EXTREMAL EXTENSIONS FOR THE SUM OF NONNEGATIVE SELFADJOINT RELATIONS

SEPPO HASSI, ADRIAN SANDOVICI, HENK DE SNOO, AND HENRIK WINKLER

(Communicated by Joseph A. Ball)

ABSTRACT. The sum $A + B$ of two nonnegative selfadjoint relations (multi-valued operators) A and B is a nonnegative relation. The class of all extremal extensions of the sum $A + B$ is characterized as products of relations via an auxiliary Hilbert space associated with A and B . The so-called form sum extension of $A + B$ is a nonnegative selfadjoint extension, which is constructed via a closed quadratic form associated with A and B . Its connection to the class of extremal extensions is investigated and a criterion for its extremality is established, involving a nontrivial dependence on A and B .

1. INTRODUCTION

Let A and B be nonnegative selfadjoint relations (multi-valued operators) in a Hilbert space $\mathfrak{H}$. Their operator-like sum $A + B$, defined as

$$(1.1) \quad A + B = \{ \{f, g + h\} : \{f, g\} \in A, \{f, h\} \in B \},$$

is a nonnegative relation in $\mathfrak{H}$. Here $\text{dom}(A + B) = \text{dom } A \cap \text{dom } B$ is not necessarily dense in $\mathfrak{H}$, even in the case that A and B are densely defined operators. Furthermore, (the closure of) $A + B$ need not be an operator; consequently all selfadjoint extensions of $A + B$ can be multi-valued. Standard results concerning (nonnegative) selfadjoint extensions of closed symmetric operators do not directly apply for studying selfadjoint extensions of $A + B$. The recent factorization approach developed in its most general setting in [9] turns out to be particularly useful for studying nonnegative selfadjoint extensions of $A + B$. The idea is based on certain factorizations of the extremal extensions via an auxiliary Hilbert space $\mathfrak{H}_S$ constructed from the initial, not necessarily closable, nonnegative operator (or relation) S . Then suitable closable operators $R : \mathfrak{H} \rightarrow \mathfrak{H}_S$ are defined such that R^*R^{**} extends S (see Theorem 2.3 below; cf. [9]). Since the methods needed here do not become essentially more involved, the most general situation is taken up by allowing the original A and B to be multi-valued. Moreover, the so-called form sum that can always be constructed for $A + B$ via the nonnegative quadratic forms

Received by the editors March 27, 2006 and, in revised form, June 15, 2006.

2000 *Mathematics Subject Classification*. Primary 47A57, 47B25; Secondary 47A55, 47B65.

Key words and phrases. Nonnegative selfadjoint relation, Friedrichs extension, Krein-von Neumann extension, extremal extension, form sum extension.

The fourth author was supported by the “Fond zur Förderung der wissenschaftlichen Forschung” (FWF, Austria), grant number P15540-N05.

©2007 American Mathematical Society
 Reverts to public domain 28 years from publication

associated with A and B will be analyzed as a nonnegative selfadjoint (relation) extension of $A + B$.

To outline some of the results more precisely consider the auxiliary Hilbert space $\mathfrak{H}_{A+B}$, which is obtained by providing the quotient space

$$\text{ran}(A + B)/\text{mul}(A + B)$$

with a suitable inner product and then completing it. Applying the recent results in [9] all extremal extensions of $A + B$ can then be factorized via $\mathfrak{H}_{A+B}$ in the form R^*R^{**} with some closable operator $R : \mathfrak{H} \rightarrow \mathfrak{H}_{A+B}$. In particular, the Kreĭn-von Neumann extension $(A + B)_N$ and the Friedrichs extension $(A + B)_F$ of $A + B$ can be factorized in this way. Here R need not be densely defined. Necessarily $\text{dom } R \subset \text{dom}(A + B)_N^{\frac{1}{2}}$, but also $\text{dom}(A + B)_N^{\frac{1}{2}}$ need not be dense in $\mathfrak{H}$, which reflects the fact that $A + B$ need not be positively closable. It is shown that the space $\mathfrak{H}_{A+B}$ can be identified as a closed subspace $\mathfrak{E}$ of the Cartesian product space $\mathfrak{H} \times \mathfrak{H}$. This leads to more explicit formulas for the class of extremal extensions; all of them are factorized via $\mathfrak{H} \times \mathfrak{H}$ with closable operators $T : \mathfrak{H} \rightarrow \mathfrak{H} \times \mathfrak{H}$ with $\text{ran } T \subset \mathfrak{E}$ (see Theorems 3.3, 3.4). The explicit factorizations constructed via the space $\mathfrak{H} \times \mathfrak{H}$ make it natural to study the above mentioned form sum extension and its relation to extremal extensions of $A + B$. It is shown that the form sum extension need not be extremal (see Example 5.2); a criterion for its extremality is established in Theorem 4.1, indicating a nontrivial dependence on A and B .

The factorization of nonnegative selfadjoint extensions of a nonnegative operator S over an auxiliary Hilbert space related to S is a procedure which goes back to Z. Sebestyén and J. Stochel [10], [11]. It was further developed in [1], [9]. The factorization of the form sum extension in the case that A , B , and $A + B$ are densely defined operators goes back to B. Farkas and M. Matolcsi [4], [5]; in the case that $A + B$ is a nondensely defined operator, see [8].

2. PRELIMINARY RESULTS

2.1. Some terminology. A linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$ is a linear subspace of the Cartesian product $\mathfrak{H} \times \mathfrak{K}$. The following self-explanatory notions domain, range, kernel, and multi-valued part of A will be used throughout the paper:

$$\begin{aligned} \text{dom } A &= \{f \in \mathfrak{H} : \{f, f'\} \in A\}, & \text{ran } A &= \{f' \in \mathfrak{K} : \{f, f'\} \in A\}, \\ \ker A &= \{f \in \mathfrak{H} : \{f, 0\} \in A\}, & \text{mul } A &= \{f' \in \mathfrak{K} : \{0, f'\} \in A\}. \end{aligned}$$

The closures of $\text{dom } A$ and $\text{ran } A$ in $\mathfrak{H}$ and $\mathfrak{K}$, respectively, will be denoted by $\overline{\text{dom } A}$ and $\overline{\text{ran } A}$. The formal inverse A^{-1} is defined as $A^{-1} = \{\{f', f\} : \{f, f'\} \in A\}$; it is a linear relation from $\mathfrak{K}$ to $\mathfrak{H}$. Observe the following formal identities $\text{dom } A^{-1} = \text{ran } A$ and $\ker A^{-1} = \text{mul } A$. The relation A is closed if it is closed as a subspace of $\mathfrak{H} \times \mathfrak{K}$; the closure of the relation A is the closure of the subspace A in $\mathfrak{H} \times \mathfrak{K}$. If A is closed, then the subspaces $\ker A$ and $\text{mul } A$ are closed. A linear relation A is the graph of an operator if and only if $\text{mul } A = \{0\}$. In the present context a linear operator A from $\mathfrak{H}$ to $\mathfrak{K}$ is identified with its graph. It is said to be closable if its closure is the graph of an operator.

