HYPERCYCLIC PROPERTY OF WEIGHTED COMPOSITION OPERATORS

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Dedicated to the memory of Professor K. Seddighi

ABSTRACT. In the present paper we investigate conditions under which a holomorphic self-map of the open unit disk induces a hypercyclic weighted composition operator in the space of holomorphic functions.

1. INTRODUCTION

Let U be the open unit disk of the plane. Then the space $H(U)$ of all complex-valued functions holomorphic on U can be made into an F-space by a complete metric for which a sequence $\{f_n\}$ in $H(U)$ converges to $f \in H(U)$ if and only if $f_n \rightarrow f$ uniformly on every compact subset of U . Each $\varphi \in H(U)$ and holomorphic self-map ψ of U induces a linear weighted composition operator $C_{\varphi,\psi} : H(U) \rightarrow H(U)$ by $C_{\varphi,\psi}(f)(z) = \varphi(z)f(\psi(z))$ for every $f \in H(U)$ and $z \in U$. Indeed, $C_{\varphi,\psi} = M_\varphi C_\psi$, where M_φ denotes the operator of multiplication by φ and C_ψ is a composition operator by means of the definition $C_\psi(f) = f \circ \psi$ for every $f \in H(U)$. Good sources of background information on composition operators include [4], [7] and [17].

A bounded linear operator T on an F-space X is said to be hypercyclic if there exists a vector $x \in X$ for which the orbit $Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$ is dense in X and in this case we refer to x as a hypercyclic vector for T . Backward and bilateral shifts ([9, 15, 16]), translations and differentiation operators ([1, 5]), and adjoints of multiplication operators ([10]) contain hypercyclic operators. For other useful references one can see [8, 11, 12]. The hypercyclicity of composition operators also has been considered by Bourdon and Shapiro in ([2, 3]). They studied the hypercyclicity of composition operators on the Hardy space H^2 . Also, Shapiro gave a complete characterization of hypercyclic composition operators on $H(U)$ as follows:

Theorem 1.1 ([18]). *Suppose that ψ is a univalent holomorphic self-map of U . Then C_ψ is hypercyclic if and only if ψ has no fixed point in U .*

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Studying the dynamics of weighted composition operators entails a study of the iterate behavior of holomorphic self-maps. The holomorphic self-maps of U are divided into classes of elliptic and nonelliptic. The elliptic type is an automorphism and has a fixed point in U . It is well known that this map is conjugate to a rotation $z \rightarrow \lambda z$ for some complex number λ with $|\lambda| = 1$. The maps that are not elliptic are called of nonelliptic type. The iterate of a nonelliptic map can be characterized by the Denjoy-Wolff Iteration Theorem ([17]).

For simplicity, throughout this paper we use the notation “ $\xrightarrow{k}$ ” for indicating uniform convergence on compact subsets of U . Also by ψ_n we denote the n th iterate of ψ and by $\psi'(w)$ we denote the angular derivative of ψ at $w \in \partial U$. Note that if $w \in U$, then $\psi'(w)$ has the natural meaning of derivative.

Denjoy-Wolff iteration theorem. Suppose ψ is a holomorphic self-map of U that is not an elliptic automorphism.

- (i) If ψ has a fixed point $w \in U$, then $\psi_n \xrightarrow{k} w$ and $|\psi'(w)| < 1$.
- (ii) If ψ has no fixed point in U , then there is a point $w \in \partial U$ such that $\psi_n \xrightarrow{k} w$ and the angular derivative of ψ exists at w , with $0 < \psi'(w) \leq 1$.

We call the unique attracting point w , the Denjoy-Wolff point of ψ . By the Denjoy-Wolff Iteration Theorem, a general classification for nonelliptic holomorphic self-maps of U can be given: let w be the Denjoy-Wolff point of a holomorphic self-map ψ of U . We say ψ is of *dilation type* if $w \in U$, of *hyperbolic type* if $w \in \partial U$ and $\psi'(w) < 1$, and of *parabolic type* if $w \in \partial U$ and $\psi'(w) = 1$.

