THE LIMITING CASE OF THE MARCINKIEWICZ INTEGRAL: GROWTH FOR CONVEX SETS

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ABSTRACT. The Marcinkiewicz integral

$$I_\lambda(x) = \int_{\Omega} \frac{(\text{dist}(y, \mathbb{R}^n \setminus \Omega))^\lambda}{|x - y|^{n+\lambda}} dy, \text{ where } \lambda > 0,$$

plays a well-known and prominent role in harmonic analysis. In this paper, we estimate the growth of it in the limiting case $\lambda \rightarrow 0$. Throughout, we assume that Ω is convex; it is interesting that this condition cannot be dropped.

1. INTRODUCTION

Our aim in this paper is to prove the following theorem:

Theorem 1. *Let Ω be a bounded open convex subset of $\mathbb{R}^n$. For $x \in \mathbb{R}^n \setminus \Omega$, put*

$$(1.1) \quad F_\Omega(x) = \int_{\Omega} \frac{1}{|x - y|^n} dy.$$

Let $\Omega_t = \{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) < t\}$. Then

$$(1.2) \quad \sup_{x \in \mathbb{R}^n \setminus \Omega_t} F_\Omega(x) \leq c(n) \frac{|\Omega|}{|\Omega_t \setminus \Omega|},$$

where $c = c(n) > 0$ is a constant dependent only on the dimension n .

The most interesting feature of this result is that the constant in (1.2) is *independent* of the geometry of the convex set Ω . It should be noted that the convexity of Ω is essential. In §4 we present an example of a bounded domain Ω for which the constant c in (1.2) is infinite. It is also possible to construct a bounded domain Ω with a smooth boundary and with an arbitrarily large constant c in (1.2). Moreover, the estimate (1.2) is sharp in the sense that both sides of it are equivalent to t^{-n} for large t .

Theorem 1 will be proved in §§2-3; here we comment on the question of why the function (1.1) is interesting.

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First of all this function can be considered as a bordering case between the Marcinkiewicz integral

$$I_\lambda(x) = \int_{\Omega} \frac{(\text{dist}(y, \mathbb{R}^n \setminus \Omega))^\lambda}{|x - y|^{n+\lambda}} dy, \quad \text{where } \lambda > 0,$$

and the Riesz potential operator

$$R(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy, \quad \alpha > 0$$

for $f = \chi_\Omega$.

Moreover, the function (1.1) arises naturally in estimates of Poisson integral (see [8]), the moment problem, operator theory and computerized tomography (see, for example, the survey [7] and [1], [3]). It is also connected with an analytic continuation of Cauchy transform, free boundary problems and with fluid mechanics (Hele-Shaw flow) (see [2]).

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2. AUXILIARY LEMMAS

In this section we prove some statements to be used subsequently.

2.1. Independence of a metric in $\mathbb{R}^n$. Put

$$(2.1) \quad \varphi(t) = \sup_{x \in \mathbb{R}^n \setminus \Omega_t} F_\Omega(x), \text{ for } t > 0.$$

In the definition $\Omega_t = \{x \in \mathbb{R}^n \mid \text{dist}(x, \Omega) < t\}$ of Ω_t we assume that

$$(2.2) \quad \text{dist}(x, \Omega) = \inf_{y \in \Omega} \left(\sum_{i=1}^n (x_i - y_i)^2 \right)^{\frac{1}{2}}.$$

In what follows, it will be convenient to use the metric

$$(2.3) \quad \rho(x, y) = \max_{i=1, \dots, n} |x_i - y_i|$$

in place of the Euclidean metric in $\mathbb{R}^n$. Consequently, (2.2) is replaced by

$$(2.4) \quad \rho(x, \Omega) = \inf_{y \in \Omega} \rho(x, y).$$

We show that it suffices to prove the version of Theorem 1 in which the distance (2.4) is used in place of (2.2) throughout, i.e., Ω_t and $\varphi(t)$ are replaced by

$$(2.5) \quad \Omega_{t,\rho} = \{x \in \mathbb{R}^n \mid \rho(x, \Omega) < t\},$$

$$(2.6) \quad \varphi_\rho(t) = \sup_{x \in \mathbb{R}^n \setminus \Omega_{t,\rho}} F_\Omega(x).$$