The adjoint of a linear relation A from $\mathfrak{H}$ to $\mathfrak{K}$ is the closed linear relation A^* from $\mathfrak{K}$ to $\mathfrak{H}$ defined by

$$A^* = \{\{f, f'\} \in \mathfrak{K} \times \mathfrak{H} : (f', h) = (f, h') \text{ for all } \{h, h'\} \in A\}.$$

Observe that $(A^{-1})^* = (A^*)^{-1}$, so that $(\operatorname{dom} A)^\perp = \operatorname{mul} A^*$ and $(\operatorname{ran} A)^\perp = \ker A^*$. Clearly the double adjoint A^{**} is the closure of the relation A . A linear relation A in a Hilbert space $\mathfrak{H}$ (i.e., from $\mathfrak{H}$ to $\mathfrak{H}$) is said to be symmetric if $A \subset A^*$, or equivalently, if $(f', f) \in \mathbb{R}$ for all $\{f, f'\} \in A$. A linear relation A in a Hilbert space $\mathfrak{H}$ is said to be nonnegative if $(f', f) \geq 0$ for all $\{f, f'\} \in A$. A linear relation A in a Hilbert space $\mathfrak{H}$ is said to be selfadjoint if $A^* = A$ (so that it is automatically closed); it is said to be essentially selfadjoint if its closure is equal to A^* . For any closed linear relation A the relation $A_\infty = \{0\} \times \operatorname{mul} A$ is selfadjoint in the Hilbert space $\operatorname{mul} A$. A selfadjoint relation can always be orthogonally decomposed as $A = A_s \oplus A_\infty$, where A_s is a (densely defined) selfadjoint operator in $\mathfrak{H} \ominus \operatorname{mul} A = \overline{\operatorname{dom} A}$.

2.2. Some block representations with unbounded and multi-valued entries. Let $\mathfrak{H}$, $\mathfrak{K}$, $\mathfrak{H}_i$, and $\mathfrak{K}_i$, $i = 1, 2$, be Hilbert spaces. Let A be a linear relation from $\mathfrak{H}$ to $\mathfrak{K}_1$, let B be a linear relation from $\mathfrak{H}$ into $\mathfrak{K}_2$, and define a linear relation from $\mathfrak{H}$ into $\mathfrak{K}_1 \times \mathfrak{K}_2$ by

$$(2.1) \quad \operatorname{col}(A, B) := \begin{pmatrix} A \\ B \end{pmatrix} = \{ \{h, k_1 \oplus k_2\} : \{h, k_1\} \in A, \{h, k_2\} \in B \}.$$

Then the following relations hold:

$$\begin{aligned} \operatorname{dom} \operatorname{col}(A, B) &= \operatorname{dom} A \cap \operatorname{dom} B, \\ \ker \operatorname{col}(A, B) &= \ker A \cap \ker B, \\ \operatorname{ran} \operatorname{col}(A, B) &= \{ A(h) \times B(h) : h \in \operatorname{dom} A \cap \operatorname{dom} B \}, \\ \operatorname{mul} \operatorname{col}(A, B) &= \operatorname{mul} A \times \operatorname{mul} B. \end{aligned}$$

Here the image of $\mathfrak{M}$ under $A : \mathfrak{H} \rightarrow \mathfrak{K}$ is denoted by $A(\mathfrak{M}) = \{ k \in \mathfrak{K} : \{h, k\} \in A \text{ for some } h \in \mathfrak{M} \}$. Moreover, let C be a linear relation from $\mathfrak{H}_1$ to $\mathfrak{K}$, let D be a linear relation from $\mathfrak{H}_2$ into $\mathfrak{K}$, and define a linear relation from $\mathfrak{H}_1 \times \mathfrak{H}_2$ to $\mathfrak{K}$ by

$$(2.2) \quad \begin{pmatrix} C & D \end{pmatrix} = \{ \{h_1 \oplus h_2, k_1 + k_2\} : \{h_1, k_1\} \in C, \{h_2, k_2\} \in D \}.$$

Now the following relations hold:

$$\begin{aligned} \operatorname{dom} \begin{pmatrix} C & D \end{pmatrix} &= \operatorname{dom} C \times \operatorname{dom} D, \\ \ker \begin{pmatrix} C & D \end{pmatrix} &= \{ C^{-1}(k) \times D^{-1}(-k) : k \in \operatorname{ran} C \cap \operatorname{ran} D \}, \\ \operatorname{ran} \begin{pmatrix} C & D \end{pmatrix} &= \operatorname{ran} C + \operatorname{ran} D, \\ \operatorname{mul} \begin{pmatrix} C & D \end{pmatrix} &= \operatorname{mul} C + \operatorname{mul} D. \end{aligned}$$

Here $A^{-1}(\mathfrak{L})$ stands for the preimage of $\mathfrak{L}$ under $A : \mathfrak{H} \rightarrow \mathfrak{K}$:

$$A^{-1}(\mathfrak{L}) = \{ h \in \mathfrak{H} : \{h, k\} \in A \text{ for some } k \in \mathfrak{L} \}.$$

Proposition 2.1. *Let A , B , C , and D be relations as above. Then:*

(i) *the adjoint of the linear relation in (2.1) satisfies*

$$\begin{pmatrix} A \\ B \end{pmatrix}^* \supset (A^* \quad B^*);$$

(ii) *the adjoint of the linear relation in (2.2) satisfies*

$$\begin{pmatrix} C & D \end{pmatrix}^* = \begin{pmatrix} C^* \\ D^* \end{pmatrix};$$

(iii) *if A (or B) is bounded and densely defined with $\operatorname{dom} B \subset \operatorname{dom} A$ (with $\operatorname{dom} A \subset \operatorname{dom} B$, respectively), then equality holds in (i).*

Proof. (i) Let $\{f_1, g_1\} \in A^*$ and $\{f_2, g_2\} \in B^*$. Then

$$(g_1 + g_2, h) - (f_1 \oplus f_2, k_1 \oplus k_2) = (g_1, h) - (f_1, k_1) + (g_2, h) - (f_2, k_2) = 0$$

holds for all $\{h, k_1\} \in A$ and $\{h, k_2\} \in B$.

(ii) Let $\{f, g_1\} \in C^*$ and $\{f, g_2\} \in D^*$. Then

$$(g_1 \oplus g_2, h_1 \oplus h_2) - (f, k_1 + k_2) = (g_1, h_1) - (f, k_1) + (g_2, h_2) - (f, k_2) = 0$$

holds for all $\{h_1, k_1\} \in C$ and $\{h_2, k_2\} \in D$, which proves the inclusion “ $\supset$ ”.

Now assume that $\{f, g_1 \oplus g_2\} \in \mathfrak{K} \times (\mathfrak{H}_1 \oplus \mathfrak{H}_2)$ belongs to $(C \cup D)^*$, so that

$$(2.3) \quad (g_1 \oplus g_2, h_1 \oplus h_2) - (f, k_1 + k_2) = 0$$

is satisfied for all $\{h_1, k_1\} \in C$ and $\{h_2, k_2\} \in D$. By taking $h_1 = k_1 = 0$ it follows from (2.3) that $\{f, g_2\} \in D^*$ and similarly by taking $h_2 = k_2 = 0$ one obtains $\{f, g_1\} \in C^*$. This proves the reverse inclusion “ $\subset$ ”.