The next section of the present paper shows that weighted composition operators with nonconstant weight functions can be hypercyclic on $H(U)$. In particular we show that the hypercyclicity of $C_{\varphi, \psi}$ depends on how freely $\varphi(z)$ is allowed to approach the unit circle as $z \rightarrow w$, where w is the Denjoy-Wolff point of ψ . In Section 3, we consider the hypercyclicity behavior of the adjoint of weighted composition operators on a Hilbert space of analytic functions. The work of Godefroy and Shapiro ([10]) shows that hypercyclicity can occur for the adjoint of a multiplication operator M_φ on Hilbert spaces of analytic functions whenever $\text{ran } \varphi$ intersects the unit circle. We consider the operator $C_{\varphi, \psi}$ on a Hilbert space of analytic functions and we will investigate the hypercyclicity of the adjoint of $C_{\varphi, \psi}$.

2. WEIGHTED COMPOSITION OPERATORS ON $H(U)$

In this section ψ will denote a univalent holomorphic self-map of U and φ is a nonzero holomorphic map on U . The following theorem severely limits the kinds of maps that can produce hypercyclic weighted composition operators.

Proposition 2.1. *If $C_{\varphi, \psi}$ is hypercyclic on $H(U)$, then*

- (i) ψ has no fixed point in U and $\varphi(z) \neq 0$ for every $z \in U$;
- (ii) ψ is univalent.

Proof. Note that for each $z \in U$, the linear functional $\Lambda_z : H(U) \rightarrow \mathbb{C}$ by $\Lambda_z(f) = f(z)$ is continuous on $H(U)$ and

$$C_{\varphi, \psi}^*(\Lambda_z)(f) = \Lambda_z \circ C_{\varphi, \psi}(f) = \Lambda_z(\varphi \cdot f \circ \psi) = \varphi(z)(f \circ \psi)(z)$$

for each $f \in H(U)$. Thus if ψ has a fixed point in U or $\varphi(z) = 0$ for some $z \in U$, the adjoint $C_{\varphi,\psi}$ has eigenvalue and so $C_{\varphi,\psi}$ is not hypercyclic. This completes the proof of part (i).

For part (ii), let $\psi(p) = \psi(q)$ and consider a hypercyclic vector f of $C_{\varphi,\psi}$. Observe that every member of $\text{Orb}(C_{\varphi,\psi}, f)$ has the following form:

$$C_{\varphi,\psi}^n(f) = \prod_{i=0}^{n-1} \varphi \circ \psi_i \cdot f \circ \psi_n$$

for some nonnegative integer n and here ψ_0 is the identity map on U . Thus the linear functionals $\frac{1}{\varphi(p)}\Lambda_p$ and $\frac{1}{\varphi(q)}\Lambda_q$ have the same value at each function in $\text{Orb}(C_{\varphi,\psi}, f)$, and this property extends to functions in the closure of the orbit. So we get $\frac{1}{\varphi(p)}\Lambda_p = \frac{1}{\varphi(q)}\Lambda_q$ and consequently $p = q$. $\square$

By using the following lemma, in Proposition 2.3, we will show that the unimodular scalar multiple of a hypercyclic composition operator is also hypercyclic. Note that we cannot use the results in [14], because they work in a Banach space setting.