Indeed, all norms on $\mathbb{R}^n$ are equivalent, therefore

$$(2.7) \quad \Omega_{c_1 t, \rho} \subset \Omega_t \subset \Omega_{c_2 t, \rho}$$

with constants $c_2 > c_1 > 0$ independent of $t > 0$. So

$$(2.8) \quad \varphi_\rho(c_2 t) \leq \varphi(t) \leq \varphi_\rho(c_1 t).$$

If Theorem 1 is proved with the metric ρ , then

$$\varphi(t) = \sup_{x \in \mathbb{R}^n \setminus \Omega_t} F_\Omega(x) \leq \sup_{x \in \mathbb{R}^n \setminus \Omega_{c_1 t, \rho}} F_\Omega(x) \leq c(n) \frac{|\Omega|}{|\Omega_{c_1 t, \rho} \setminus \Omega|} \leq c(n) \frac{|\Omega|}{|\Omega_{\frac{c_1}{c_2} t} \setminus \Omega|}.$$

Consequently, we only need to prove the inequality

$$(2.9) \quad |\Omega_{\lambda t} \setminus \Omega| \geq c |\Omega_t \setminus \Omega|$$

with a constant $c = c(n, \lambda) > 0$ independent of t . If $\lambda \geq 1$, then $\Omega_{\lambda t} \supset \Omega_t$ and we can take $c = 1$, so we can assume that $\lambda < 1$.

To prove this, take $x \in \Omega_t \setminus \Omega$. Then there exists $y \in \Omega$ such that the ball $B(y, t)$ of radius t and centered at y contains x . Since Ω is open, there is no loss of generality in assuming that $y \in \partial\Omega$ ($\partial\Omega$ is the boundary of Ω). Therefore, $\Omega_t \setminus \Omega$ is covered by certain balls of radius t whose centers lie on $\partial\Omega$. By the Wiener covering theorem (see [9, p. 9] or the so-called $5r$ theorem in [6]), we can find a finite collection $y_1, \dots, y_N \in \partial\Omega$ such that the balls $B(y_i, t)$ are mutually disjoint and the balls $B(y_i, 5t)$ cover $\Omega_t \setminus \Omega$. Therefore,

$$(2.10) \quad |\Omega_t \setminus \Omega| \leq 5^n w_n \cdot (t)^n \cdot N,$$

where w_n is the volume of the n -dimensional unit ball.

Now, consider the balls $B(y_i, \lambda t)$. For each i , a support hyperplane π_i to the convex set Ω passes through y_i , and Ω lies entirely in only one of the two half-spaces determined by π_i . The intersection of $B(y_i, \lambda t)$ with the other half-space is included in $\Omega_t \setminus \Omega$, and the balls $B(y_i, \lambda t)$ are mutually disjoint (because so are the balls $B(y_i, t)$, and $\lambda < 1$). It follows that

$$(2.11) \quad |\Omega_{\lambda t} \setminus \Omega| \geq \frac{1}{2} \lambda^n w_n t^n N.$$

Combining this with (2.10), we obtain

$$(2.12) \quad |\Omega_{\lambda t} \setminus \Omega| \geq \frac{\lambda^n}{2 \cdot 5^n} |\Omega_t \setminus \Omega|.$$

This proves (2.9).

In fact, the above argument shows that it suffices to prove Theorem 1 for any particular metric generated by some norm in $\mathbb{R}^n$.

2.2. Reduction to the case of an n -dimensional rectangle. In this subsection we show that it suffices to prove Theorem 1 only for the rectangle $(-a_1, a_1) \times \dots \times (-a_n, a_n)$ in the role of Ω .

Indeed, let Ω be an arbitrary open and bounded convex set. By the John theorem, there exists an ellipsoid E with $\Omega \subset E$ and $|E| \leq c(n) |\Omega|$. Since translations and orthogonal transformations change neither distances nor volumes (consequently, nor the values of F_Ω), we may assume that the center of E is at zero and its axes are directed along coordinate axes. Let a_i be the length of the i th semiaxis of E . Then for the rectangle

$$R = (-a_1, a_1) \times \dots \times (-a_n, a_n)$$

we have $\Omega \subset R$ and

$$(2.13) \quad |R| \leq c(n) |\Omega|.$$

Now, suppose that Theorem 1 is true for R . We show that then it is true for Ω .