(iii) Assume that for instance B is bounded and densely defined. Then B^* is bounded and everywhere defined, and $\{f_1 \oplus f_2, g\} \in (\mathfrak{K}_1 \times \mathfrak{K}_2) \times \mathfrak{H}$ belongs to $(\text{col}(A, B))^*$ if and only if

$$0 = (g, h) - (f_1 \oplus f_2, k_1 \oplus k_2) = (g, h) - (f_1, k_1) - (B^* f_2, h) = (g - B^* f_2, h) - (f_1, k_1)$$

holds for all $\{h, k_1\} \in A$ and $\{h, k_2\} \in B$. Here $\text{dom } A \cap \text{dom } B = \text{dom } A$ and therefore $\{f_1, g - B^* f_2\} \in A^*$. This proves the reverse inclusion “ $\subset$ ” in (i). $\square$

The restriction of a relation $A : \mathfrak{H} \rightarrow \mathfrak{K}$ to the subspace $\mathfrak{M} \subset \mathfrak{H}$ is defined by

$$A \upharpoonright \mathfrak{M} = \{ \{h, k\} \in A : h \in \mathfrak{M} \}.$$

The adjoint of the restriction of A to $\mathfrak{M}$ satisfies $(A \upharpoonright \mathfrak{M})^* \supset P_{\mathfrak{M}} A^*$, where $P_{\mathfrak{M}}$ is the orthogonal projection onto the closure $\text{clos } \mathfrak{M}$ of $\mathfrak{M}$. In particular,

$$(2.4) \quad \text{if } \text{dom } A \subset \mathfrak{M}, \text{ then } (A \upharpoonright \mathfrak{M})^* = P_{\mathfrak{M}} A^*, \quad (A \upharpoonright \mathfrak{M})^{**} = A^{**} \upharpoonright \text{clos } \mathfrak{M}.$$

Here the identity for the adjoints is also a special case of (ii) in Proposition 2.1 with $D = \{0, 0\}$. If $P_{\mathfrak{L}}$ is an orthogonal projection onto a closed subspace $\mathfrak{L} \subset \mathfrak{K}$, then $(P_{\mathfrak{L}} A)^* = A^* P_{\mathfrak{L}}$ holds for any linear relation A . In particular, if $\text{ran } A \subset \mathfrak{L}$ or equivalently $A = P_{\mathfrak{L}} A$, then $A^* = A^* P_{\mathfrak{L}}$ and $A^{**} = P_{\mathfrak{L}} A^{**}$. In this case A can be viewed as a relation A_r from $\mathfrak{H}$ to $\mathfrak{L}$, in which case

$$A_r^* = A^* \upharpoonright \mathfrak{L}, \quad A_r^{**} = P_{\mathfrak{L}} A^{**}.$$

Here the identity for A_r^* is also obtained from part (iii) in Proposition 2.1 with $B = 0$ (i.e., $Bf = 0$ for all $f \in \mathfrak{H}$). These simple observations will be useful in the construction of the factorizations for nonnegative selfadjoint relations in view of

$$(2.5) \quad (A \upharpoonright \mathfrak{M})^{**} (A \upharpoonright \mathfrak{M})^* = A^{**} A^*, \quad A_r^* A_r^{**} = A^* A^{**}.$$

2.3. A factorization of extremal extensions. Let S be a nonnegative linear relation in a Hilbert space $\mathfrak{H}$. Provide the linear space $\text{ran } S$ with the semi-inner product $\langle \cdot, \cdot \rangle$ defined by

$$(2.6) \quad \langle f', g' \rangle := (f', g) = (f, g'), \quad \{f, f'\}, \{g, g'\} \in S.$$

Define the linear space $\mathfrak{R}_0 \subset \text{ran } S$ by

$$(2.7) \quad \mathfrak{R}_0 = \{ f' : (f', f) = 0 \text{ for some } \{f, f'\} \in S \}.$$

Then $\mathfrak{R}_0 = \text{ran } S \cap \text{mul } S^*$; cf. [9, Lemma 4.1]. Clearly, the quotient space $\text{ran } S / \mathfrak{R}_0$ is a pre-Hilbert space with the inner product

$$(2.8) \quad \langle [f'], [g'] \rangle := (f', g) = (f, g'), \quad \{f, f'\}, \{g, g'\} \in S,$$

where $[f']$, $[g']$ denote the equivalence classes containing f' and g' . Let $\mathfrak{H}_S$ be the Hilbert space completion of $\text{ran } S / \mathfrak{R}_0$, whose inner product is again denoted by $\langle \cdot, \cdot \rangle$. The linear relation Q from $\mathfrak{H}$ to $\mathfrak{H}_S$ is defined by

$$(2.9) \quad Q = \{ \{f, [f']\} : \{f, f'\} \in S \},$$

so that actually Q is (the graph of) an operator. Define the linear relation J from $\mathfrak{H}_S$ to $\mathfrak{H}$ by

$$(2.10) \quad J = \{ \{[f'], f'\} : \{f, f'\} \in S \},$$

so that $\text{mul } J = \mathfrak{R}_0$. Since J is densely defined in $\mathfrak{H}_S$, J^* is an operator. The definitions (2.9) and (2.10) imply that $J \subset Q^*$ and $Q \subset J^*$. In particular Q^{**} , the closure of Q , is an operator. Now one can state the following factorization result established in the general case in [9, Theorem 4.3]; in the case that S is densely defined, see [1, Proposition 3.1].

Theorem 2.2 ([1, 9]). *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$ and let J and Q be defined by (2.9) and (2.10). Then the Kreĭn-von Neumann extension S_N of S is given by $S_N = J^{**}J^*$ and the corresponding closed form $\mathfrak{t}_N$ is given by*

$$\mathfrak{t}_N[f, g] = \langle J^*f, J^*g \rangle, \quad f, g \in \text{dom } J^* = \text{dom } S_N^{\frac{1}{2}}.$$

Furthermore, the Friedrichs extension S_F of S is given by $S_F = Q^*Q^{**}$ and the corresponding closed form $\mathfrak{t}_F$ is given by

$$\mathfrak{t}_F[f, g] = \langle Q^{**}f, Q^{**}g \rangle, \quad f, g \in \text{dom } Q^{**} = \text{dom } S_F^{\frac{1}{2}}.$$

A nonnegative selfadjoint extension $\tilde{A}$ of S is called *extremal* when

$$\inf \{ (f' - h', f - h) : \{h, h'\} \in S \} = 0 \quad \text{for all } \{f, f'\} \in \tilde{A};$$

see [2]. The Kreĭn-von Neumann extension S_N and the Friedrichs extension S_F are extremal extensions. Let $\mathfrak{L}$ be any subspace such that

$$(2.11) \quad \text{dom } S \subset \mathfrak{L} \subset \text{dom } J^* = \text{dom } S_N^{\frac{1}{2}},$$

and associate with $\mathfrak{L}$ the restriction operator $R_{\mathfrak{L}}$ from $\mathfrak{H}$ to $\mathfrak{H}_S$ by

$$(2.12) \quad R_{\mathfrak{L}} := J^* \upharpoonright \mathfrak{L} = \{ \{f, f'\} \in J^* : f \in \mathfrak{L} \}.$$

Since J^* is a closed operator from $\mathfrak{H}$ to $\mathfrak{H}_S$, it is clear that $R_{\mathfrak{L}}$ is a closable operator. The definition of $R_{\mathfrak{L}}$ and Theorem 2.2 lead to

$$\langle R_{\mathfrak{L}}f, R_{\mathfrak{L}}g \rangle = \langle J^*f, J^*g \rangle = \mathfrak{t}_N[f, g], \quad f, g \in \mathfrak{L}.$$

Hence, $R_{\mathfrak{L}}$ is closed if and only if the restriction to $\mathfrak{L}$ of the form $\mathfrak{t}_N[\cdot, \cdot]$ is closed. Clearly, operators of the form $R_{\mathfrak{L}}$ induce nonnegative selfadjoint relations $R_{\mathfrak{L}}^*R_{\mathfrak{L}}^{**}$, and the corresponding closed nonnegative forms $\mathfrak{t}_{\mathfrak{L}}$ are given by

$$\mathfrak{t}_{\mathfrak{L}}[f, g] = \langle R_{\mathfrak{L}}^{**}f, R_{\mathfrak{L}}^{**}g \rangle \quad f, g \in \text{dom } R_{\mathfrak{L}}^{**}.$$

This yields the following useful characterization of extremal extensions of S ; again the general case is established in [9, Theorem 6.1] and the densely defined case in [1, Proposition 4.1]. For another approach see [7].