Lemma 2.2. *Let X be the set of polynomials vanishing in the boundary fixed point w . Then X is a dense subset of $H(U)$.*

Proof. This result is well known in the Hardy space setting. Since convergence in the Hardy space $H^2(U)$ implies convergence in $H(U)$, with both spaces containing the polynomials as a dense subset, the proof is complete. $\square$

Proposition 2.3. *Suppose that ψ has no fixed point in U . Then λC_ψ is hypercyclic for every complex number λ with $|\lambda| = 1$.*

Proof. Let $T = C_\psi$ and X be the set stated in the preceding lemma. Note that for every $f \in X$, $T^n(f) = f \circ \psi_n$ tends to zero as $n \rightarrow \infty$. Now fix the open subsets U' and V' and the open neighborhood W of zero in $H(U)$. Since T is hypercyclic and the sequence $\{T^n\}$ converges pointwise to zero on the dense subset X , so there exists some positive integer n such that $T^n(U') \cap W \neq \emptyset$ and $T^n(W) \cap V' \neq \emptyset$. But for each positive integer n , $\lambda^n W = W$. This implies that $(\lambda T)^n(U') \cap W \neq \emptyset$ and $(\lambda T)^n(W) \cap V' \neq \emptyset$. Now the hypercyclicity of λT follows similarly from [10, Corollary 1.3]. $\square$

Remember that bounded operators T and S on an F-space X are quasi-similar if there exists a bounded operator V on X that has dense range and $TV = VS$. If V is invertible, then T and S are said to be similar. The similarity and quasi-similarity preserve hypercyclicity. Indeed if $x \in X$ is a hypercyclic vector for S , then $V(x)$ is a hypercyclic vector for T .

The existence of nonzero eigenvalues of a weighted composition operator $C_{\varphi,\psi}$ makes it possible to show that, at least in some cases, $C_{\varphi,\psi}$ inherits the hypercyclicity of C_ψ . In fact if λ is a nonzero eigenvalue of $C_{\varphi,\psi}$, then there is a nonzero holomorphic function g with $\varphi \cdot g \circ \psi = \lambda g$, whence $C_{\varphi,\psi}M_g = M_g(\lambda C_\psi)$. If λ is unimodular and g has no zero in U , then M_g is one-to-one and has dense range and so $C_{\varphi,\psi}$ is quasi-similar to λC_ψ , which is hypercyclic by Proposition 2.3. Thus $C_{\varphi,\psi}$ is also hypercyclic.

Definition. For any $w \in \partial U$ and real number α , we denote by $\text{Lip}_\alpha(w)$ the class of holomorphic functions φ such that there exist some neighborhood G of w in ∂U

and a positive constant M with

$$|\varphi(z) - \varphi(w)| \leq M|z - w|^\alpha \quad (z \in G).$$

For example, if an analytic function φ on U is also analytic at $w \in \partial U$, then $\varphi \in Lip_\alpha(w)$ whenever $0 \leq \alpha \leq 1$. Moreover, if $\varphi'(w) = 0$, then $\varphi \in Lip_\alpha(w)$ whenever $0 \leq \alpha \leq 2$.

Proposition 2.4. *Let w be the Denjoy-Wolff fixed point of ψ , $\varphi \in Lip_\beta(w)$ for some real number β , and $\varphi(w) \neq 0$. Then $\varphi(w)$ is an eigenvalue for $C_{\varphi, \psi}$, whenever ψ is hyperbolic and $\beta > 0$ or ψ is a parabolic automorphism and $\beta > 1$. Moreover, if φ never vanishes on U , then the same is true of the eigenfunction corresponding to $\varphi(w)$. Also, $C_{\varphi, \psi}$ is hypercyclic if $|\varphi(w)| = 1$.*

Before we prove the above proposition, we need the following lemma.

