x_1, x_2, v_1, v_2) + t(v_1\partial_1 + v_2\partial_2)f(x_1, x_2, v_1, v_2) + t^2(\cdots). \end{aligned}$$

Since v_1, v_2 are irreducible elements and distinct, we may conclude that

$$v_1\partial_1f + v_2\partial_2f = 0.$$

Therefore, since we are in a unique factorization domain, it follows that

$$(1) \quad \partial_1f = -v_2p, \quad \partial_2f = v_1p$$

for some polynomial p . The key observation to make is that the operator $x_1\partial_1 + x_2\partial_2$ acts as scalar multiplication by n on a homogeneous polynomial in x_1, x_2 of degree

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n . Now write

$$f = \sum_{n \geq 0} f_n, \quad p = \sum_{n \geq 0} p_n$$

as homogeneous expansions in x_1, x_2 over $\mathbb{R}[v_1, v_2]$. Equation (1) implies that

$$(x_1 \partial_1 + x_2 \partial_2) f = (x_2 v_1 - x_1 v_2) p$$

or, expanding, that

$$(x_1 \partial_1 + x_2 \partial_2) \sum_{n \geq 0} f_n = \sum_{n \geq 1} n f_n = \sum_{n \geq 0} p_n (x_2 v_1 - x_1 v_2).$$

Comparing coefficients yields

$$f_n = \frac{p_{n-1}}{n} (x_2 v_1 - x_1 v_2)$$

so that

$$f = f_0 + (x_2 v_1 - x_1 v_2) \left(\frac{p_0}{1} + \frac{p_1}{2} + \cdots \right).$$

Since f , f_0 and $x_2 v_1 - x_1 v_2$ are all invariants, it follows that

$$\left(\frac{p_0}{1} + \frac{p_1}{2} + \cdots \right)$$

is as well. However, in x_1 and x_2 it is of degree one less. Hence, by induction, f is a polynomial in v_1, v_2 and j . $\square$

Now consider the function f on the phase space P defined by

$$f(x_1, x_2, v_1, v_2) = \frac{e^{-1/j^2}}{v_1^2 + v_2^2}$$

and $f = 0$ when $j = 0$. Since $j^2 \leq (x_1^2 + x_2^2)(v_1^2 + v_2^2)$, f is smooth on phase space. However, f , thought of as a function on the image of the invariants map $J : (x_1, x_2, v_1, v_2) \rightarrow (v_1, v_2, j)$, is not a smooth function as it has no continuous extension to any neighbourhood of the origin. Note that the image of J is all of $\mathbb{R}^3$ with the j axis removed and the origin stuck back in. It is the peculiar form of the image of J that explains why it is possible for the pullback $J^* f$ to be smooth.

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REFERENCES

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