Theorem 2.3 ([1, 9]). *Let S be a nonnegative relation in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

- (i) $\tilde{A}$ is an extremal extension of S ;
- (ii) $\tilde{A} = R_{\mathfrak{L}}^* R_{\mathfrak{L}}^{**}$ for some subspace $\mathfrak{L}$ such that $\text{dom } S \subset \mathfrak{L} \subset \text{dom } S_N^{\frac{1}{2}}$;
- (iii) $\tilde{A}$ is a nonnegative selfadjoint extension of S whose corresponding form $\tilde{\mathfrak{t}}$ satisfies $\tilde{\mathfrak{t}} \subset \mathfrak{t}_N$.

As an example consider the purely multi-valued relation S in $\mathfrak{H}$ defined by

$$(2.13) \quad S = \{0\} \times \mathfrak{K},$$

where $\mathfrak{K}$ is a not necessarily closed subspace of the Hilbert space $\mathfrak{H}$. Then S is closed if and only if $\mathfrak{K}$ is closed. The adjoint S^* is given by

$$S^* = \mathfrak{K}^{\perp} \times \mathfrak{H},$$

so that $\text{mul } S^* = \mathfrak{H}$. Since S is nonnegative, the above construction is applicable and $\mathfrak{R}_0 = \text{ran } S = \mathfrak{K}$, $\mathfrak{H}_S = \{[0]\}$, $Q = \{[0, [0]]\}$, and $J = \{[0]\} \times \mathfrak{K}$, so that $Q^* = \{[0]\} \times \mathfrak{H}$, $Q^{**} = \{0\} \times \{[0]\}$, $J^* = \mathfrak{K}^{\perp} \times \{[0]\}$, and $J^{**} = \{[0]\} \times \overline{\mathfrak{K}}$. Therefore, according to Theorem 2.2, the Kreĭn-von Neumann extension $S_N = J^{**} J^*$ and the Friedrichs extension $S_F = Q^* Q^{**}$ are given by

$$(2.14) \quad S_N = \mathfrak{K}^{\perp} \times \overline{\mathfrak{K}}, \quad S_F = \{0\} \times \mathfrak{H};$$

cf. [3]. The form corresponding to S_N is the null form defined on all of $\mathfrak{K}^{\perp}$, and the form corresponding to S_F is the null form on $\{0\}$. This gives the following result.

Corollary 2.4. *Let the nonnegative relation S be given by (2.13). Then there exists a one to-one-correspondence between the class of all extremal extensions $\tilde{A}$ of S and the set of all closed subspaces $\mathfrak{L}$ of $\mathfrak{K}^{\perp}$. The correspondence is given by*

$$(2.15) \quad \tilde{A} = \mathfrak{L} \times \mathfrak{L}^{\perp},$$

and the associated form $\tilde{\mathfrak{t}}$ is the null form on $\mathfrak{L}$.

3. CONSTRUCTION OF EXTREMAL EXTENSIONS FOR THE SUM

3.1. Pairs of nonnegative selfadjoint relations. Let A and B be nonnegative selfadjoint relations in a Hilbert space $\mathfrak{H}$. Then the sum $A + B$ is the relation in $\mathfrak{H}$ defined by (1.1). Note that

$$(3.1) \quad \text{mul } (A + B) = \text{mul } A + \text{mul } B.$$

If A and B are decomposed as $A = A_s \oplus A_{\infty}$ and $B = B_s \oplus B_{\infty}$, where $A_{\infty} = \{0\} \times \text{mul } A$, $B_{\infty} = \{0\} \times \text{mul } B$, and A_s and B_s are densely defined nonnegative selfadjoint operators (defined as orthogonal complements in the graph sense), then the uniquely determined square roots of A and B are given by

$$A^{\frac{1}{2}} = A_s^{\frac{1}{2}} \oplus A_{\infty}, \quad B^{\frac{1}{2}} = B_s^{\frac{1}{2}} \oplus B_{\infty}.$$

Associate with A and B the relation Φ from $\mathfrak{H} \times \mathfrak{H}$ to $\mathfrak{H}$, defined by

$$(3.2) \quad \Phi = \left\{ \{f \oplus g, f' + g'\} : \{f, f'\} \in A^{\frac{1}{2}}, \{g, g'\} \in B^{\frac{1}{2}} \right\}.$$

Clearly, the domain and the multi-valued part of Φ are given by

$$(3.3) \quad \text{dom } \Phi = \text{dom } A^{\frac{1}{2}} \times \text{dom } B^{\frac{1}{2}}, \quad \text{mul } \Phi = \text{mul } A + \text{mul } B.$$

Lemma 3.1. *The adjoint Φ^* of Φ is the relation from $\mathfrak{H}$ to $\mathfrak{H} \times \mathfrak{H}$, given by*

$$(3.4) \quad \Phi^* = \left\{ \{h, k \oplus l\} : \{h, k\} \in A^{\frac{1}{2}}, \{h, l\} \in B^{\frac{1}{2}} \right\}.$$

Proof. Since $\Phi = \begin{pmatrix} A^{\frac{1}{2}} & B^{\frac{1}{2}} \end{pmatrix}$, (3.4) is obtained from part (ii) in Proposition 2.1. $\square$

Clearly, $\text{dom } \Phi^* = \text{dom } A^{\frac{1}{2}} \cap \text{dom } B^{\frac{1}{2}}$, $\text{mul } \Phi^* = \text{mul } A \times \text{mul } B$, and one can define the (orthogonal) operator part $(\Phi^*)_s$ of Φ^* by

$$(3.5) \quad (\Phi^*)_s = \begin{pmatrix} A_s^{\frac{1}{2}} \\ B_s^{\frac{1}{2}} \end{pmatrix} = \left\{ \left\{ h, A_s^{\frac{1}{2}} h \oplus B_s^{\frac{1}{2}} h \right\} : h \in \text{dom } A^{\frac{1}{2}} \cap \text{dom } B^{\frac{1}{2}} \right\}.$$

Now define the relation Ψ from $\mathfrak{H}$ to $\mathfrak{H} \times \mathfrak{H}$ by

$$(3.6) \quad \Psi = \left\{ \left\{ h, A_s^{\frac{1}{2}} h \oplus B_s^{\frac{1}{2}} h \right\} : h \in \text{dom } A \cap \text{dom } B \right\} \subset \mathfrak{H} \times (\mathfrak{H} \times \mathfrak{H}).$$