Lemma 2.5. *Suppose ψ has no fixed point in U . Then $\sum_{i=1}^{+\infty} (1 - |\psi_i(z)|^2)^\alpha$ converges uniformly on compact subsets of U whenever ψ is hyperbolic and $\alpha > 0$ or ψ is a parabolic automorphism and $\alpha > \frac{1}{2}$.*

Proof. Let w be the Denjoy-Wolff fixed point of ψ . By the Denjoy-Wolff Iteration Theorem, the angular derivative of ψ exists at w , and $0 < \psi'(w) \leq 1$. Setting $\delta = \psi'(w)$, it can follow from the Julia-Carathéodory Theorem ([3, Theorem 1.3]) that

$$\frac{|\psi(z) - w|^2}{1 - |\psi(z)|^2} < \delta \frac{|z - w|^2}{1 - |z|^2}.$$

By substituting $\psi_n(z)$ for $\psi(z)$, in the above inequality we get

$$\frac{|\psi_n(z) - w|^2}{1 - |\psi_n(z)|^2} < \delta^n \frac{|z - w|^2}{1 - |z|^2}$$

for every $z \in U$ and for all nonnegative integers n . Now if K is a compact subset of U , then there exists a constant $M > 0$ such that $\frac{4|z - w|^2}{1 - |z|^2} < M$ for all z in K . So it follows that

$$\begin{aligned} 1 - |\psi_n(z)|^2 &= (1 - |\psi_n(z)|)(1 + |\psi_n(z)|) \leq 2(1 - |\psi_n(z)|) \leq 2|\psi_n(z) - w| \\ &\leq 4 \frac{|\psi_n(z) - w|^2}{1 - |\psi_n(z)|^2} \leq 4\delta^n \frac{|z - w|^2}{1 - |z|^2} < M\delta^n \end{aligned}$$

and consequently $(1 - |\psi_n(z)|^2)^\alpha \leq M^\alpha \delta^{\alpha n}$. Thus in the hyperbolic case with $\alpha > 0$, $\sum_{i=0}^{\infty} (1 - |\psi_i(z)|^2)^\alpha$ converges uniformly on compact subsets of U . Now assume that ψ is a parabolic automorphism and $\alpha > \frac{1}{2}$. Upon conjugating by an appropriate rotation we may assume $w = 1$. Let

$$\sigma(z) = \frac{1+z}{1-z} \text{ and } \Psi = \sigma \circ \psi \circ \sigma^{-1}.$$

Then σ is a linear-fractional mapping of U onto the open right half-plane P , and one easily checks that $\Psi(p) = p + a$, where $\operatorname{Re} a \geq 0$ and $p \in P$. But for each $p \in P$ we have

$$1 - |\sigma^{-1}(p)|^2 = \frac{4 \operatorname{Re} p}{|1 + p|^2}.$$

So for each positive integer n and $z \in U$,

$$1 - |\sigma^{-1} \circ \Psi_n \circ \sigma(z)|^2 = \frac{4 \operatorname{Re} \Psi_n \circ \sigma(z)}{|1 + \Psi_n \circ \sigma(z)|^2}$$

and

$$1 - |\psi_n(z)|^2 = \frac{4 \operatorname{Re}(\sigma(z) + na)}{|1 + \sigma(z) + na|^2}.$$

Also, $\operatorname{Re} a = 0$ when ψ is an automorphism. So if K is any compact subset of U , then for sufficiently large n , there exists some positive constant M such that

$$(1 - |\psi_n(z)|^2)^\alpha < \frac{M^\alpha}{n^{2\alpha}} \quad (z \in K).$$

Therefore, $\sum_{i=0}^{\infty} (1 - |\psi_n(z)|^2)^\alpha$ converges uniformly on compact subsets of U for $\alpha > \frac{1}{2}$. $\square$

Proof of Proposition 2.4. Our hypothesis shows that there exist some constant M and $\delta > 0$, such that

$$(*) \quad |\varphi(w) - \varphi(z)| < M|w - z|^\beta$$

for every z with $|z - w| < \delta$. Now fix a compact subset K of U . The Julia-Carathéodory Theorem provides a constant $c > 0$ such that

$$|w - \psi_n(z)|^2 < c(1 - |\psi_n(z)|^2)$$

and so

$$|w - \psi_n(z)|^\beta < c^{\frac{\beta}{2}}(1 - |\psi_n(z)|^2)^{\frac{\beta}{2}}$$

for each $z \in K$ and all n . In $(*)$, by substituting $\psi_n(z)$ instead of z , we get

$$|\varphi(w) - \varphi(\psi_n(z))| < M c^{\frac{\beta}{2}} (1 - |\psi_n(z)|^2)^{\frac{\beta}{2}}$$

for every $n > N$. So we get

$$|1 - \frac{1}{\varphi(w)} \varphi(\psi_n(z))| < \frac{M c^{\frac{\beta}{2}}}{|\varphi(w)|} (1 - |\psi_n(z)|^2)^{\frac{\beta}{2}}.$$

Now if ψ is hyperbolic and $\beta > 0$ or ψ is a parabolic automorphism and $\beta > 1$, then by Lemma 2.5, $\sum_{n=0}^{\infty} (1 - |\psi_n(z)|^2)^{\frac{\beta}{2}}$ and consequently $\sum_{n=0}^{\infty} |1 - \frac{1}{\varphi(w)} \varphi(\psi_n(z))|$ converges uniformly on K . Thus $\prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi(\psi_n(z))$ also converges uniformly on K . Define

$$g(z) = \prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi(\psi_n(z)).$$

Thus g is a nonzero holomorphic function on U and $\varphi(z) \cdot g(\psi(z)) = \varphi(w)g(z)$ ([6, Theorem 5.9, p. 163]). Note that if φ never vanishes on U , then g does so. Now if $|\phi(w)| = 1$, then $C_{\varphi, \psi}$ is hypercyclic. $\square$

The ideas of the proof of Proposition 2.4 may be adapted to prove an eigenvalue result for $C_{\varphi, \psi}$ when ψ has its Denjoy-Wolff point in U .

Proposition 2.6. *Let ψ be an arbitrary holomorphic self-map of U having Denjoy-Wolff point $w \in U$. If $\varphi(w) \neq 0$, then $\varphi(w)$ is an eigenvalue for the operator $C_{\varphi, \psi}$ acting on $H(U)$.*

Proof. Choose δ with $|\psi'(w)| < \delta < 1$. First suppose that $w = 0$. So $|\psi(z)| < \delta|z|$ when z is sufficiently near to zero. If K is a compact subset of U , then by the Denjoy-Wolff Theorem, $\psi_n \xrightarrow{k} 0$ on K and $|\psi_{n+k}(z)| < \delta^k |\psi_n(z)|$ for sufficiently large n and every $k \in \mathbb{N}$, and $z \in K$. This implies that $\sum_{i=0}^{\infty} |\psi_i(z)|$ converges uniformly on compact subsets of U . Since φ is bounded, an application of Schwarz's Lemma shows that there exists a constant $M > 0$ such that $|\varphi(0) - \varphi(z)| < M|z|$ for every $z \in U$. But $\varphi(0) \neq 0$; thus

$$|1 - \frac{1}{\varphi(0)}\varphi(z)| < \frac{M}{|\varphi(0)|}|z| \quad (z \in U).$$

By substituting $\psi_i(z)$ instead of z in the above inequality, we get

$$|1 - \frac{1}{\varphi(0)}\varphi(\psi_i(z))| < \frac{M}{|\varphi(0)|}|\psi_i(z)|.$$

This implies that $\sum_{i=0}^{\infty} |1 - \frac{1}{\varphi(0)}\varphi(\psi_i(z))|$ and consequently $\prod_{i=0}^{\infty} \frac{1}{\varphi(0)}\varphi(\psi_i(z))$ converges uniformly on compact subsets of U . Set $g(z) = \prod_{i=0}^{\infty} \frac{1}{\varphi(0)}\varphi(\psi_i(z))$. Then g is a nonzero holomorphic function on U . Also, note that $\varphi \cdot g \circ \psi = \varphi(0)g$. If $w \neq 0$, consider the self-maps $\Psi = \alpha_w \circ \psi \circ \alpha_w$ and $\Phi = \varphi \circ \alpha_w$, where $\alpha_w(z) = \frac{w-z}{1-\bar{w}z}$. Clearly we can see that $\Psi(0) = 0$ and $\Phi(0) \neq 0$, so the first step of the theorem shows that there exists a nonzero holomorphic map G on U such that $\Phi \cdot G \circ \Psi = \Phi(0)G$. Thus