Then Ψ is an operator with $\text{dom } \Psi = \text{dom } A \cap \text{dom } B$. In fact $\Psi \subset (\Phi^*)_s$, so that Ψ is a closable operator. Denote by $\mathfrak{F}_0$ the range of $(\Phi^*)_s$ and let $\mathfrak{E}_0$ be the range of Ψ and let their closures in $\mathfrak{H} \times \mathfrak{H}$ be denoted by $\mathfrak{F}$ and $\mathfrak{E}$, respectively. Then

$$(3.7) \quad \mathfrak{E}_0 = \text{ran } \Psi \subset \text{ran } (\Phi^*)_s = \mathfrak{F}_0 \text{ and } \mathfrak{E} \subset \mathfrak{F}.$$

It follows from $\overline{\text{dom } \Psi^*} = (\text{mul } \Psi^{**})^\perp$ and $\text{mul } \Psi^* = (\text{dom } \Psi)^\perp$ that

$$\overline{\text{dom } \Psi^*} = \mathfrak{H}, \quad \text{mul } \Psi^* = (\text{dom } A \cap \text{dom } B)^\perp.$$

Next define the linear relation K from $\mathfrak{H} \times \mathfrak{H}$ to $\mathfrak{H}$ by

$$(3.8) \quad K = \left\{ \left\{ A_s^{\frac{1}{2}} f \oplus B_s^{\frac{1}{2}} f, f' + f'' \right\} : \left\{ A_s^{\frac{1}{2}} f, f' \right\} \in A^{\frac{1}{2}}, \left\{ B_s^{\frac{1}{2}} f, f'' \right\} \in B^{\frac{1}{2}} \right\}.$$

Clearly, the domain and the multi-valued part of K are given by

$$(3.9) \quad \text{dom } K = \mathfrak{E}_0, \quad \text{mul } K = \text{mul } (A + B).$$

Lemma 3.2. *The relations K , Φ , and Ψ satisfy the following inclusions:*

$$K \subset \Phi \subset \Psi^*, \quad \Psi \subset \Phi^* \subset K^*.$$

Proof. To see this note that $K \subset \Phi$ follows from (3.2) and (3.8), and that $\Psi \subset \Phi^*$ follows from (3.4) and (3.6). Therefore, also $\Phi^* \subset K^*$ and $\Phi \subset \Phi^{**} \subset \Psi^*$. $\square$

3.2. Factorization of extremal extensions via $\mathfrak{H} \times \mathfrak{H}$. Let A and B be non-negative selfadjoint relations and let $\mathfrak{R}_0$ be defined by (2.7) with $S = A + B$. Then $\mathfrak{R}_0 = \text{mul } (A + B)$, and the corresponding quotient space is

$$(3.10) \quad \text{ran } S/\mathfrak{R}_0 = \text{ran } (A + B)/\text{mul } (A + B),$$

with the inner product given by (2.6) with $S = A + B$. Denote by $\mathfrak{H}_{A+B}$ the completion of the space in (3.10) with respect to the norm $\|\cdot\|_{A+B}$ induced by the inner product (2.6) with $S = A + B$. Define the linear relation Z_0 by

$$(3.11) \quad Z_0 = \left\{ \left\{ A_s^{\frac{1}{2}} h \oplus B_s^{\frac{1}{2}} h, [h' + h''] \right\} : \{h, h'\} \in A, \{h, h''\} \in B \right\}.$$

Then

$$\left\| A_s^{\frac{1}{2}} h \oplus B_s^{\frac{1}{2}} h \right\| = \|[h' + h'']\|_{A+B}, \quad \{h, h'\} \in A, \{h, h''\} \in B,$$

so that the relation Z_0 is isometric from $\mathfrak{E}_0$ onto $\text{ran } (A + B)/\text{mul } (A + B)$; and, in fact, Z_0 is an operator. Hence the closure Z of Z_0 is a closed isometric operator from the Hilbert space $\mathfrak{E}$, the closure of $\mathfrak{E}_0$, onto the Hilbert space $\mathfrak{H}_{A+B}$.

Recall that Ψ in (3.6) acts from $\mathfrak{H}$ to $\mathfrak{H} \times \mathfrak{H}$ with $\text{ran } \Psi \subset \mathfrak{E} = \overline{\text{ran } \Psi}$. Hence Ψ can be viewed as an operator Ψ_e from $\mathfrak{H}$ to $\mathfrak{E}$, so that $\Psi_e \subset \Psi$ are the same sets, interpreted in different spaces. Since Z is an isometry from $\mathfrak{E}$ to $\mathfrak{H}_{A+B}$, the product $Z\Psi_e$ is well-defined and

$$(3.12) \quad Z\Psi_e = \left\{ \left\{ h, Z(A_s^{\frac{1}{2}}h \oplus B_s^{\frac{1}{2}}h) \right\} : h \in \text{dom } A \cap \text{dom } B \right\}.$$

Since $Z(A_s^{\frac{1}{2}}h \oplus B_s^{\frac{1}{2}}h) = [h' + h'']$ by (3.11), and $\{h, h' + h''\} \in A + B$, it follows from (2.9) and (3.12) that the operators Q and Ψ_e are connected by

$$(3.13) \quad Q = Z\Psi_e.$$

Likewise, the relation K in (3.8) is defined as a relation from $\mathfrak{H} \times \mathfrak{H}$ to $\mathfrak{H}$, and $\text{dom } K \subset \mathfrak{E}$; cf. (3.9). Hence K can be viewed as a relation K_e from $\mathfrak{E}$ to $\mathfrak{H}$, so that $K_e \subset K$ are the same sets, interpreted in different spaces. Since Z is an isometry from $\mathfrak{E}$ to $\mathfrak{H}_{A+B}$, the product $K_e Z^*$ is well-defined and

$$K_e Z^* = \{ \{ \varphi, \psi \} : \{ \varphi, \chi \} \in Z^*, \{ \chi, \psi \} \in K_e \text{ for some } \chi \in \mathfrak{E} \};$$

note that $\{ \varphi, \chi \} \in Z^*$ if and only if $\varphi = Z\chi$. In view of (3.8) this leads to

$$K_e Z^* = \left\{ \left\{ Z(A_s^{\frac{1}{2}}h \oplus B_s^{\frac{1}{2}}h), h' + h'' \right\} : \left\{ A_s^{\frac{1}{2}}h, h' \right\} \in A^{\frac{1}{2}}, \left\{ B_s^{\frac{1}{2}}h, h'' \right\} \in B^{\frac{1}{2}} \right\}.$$

This means that the linear relations J in (2.10) and K_e are connected by

$$(3.14) \quad J = K_e Z^*.$$

With these identifications of Q and J in terms of Ψ_e and K_e , the Kreĭn-von Neumann and the Friedrichs extensions of $A + B$ can now be expressed in terms of K and Ψ .

Theorem 3.3. *Let A and B be nonnegative selfadjoint relations. Then:*

- (i) *the Kreĭn-von Neumann extension $(A + B)_N$ is given by $K^{**}K^*$;*
- (ii) *the Friedrichs extension $(A + B)_F$ is given by $\Psi^*\Psi^{**}$.*

Proof. (i) Since Z is an isometric operator from $\mathfrak{E}$ onto $\mathfrak{H}_{A+B}$, (3.14) implies that

$$(3.15) \quad Z(K_e)^* = J^*, \quad (K_e)^{**}Z^* = J^{**}.$$

Multiplication of the relations in (3.15) shows that

$$(3.16) \quad (K_e)^{**}(K_e)^* = (K_e)^{**}Z^*Z(K_e)^* = J^{**}J^*.$$

Since $(K_e)^{**}(K_e)^* = K^{**}K^*$, see (2.5), Theorem 2.2 gives $K^{**}K^* = (A + B)_N$.

(ii) Now it follows from (3.13) that

$$(3.17) \quad (\Psi_e)^*Z^* = Q^*, \quad Z(\Psi_e)^{**} = Q^{**},$$

and multiplication of the relations in (3.17) shows that

$$(3.18) \quad (\Psi_e)^*(\Psi_e)^{**} = (\Psi_e)^*Z^*Z(\Psi_e)^{**} = Q^*Q^{**}.$$

By (2.5) $(\Psi_e)^*(\Psi_e)^{**} = \Psi^*\Psi^{**}$ and now Theorem 2.2 gives $\Psi^*\Psi^{**} = (A + B)_F$. $\square$

Since $\text{mul } (A + B)_N = \text{mul } K^{**}$ the relation $A + B$ is positively closable (i.e., the Kreĭn-von Neumann extension $(A + B)_N$ is an operator; cf. [9, Subsection 5.4]) if and only if the relation K is closable. If either A or B has a positive lower bound, then $A + B$ is positively closable if and only if $A + B$ is closable; cf. [9, Subsection 5.4]. Likewise, $\text{mul } (A + B)_F = \text{mul } \Psi^* = (\text{dom } A \cap \text{dom } B)^\perp$.

Since $\text{dom } K_e = \mathfrak{E}_0$ is dense in $\mathfrak{E}$, K_e^* is an operator from $\mathfrak{H}$ to $\mathfrak{E}$. For any subspace $\mathfrak{L}$ with $\text{dom } \Psi_e \subset \mathfrak{L} \subset \text{dom } K_e^*$ define the operator $T_{\mathfrak{L}e}$ from $\mathfrak{H}$ to $\mathfrak{E}$ by

$$T_{\mathfrak{L}e} = K_e^* \upharpoonright \mathfrak{L}.$$

By (3.15) this leads to

$$ZT_{\mathfrak{L}e} = ZK_e^* \upharpoonright \mathfrak{L} = J^* \upharpoonright \mathfrak{L} = R_{\mathfrak{L}},$$

cf. (2.12), so that $T_{\mathfrak{L}e} = Z^*R_{\mathfrak{L}}$ and

$$(3.19) \quad T_{\mathfrak{L}e}^* = R_{\mathfrak{L}}^*Z, \quad T_{\mathfrak{L}e}^{**} = Z^*R_{\mathfrak{L}}^{**}.$$

Denote by $T_{\mathfrak{L}}$ the operator $T_{\mathfrak{L}e}$ viewed as an operator from $\mathfrak{H}$ to $\mathfrak{H} \times \mathfrak{H}$. Then (2.5) and (3.19) imply that

$$(3.20) \quad T_{\mathfrak{L}}^*T_{\mathfrak{L}}^{**} = T_{\mathfrak{L}e}^*T_{\mathfrak{L}e}^{**} = R_{\mathfrak{L}}^*R_{\mathfrak{L}}^{**}.$$

Recall that $\text{dom } \Psi_e = \text{dom } A \cap \text{dom } B = \text{dom } (A + B)$ and $\text{dom } K_e^* = \text{dom } J^*$. Hence (3.20) and Theorem 2.3 (i), (ii) give the following result.

Theorem 3.4. *Let A and B be nonnegative relations in a Hilbert space $\mathfrak{H}$. Then the following statements are equivalent:*

- (i) $\tilde{A}$ is a nonnegative selfadjoint extremal extension of $A + B$;
- (ii) $\tilde{A} = T_{\mathfrak{L}}^*T_{\mathfrak{L}}^{**}$ for some $\mathfrak{L}$ such that $\text{dom } (A + B) \subset \mathfrak{L} \subset \text{dom } (A + B)_{\mathcal{N}}^{\frac{1}{2}}$.

4. THE FORM SUM CONSTRUCTION

The nonnegative selfadjoint relations A and B generate the following closed nonnegative form:

$$(4.1) \quad (A_s^{\frac{1}{2}}h, A_s^{\frac{1}{2}}k) + (B_s^{\frac{1}{2}}h, B_s^{\frac{1}{2}}k), \quad h, k \in \text{dom } A^{\frac{1}{2}} \cap \text{dom } B^{\frac{1}{2}}.$$

Observe that the restriction of this form to $\text{dom } \Psi^{**}$ is equal to

$$(4.2) \quad (\Psi^{**}h, \Psi^{**}k) = (A_s^{\frac{1}{2}}h, A_s^{\frac{1}{2}}k) + (B_s^{\frac{1}{2}}h, B_s^{\frac{1}{2}}k), \quad h, k \in \text{dom } \Psi^{**},$$

since $\Psi^{**} \subset (\Phi^*)_s$; cf. (3.5). In fact, this shows that the domain $\text{dom } \Psi^{**}$ is equal to the energy space $\mathfrak{H}[A + B]$ associated with $A + B$; cf. [3]. Thus the form in (4.2) has a natural domain which is in general larger than $\text{dom } \Psi^{**}$.

Theorem 4.1. *Let A and B be nonnegative selfadjoint relations. The nonnegative selfadjoint relation $\Phi^{**}\Phi^*$ is an extension of the nonnegative relation $A + B$, which corresponds to the closed nonnegative form in (4.1). The following statements are equivalent:*

- (i) $\Phi^{**}\Phi^*$ is an extremal extension of $A + B$;
- (ii) $\mathfrak{E} = \mathfrak{F}$, i.e., $\overline{\text{ran}} \Psi = \overline{\text{ran}} (\Phi^*)_s$.

Moreover, $\Phi^{**}P_{\mathfrak{E}}(P_{\mathfrak{E}}\Phi^*)^{**} = (\Phi^{**}P_{\mathfrak{E}}\Phi^*)_F$ is an extremal extension of $A + B$.

Proof. The form sum (4.1) can be rewritten as

$$((\Phi^*)_s h, (\Phi^*)_s k), \quad h, k \in \text{dom } (\Phi^*)_s = \text{dom } A^{\frac{1}{2}} \cap \text{dom } B^{\frac{1}{2}},$$

so that $\Phi^{**}\Phi^*$ is the nonnegative selfadjoint relation which corresponds to (4.1); cf. [8, Proposition 5.2]. To show that $\Phi^{**}\Phi^*$ extends $A + B$, let $\{h, h'_A + h'_B\} \in (A + B)$ for some $\{h, h'_A\} \in A$ and $\{h, h'_B\} \in B$, so that $h \in \text{dom } A \cap \text{dom } B$. Clearly,

$\{h, A_s^{\frac{1}{2}}h \oplus B_s^{\frac{1}{2}}h\} \in \Phi^*$. Moreover, $\{A_s^{\frac{1}{2}}h \oplus B_s^{\frac{1}{2}}h, h'_A + h'_B\} \in \Phi^{**}$, as can be verified directly:

$$(h'_A + h'_B, \varphi) - (A_s^{\frac{1}{2}}h \oplus B_s^{\frac{1}{2}}h, A_s^{\frac{1}{2}}\varphi \oplus B_s^{\frac{1}{2}}\varphi) = 0, \quad \varphi \in \operatorname{dom} A \cap \operatorname{dom} B.$$

Therefore $\{h, h'_A + h'_B\} \in \Phi^{**}\Phi^*$. Hence $A + B \subset \Phi^{**}\Phi^*$.