$$\varphi \circ \alpha_w(z) \cdot G \circ (\alpha_w \circ \psi \circ \alpha_w)(z) = (\varphi \circ \alpha_w(0))G(z) = \varphi(w)G(z).$$

Note that $\alpha_w \circ \alpha_w(z) = z$ and substitute $\alpha_w(z)$ instead of z in the above equality. So we get $\varphi \cdot g \circ \psi = \varphi(w)g$ where $g = G \circ \alpha_w$ belongs to $H(U)$. Thus in each case $\varphi(w)$ is indeed an eigenvalue for $C_{\varphi, \psi}$ and so the proof is complete. $\square$

3. ADJOINT OF WEIGHTED COMPOSITION OPERATORS ON HILBERT SPACES OF ANALYTIC FUNCTIONS

Throughout this section let H be a Hilbert space of holomorphic functions on the open unit disk U such that for each $\lambda \in U$, the evaluation function $e_\lambda : H \rightarrow \mathbb{C}$ defined by $e_\lambda(f) = f(\lambda)$ is bounded on H . By the Riesz Representation Theorem there is a vector $k_z \in H$ such that $f(z) = \langle f, k_z \rangle$ for every $z \in U$. A complex-valued function φ on U for which $\varphi H \subseteq H$ is called a multiplier of H . The set of all multipliers of H is denoted by $M(H)$, and it is well known that $M(H) \subseteq H^\infty(U)$ ([19]). In this section we assume that H contains the constant functions and $M(H) = H^\infty(U)$. Note that H may be, for example, the classical Bergman and Hardy Hilbert space of U . Moreover, we suppose that ψ is a holomorphic self-map of U such that C_ψ acts boundedly on H , and φ is a multiplier of H . The adjoint of a composition operator and a multiplication operator has not yet been well characterized on any spaces of holomorphic functions. Nevertheless their action on reproducing kernels is determined. In fact $C_\psi^*(k_z) = k_{\psi(z)}$ and $M_\varphi^*(k_z) = \overline{\varphi(z)}k_z$ for every $z \in U$. Thus for each f in H and $z \in U$, we have

$$C_{\varphi, \psi}^* k_z = C_\psi^* M_\varphi^* k_z = \overline{\varphi(z)} C_\psi^* k_z = \overline{\varphi(z)} k_{\psi(z)}.$$

Theorem 3.1. *Let ψ be a holomorphic self-map of U such that $\|\psi_n\|_U < 1$ for some $n \geq 1$, and let w be the Denjoy-Wolff point of ψ . If $\varphi(w) \neq 0$, then $\varphi(w)$ is an eigenvalue for the operator $C_{\varphi, \psi}$ acting on H . Hence, $C_{\varphi, \psi}^*$ is not hypercyclic.*

Proof. Note that $w \in U$ since $\|\psi_n\|_U < 1$. Assume that $\varphi(w) \neq 0$. By Proposition 2.6, we know that there is a function $g \in H(U)$ such that

$$(**) \quad C_{\varphi, \psi} g = \varphi(w)g.$$

Hence

$$\prod_{j=0}^{n-1} \varphi(\psi_j)g \circ \psi_n = \varphi(w)^n g,$$

so that

$$g = \varphi(w)^{-n} \prod_{j=0}^{n-1} \varphi(\psi_j)g \circ \psi_n.$$

Because $\|\psi_n\|_U < 1$, the function $g \circ \psi_n \in H^\infty(U)$; moreover, each factor of $\prod_{j=0}^{n-1} \varphi(\psi_j)$ belongs to $H^\infty(U)$ since φ belongs to $H^\infty(U)$. Thus $g \in H^\infty(U) \subseteq H$, and thus (**) shows that $\varphi(w)$ is indeed an eigenvalue for $C_{\varphi, \psi} : H \rightarrow H$, as desired. But it is well known that the adjoint of a hypercyclic operator has no eigenvector; hence $C_{\varphi, \psi}^*$ fails to be hypercyclic and so the proof is complete. $\square$

Theorem 3.2. *Let w be the Denjoy-Wolff fixed point of ψ , $\varphi \in Lip_\beta(w)$ for some real number β , and $\varphi(w) \neq 0$. Also, suppose that $\|\varphi\|_U \leq |\varphi(w)|$. Then $\varphi(w)$ is an eigenvalue for the operator $C_{\varphi, \psi}$ acting on H whenever ψ is hyperbolic and $\beta > 0$, and thus, in this case, $C_{\varphi, \psi}^*$ is not hypercyclic.*

Proof. As we saw in the proof of Theorem 2.4, the map $g(z) = \prod_{n=0}^{\infty} \frac{1}{\varphi(w)} \varphi(\psi_n(z))$ is a bounded nonzero holomorphic function satisfying $\varphi(z) \cdot g(\psi(z)) = \varphi(w)g(z)$ for every $z \in U$. Now as before we can conclude that $g \in M(H) \subseteq H$, and so the proof is complete. $\square$

Remark. There is no $\varphi \in H(U)$ satisfying the hypothesis of Theorem 3.2 if $\varphi \in Lip_\beta(w)$ for some real number $\beta > 1$. The point is that $\|\varphi\|_U \leq |\varphi(w)|$ implies that $\frac{1}{\varphi(w)}\varphi$ is a self-map of U (assuming φ nonconstant). The assumption that $\varphi \in Lip_\beta(w)$ and $\beta > 1$ implies that the self-map $\frac{1}{\varphi(w)}\varphi$ has angular derivative 0 at w , which is impossible.

A nice condition for hypercyclicity is the Hypercyclicity Criterion which was developed independently by Kitai [13] and Gethner and Shapiro [9]. This criterion has been used to show that certain classes of many operators are hypercyclic. Some reformulation of this criterion is given in [20]. We use this criterion for indicating the hypercyclicity of the adjoint of $C_{\varphi, \psi}$.

Hypercyclicity Criterion. Suppose that X is a separable Banach space and that T is a continuous linear mapping on X . If there exist two dense subsets Y and Z in X and a sequence $\{n_k\}$ such that:

1. $T^{n_k}y \rightarrow 0$ for every $y \in Y$,
2. there exist functions $S_{n_k} : Z \rightarrow X$ such that for every $z \in Z$, $S_{n_k}z \rightarrow 0$, and $T^{n_k}S_{n_k}z \rightarrow z$,

then T is hypercyclic.

Theorem 3.3. *Let ψ be an elliptic automorphism with interior fixed point p and $\varphi : U \rightarrow \mathbb{C}$ satisfies the inequality: $|\varphi(p)| < 1 < \lim_{|z| \rightarrow 1^-} \inf |\varphi(z)|$. Then $C_{\varphi, \psi}^*$ is hypercyclic.*

Proof. Put $\Psi = \alpha_p \circ \psi \circ \alpha_p$ and $\Phi = \varphi \circ \alpha_p$, where $\alpha_p(z) = \frac{p-z}{1-\bar{p}z}$. Since Ψ is an automorphism with $\Psi(0) = 0$, thus Ψ is a rotation $z \rightarrow e^{i\theta}z$ for some $\theta \in [0, 2\pi]$ and every $z \in U$. This statement should be cancelled (deleted). Since $|\alpha_p(z)| \rightarrow 1^-$ when $|z| \rightarrow 1^-$, so

$$\lim_{|z| \rightarrow 1^-} \inf |\varphi(z)| \leq \lim_{|z| \rightarrow 1^-} \inf |\varphi \circ \alpha_p(z)|.$$