(i) $\Rightarrow$ (ii) In view of (3.7) it is enough to prove that $\mathfrak{F} \subset \mathfrak{E}$. Assume that $\Phi^{**}\Phi^*$ is extremal. Then by Theorem 3.4 there exists a subspace $\mathfrak{L}$ such that

$$(4.3) \quad ((\Phi^*)_s)^*(\Phi^*)_s = \Phi^{**}\Phi^* = T_{\mathfrak{L}}^*T_{\mathfrak{L}}^{**} = T_{\mathfrak{L}^e}^*T_{\mathfrak{L}^e}^{**}.$$

By Lemma 3.2 $(\Phi^*)_s \subset \Phi^* \subset K^*$ and therefore $P_{\mathfrak{E}}(\Phi^*)_s \subset P_{\mathfrak{E}}K^* = K_e^*$ with $\operatorname{dom} P_{\mathfrak{E}}(\Phi^*)_s = \operatorname{dom}(\Phi^*)_s = \operatorname{dom} T_{\mathfrak{L}}^{**} = \operatorname{dom} T_{\mathfrak{L}^e}^{**}$; see (2.4). Since $P_{\mathfrak{E}}(\Phi^*)_s$ and $T_{\mathfrak{L}^e}^{**}$ are restrictions of the operator K_e^* it follows that

$$P_{\mathfrak{E}}(\Phi^*)_s = T_{\mathfrak{L}^e}^{**}, \quad ((\Phi^*)_s)^*P_{\mathfrak{E}} = T_{\mathfrak{L}^e}^*.$$

Now (4.3) implies that

$$((\Phi^*)_s)^*(\Phi^*)_s = T_{\mathfrak{L}^e}^*T_{\mathfrak{L}^e}^{**} = ((\Phi^*)_s)^*P_{\mathfrak{E}}T_{\mathfrak{L}^e}^{**} = ((\Phi^*)_s)^*T_{\mathfrak{L}^e}^{**}.$$

Hence for every $f \in \operatorname{dom}((\Phi^*)_s)^*(\Phi^*)_s$ one has $(\Phi^*)_sf - T_{\mathfrak{L}^e}^{**}f \in \ker((\Phi^*)_s)^*$, and since $(\Phi^*)_sf - T_{\mathfrak{L}^e}^{**}f \in \mathfrak{F} = \overline{\operatorname{ran}}(\Phi^*)_s = (\ker((\Phi^*)_s)^*)^{\perp}$, one concludes that $(\Phi^*)_sf - T_{\mathfrak{L}^e}^{**}f = 0$. Therefore $(\Phi^*)_s(\operatorname{dom}((\Phi^*)_s)^*(\Phi^*)_s) \subset \operatorname{ran} T_{\mathfrak{L}^e}^{**} \subset \mathfrak{E}$. Since $\operatorname{dom}((\Phi^*)_s)^*(\Phi^*)_s$ is a core for the corresponding closed form, or equivalently, the closure of $(\Phi^*)_s \upharpoonright \operatorname{dom}((\Phi^*)_s)^*(\Phi^*)_s$ is equal to $(\Phi^*)_s$, the claim follows:

$$\mathfrak{F} = \overline{\operatorname{ran}}(\Phi^*)_s \subset \mathfrak{E}.$$

(ii) $\Rightarrow$ (i) Assume that $\mathfrak{E} = \mathfrak{F}$. Then $\operatorname{ran}(\Phi^*)_s = \mathfrak{F}_0 \subset \mathfrak{E}$ and $(\Phi^*)_s \subset \mathfrak{H} \times \mathfrak{E}$; see (3.5). Denote by $(\Phi^*)_{se}$ the operator $(\Phi^*)_s$ viewed as an operator from $\mathfrak{H}$ to $\mathfrak{E}$. Then $(\Phi^*)_{se} = P_{\mathfrak{E}}(\Phi^*)_s \subset P_{\mathfrak{E}}K^* = K_e^*$, so that $(\Phi^*)_{se} = T_{\mathfrak{L}}$ with

$$\mathfrak{L} = \operatorname{dom}(\Phi^*)_{se} = \operatorname{dom} A^{\frac{1}{2}} \cap \operatorname{dom} B^{\frac{1}{2}}.$$

Hence,

$$\Phi^{**}\Phi^* = ((\Phi^*)_s)^*(\Phi^*)_s = ((\Phi^*)_{se})^*(\Phi^*)_{se} = T_{\mathfrak{L}}^*T_{\mathfrak{L}}^{**},$$

which shows that $\Phi^{**}\Phi^*$ is extremal; cf. Theorem 3.4.

As to the last statement, note that $P_{\mathfrak{E}}(\Phi^*)_s \subset P_{\mathfrak{E}}K^* = K_e^*$. Since $P_{\mathfrak{E}}(\Phi^*)_s = P_{\mathfrak{E}}\Phi^*$ and hence $((\Phi^*)_s)^*P_{\mathfrak{E}} = \Phi^{**}P_{\mathfrak{E}}$, the selfadjoint relation $\Phi^{**}P_{\mathfrak{E}}(P_{\mathfrak{E}}\Phi^*)^{**} = (\Phi^{**}P_{\mathfrak{E}}\Phi^*)_F$ is an extremal extension of $A + B$ by Theorem 3.4. $\square$

The nonnegative selfadjoint relation $\Phi^{**}\Phi^*$ is called the *form sum extension* of the nonnegative relation $A + B$ (induced by the form (4.1)). Its multi-valued part is given by $\operatorname{mul} \Phi^{**}\Phi^* = \operatorname{mul} \Phi^{**} = (\operatorname{dom} \Phi^*)^{\perp} = (\operatorname{dom} A^{\frac{1}{2}} \cap \operatorname{dom} B^{\frac{1}{2}})^{\perp}$. The orthogonal operator part of $\Phi^{**}\Phi^*$ is the nonnegative selfadjoint operator which corresponds to the form sum (4.1) restricted to the closure of $\operatorname{dom} A^{\frac{1}{2}} \cap \operatorname{dom} B^{\frac{1}{2}}$. Theorem 4.1 indicates that the extremality of the form sum extension depends on the relation between the sets $\operatorname{dom} A \cap \operatorname{dom} B$ and $\operatorname{dom} A^{\frac{1}{2}} \cap \operatorname{dom} B^{\frac{1}{2}}$; cf. also Example 5.2 below. It also shows how to construct an extremal extension of $A + B$ from the form sum extension.

5. EXAMPLES

Example 5.1. In the Hilbert space $\mathfrak{H} = L^2(1, \infty) \oplus L^2(1, \infty)$ consider the multiplication operators

$$A = \begin{pmatrix} 1 & t \\ t & t^2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -t \\ -t & t^2 \end{pmatrix},$$

each defined on its maximal domain. Note that the operators A and B are non-negative and selfadjoint. The sum $A + B$ is given by the multiplication operator

$$(5.1) \quad \begin{pmatrix} 2 & 0 \\ 0 & 2t^2 \end{pmatrix}$$

on the domain $\text{dom } A \cap \text{dom } B$. The function $f(t) = (1/t \ 0)^\top \in \mathfrak{H}$ does not belong to either $\text{dom } A$ or to $\text{dom } B$. However for each $n \in \mathbb{N}$ the function $f_n = \chi_{(1,n)} f$ belongs to $\text{dom } A$ and to $\text{dom } B$, and $f_n \rightarrow 2f$ and $(A + B)f_n \rightarrow 2f$ in $\mathfrak{H}$ as $n \rightarrow \infty$. Therefore the operator $A + B$ is not closed. However, the operator $A + B$ defined on $\text{dom } A \cap \text{dom } B$ has a natural selfadjoint operator extension given by (5.1) defined on its natural domain. In fact, it is not difficult to see that $A + B$ is essentially selfadjoint, so that there is precisely one selfadjoint extension of $A + B$ which is automatically nonnegative. Hence, the Kreĭn-von Neumann extension, the Friedrichs extension, and the form sum extension coincide. This example is due to Manfred Möller.