Thus

$$|\Phi(0)| < 1 < \lim_{|z| \rightarrow 1^-} \inf |\Phi(z)|.$$

Therefore there exist some constants λ_1 and λ_2 and positive numbers $\delta_1 < 1$ and $\delta_2 < 1$ such that $|\Phi(z)| < \lambda_1 < 1$ when $|z| < \delta_1$, and $|\Phi(z)| > \lambda_2 > 1$ when $|z| > 1 - \delta_2$. Set

$$H_1 = \text{span}\{k_z : |z| < \delta_1\}$$

and

$$H_2 = \text{span}\{k_z : |z| > 1 - \delta_2\},$$

where $\text{span}\{.\}$ is the set of finite linear combinations of $\{.\}$. So H_1 and H_2 are dense subsets of H (see Proposition 4.2 in [10]). Put $T = C_{\Phi, \Psi}^*$. Then for every $z \in U$ we get

$$T(k_z) = C_{\Phi, \Psi}^*(k_z) = C_{\Psi}^* M_{\Phi}^*(k_z) = C_{\Psi}^*(\overline{\Phi(z)}k_z) = \overline{\Phi(z)}C_{\Psi}^*(k_z) = \overline{\Phi(z)}k_{\Psi(z)}$$

and

$$T^2(k_z) = \overline{\Phi(z)} \cdot \overline{\Phi(\Psi(z))} k_{\Psi_2(z)}.$$

In general, for every positive integer n and $z \in U$ we have

$$T^n(k_z) = \left[\prod_{i=0}^{n-1} \overline{\Phi(\Psi_i(z))} \right] k_{\Psi_n(z)}.$$

Note that for each positive integer n , $|z| = |\Psi_n(z)|$ since Ψ_n is a rotation. So if $|z| < \delta_1$, then for each positive integer n , $|\Phi(\Psi_n(z))| < \lambda_1 < 1$, and if $|z| > 1 - \delta_2$, $|\Phi(\Psi_n(z))| > \lambda_2 > 1$. Now if $|z| < \delta_1$, then $|T^n k_z| \rightarrow 0$ as $n \rightarrow \infty$. Therefore the sequence $\{T^n\}$ converges pointwise to zero on the dense subset H_1 . Define a sequence of linear maps $S_n : H_2 \rightarrow H_2$ by

$$S_n k_z = \left[\prod_{i=1}^n \overline{(\Phi(\Psi_i^{-1}(z)))}^{-1} \right] k_{\Psi_n^{-1}(z)}$$

for every $z \in U$. Note that if $|z| > 1 - \delta_2$, then $S_n k_z \rightarrow 0$ as $n \rightarrow \infty$. Thus the sequence $\{S_n\}$ converges pointwise to zero on the dense subset H_2 . Also note that $T^n S_n k_z = k_z$ on H_2 , which implies that $T^n S_n$ is the identity on the dense subset H_2 . Therefore T satisfies the hypothesis of the Hypercyclicity Criterion and so T is hypercyclic. But $C_{\varphi, \psi} = C_{\alpha_p} T^* C_{\alpha_p}^{-1}$; thus T is similar to $C_{\varphi, \psi}^*$, which implies that $C_{\varphi, \psi}^*$ is also hypercyclic. This completes the proof. $\square$

Example. Let $\varphi(z) = 2z$ and $\psi(z) = e^{i\theta}z$. Then by Theorem 3.3 the operator $C_{\phi, \psi}^*$ is hypercyclic because 0 is an interior fixed point of ψ , $\varphi(0) = 0$ and

$$\lim_{|z| \rightarrow 1^-} \inf |\varphi(z)| > 1.$$

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