Observe that there exist nonnegative selfadjoint operators A and B for which

$$\text{dom } A \cap \text{dom } B = \{0\}, \quad \text{dom } A^{\frac{1}{2}} \cap \text{dom } B^{\frac{1}{2}} \text{ dense in } \mathfrak{H}.$$

It suffices to recall the existence of bounded nonnegative operators C and D for which $\text{ran } C \cap \text{ran } D = \{0\}$, $\text{ran } C^{\frac{1}{2}} \cap \text{ran } D^{\frac{1}{2}}$ dense in $\mathfrak{H}$; cf. [6]. Then $\ker C = \ker D = \{0\}$ and one can take $A = C^{-1}$ and $B = D^{-1}$.

Example 5.2. Let A and B be nonnegative selfadjoint relations in a Hilbert space $\mathfrak{H}$, which satisfy $\text{dom } A \cap \text{dom } B = \{0\}$. Then it is easily seen that the sum $S = A + B$ is a purely multi-valued relation $S = \{0\} \times \text{mul}(A + B)$. The Kreĭn-von Neumann extension S_N and the Friedrichs extension S_F are given by $S_N = \mathfrak{K}^\perp \times \mathfrak{K}$ and $S_F = \{0\} \times \mathfrak{H}$, where $\mathfrak{K}$ is the closure of $\text{mul}(A + B)$. The associated form $\mathfrak{t}_N$ is the null form on $\mathfrak{K}^\perp$, and the associated form $\mathfrak{t}_F$ is the null form on $\{0\}$. There exists a one-to-one correspondence between the set of all extremal extensions $\tilde{A}$ of S and the set of all closed subspaces $\mathfrak{L}$ of $\mathfrak{K}^\perp$; cf. (2.14) and Corollary 2.4. This correspondence is given by $\tilde{A} = \mathfrak{L} \times \mathfrak{L}^\perp$, and the associated form $\tilde{\mathfrak{t}}$ is the null form on $\mathfrak{L}$. If, in particular, A and B are nonnegative selfadjoint operators with $\text{dom } A \cap \text{dom } B = \{0\}$, then $\text{mul}(A + B) = \{0\}$, $S = \{0, 0\}$, and the form $\mathfrak{t}_N$ is the null form on $\mathfrak{H}$. Any extremal extension has a corresponding form which must be the null form on its domain. However, if $\text{dom } A^{\frac{1}{2}} \cap \text{dom } B^{\frac{1}{2}} \neq \{0\}$, then the form sum is nonzero on its domain, since $\ker A^{\frac{1}{2}} \cap \ker B^{\frac{1}{2}} \subset \text{dom } A \cap \text{dom } B = \{0\}$. Therefore, in this case the form sum is extremal only if $\text{dom } A^{\frac{1}{2}} \cap \text{dom } B^{\frac{1}{2}} = \{0\}$ and then it coincides with the Friedrichs extension. Finally, observe that if $\text{dom } A^{\frac{1}{2}} \cap \text{dom } B^{\frac{1}{2}} \neq \{0\}$, then the form corresponding to $\Phi^{**} P_{\mathfrak{E}} (P_{\mathfrak{E}} \Phi^*)^{**} = (\Phi^{**} P_{\mathfrak{E}} \Phi^*)_F$ in Theorem 4.1 is the null form on the closure of $\text{dom } A^{\frac{1}{2}} \cap \text{dom } B^{\frac{1}{2}}$.

REFERENCES

- [1] Yu.M. Arlinskiĭ, S. Hassi, Z. Sebestyén, and H.S.V. de Snoo, “On the class of extremal extensions of a nonnegative operator”, *Oper. Theory: Adv. Appl. (B. Sz.-Nagy memorial volume)*, 127 (2001), 41–81. MR1902794 (2003d:47028)
- [2] Yu.M. Arlinskiĭ and E.R. Tsekanovskiĭ, “Quasi selfadjoint contractive extensions of Hermitian contractions”, *Teor. Funkts., Funkts. Anal. Prilozhen*, 50 (1988), 9–16. MR0975668 (90b:47014)
- [3] E.A. Coddington, *Extension theory of formally normal and symmetric subspaces*, *Mem. Amer. Math. Soc.*, 134, 1973. MR0477855 (57:17357)
- [4] B. Farkas and M. Matolcsi, “Commutation properties of the form sum of positive, symmetric operators”, *Acta Sci. Math. (Szeged)*, 6 (2001), 777–790. MR1876466 (2003a:47050)
- [5] B. Farkas and M. Matolcsi, “Positive forms on Banach spaces”, *Acta Math. Hungar.*, 99 (2003), 43–55. MR1973084 (2004d:47004)
- [6] P.A. Fillmore and J.P. Williams, “On operator ranges”, *Adv. Math.*, 7 (1971), 254–281. MR0293441 (45:2518)
- [7] S. Hassi, M. M. Malamud, and H. S. V. de Snoo, “On Kreĭn’s extension theory of nonnegative operators”, *Math. Nachr.*, 274/275 (2004), 40–73.
- [8] S. Hassi, A. Sandovici, H.S.V. de Snoo, and H. Winkler, “Form sums of nonnegative selfadjoint operators”, *Acta Math. Hungar.*, 111 (2006), 81–105. MR2188974
- [9] S. Hassi, A. Sandovici, H.S.V. de Snoo, and H. Winkler, “A general factorization approach to the extension theory of nonnegative operators and relations”, *J. Operator Theory*, to appear.
- [10] Z. Sebestyén and J. Stochel, “Restrictions of positive self-adjoint operators”, *Acta Sci. Math. (Szeged)*, 55 (1991), 149–154. MR1124953 (92i:47024)
- [11] Z. Sebestyén and J. Stochel, “On products of unbounded operators”, *Acta Math. Hungar.*, 100 (1–2) (2003), 105–129. MR1984863 (2004c:47003)

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF VAASA, P.O. BOX 700, 65101 VAASA, FINLAND

E-mail address: sha@uwasa.fi

COLEGIUL NAȚIONAL “PETRU RAREȘ”, 610101, STR. ȘTEFAN CEL MARE, NR. 4, PIATRA NEAMȚ, ROMANIA

E-mail address: adrian.sandovici@yahoo.com

DEPARTMENT OF MATHEMATICS AND COMPUTING SCIENCE, UNIVERSITY OF GRONINGEN, P.O. BOX 800, 9700 AV GRONINGEN, NEDERLAND

E-mail address: desnoo@math.rug.nl

INSTITUT FÜR MATHEMATIK, MA 6-4, TECHNISCHE UNIVERSITÄT BERLIN, STRASSE DES 17. JUNI 136, 10623 BERLIN, DEUTSCHLAND

E-mail address: winkler@math.tu-berlin.de