Indeed, since $\Omega \subset R$ and Ω and R are bounded and convex, we have

$$(2.14) \quad |\Omega_t \setminus \Omega| \leq |R_t \setminus R|$$

(see [4]). Put

$$\varphi_R(t) = \sup_{x \in \mathbb{R}^n \setminus R_t} F_R(x).$$

It suffices to prove the estimate

$$(2.15) \quad \varphi(t) \leq c(n) \varphi_R(\lambda t),$$

with a constant $\lambda = \lambda(n)$ depending only on n .

Indeed, if Theorem 1 is true for R , then (2.14) and (2.13) imply¹

$$\varphi_R(t) \leq c(n) \frac{|R|}{|R_t \setminus R|} \leq c(n) \frac{|\Omega|}{|R_t \setminus R|} \leq c(n) \frac{|\Omega|}{|\Omega_t \setminus \Omega|}.$$

Consequently, if we prove (2.15), then from (2.9) we obtain

$$\varphi(t) \leq c(n) \varphi_R(\lambda t) \leq c(n) \frac{|\Omega|}{|\Omega_{\lambda t} \setminus \Omega|} \leq c(n) \frac{|\Omega|}{|\Omega_t \setminus \Omega|}.$$

Inequality (2.8) shows that it suffices to prove (2.15) in the case of the metric $\rho(x, y) = \max_{i=1, \dots, n} |x_i - y_i|$, i.e., under the assumption that the sets Ω_t, R_t and the functions φ, φ_R are defined on the basis of the metric ρ .

To do this, we consider a point $x \in \mathbb{R}^n \setminus \Omega_t$, i.e., $\rho(x, \Omega) \geq t$, and estimate the value $F_\Omega(x)$. Let $y \in \partial\Omega$ be a point nearest to x , i.e., $\rho(x, y) = \rho(x, \Omega)$. Let $\sigma_1, \sigma_2, \dots, \sigma_{2^n}$ denote the vertices of the rectangle $\bar{R} = [-a_1, a_1] \times \dots \times [-a_n, a_n]$, and let $\bar{R}_i$ be the translate of $\bar{R}$ such that σ_i is translated to y . Since $y \in \bar{R}$, it is easily seen that $\bar{R} \subset \bigcup_i \bar{R}_i$ and, consequently,

$$(2.16) \quad \Omega \subset \bigcup_i \bar{R}_i.$$

Among the $\bar{R}_i$, we select those that intersect Ω , i.e., we consider the set $I = \{i \mid \Omega \cap \bar{R}_i \neq \emptyset\}$. Then (2.16) and the definition of $F_\Omega(x)$ imply

$$(2.17) \quad F_\Omega(x) \leq \sum_{i \in I} F_{R_i}(x).$$

If we show that $\rho(x, R_i) \geq t$ for $i \in I$, then we obtain

$$F_\Omega(x) \leq \sum_{i \in I} \varphi_{R_i}(t) \leq |I| \varphi_R(t) \leq 2^n \varphi_R(t)$$

and, consequently,

$$\varphi(t) \leq 2^n \varphi_R(t).$$

Thus, it remains to prove that the relations $\rho(x, \Omega) = \rho(x, y) \geq t$ imply $\rho(x, R_i) \geq t$ for all $i \in I$. Suppose the contrary, i.e., there is $z \in R_i$ ($i \in I$) with $\rho(x, z) < t$. Now, we observe that all points u lying in R_i near y can be written in the form

$$\begin{aligned} u &= (y_1, y_2, \dots, y_n) + (\alpha_1(z_1 - y_1), \dots, \alpha_n(z_n - y_n)) \\ &= (\alpha_1 z_1 + (1 - \alpha_1)y_1, \dots, \alpha_n z_n + (1 - \alpha_n)y_n) \end{aligned}$$

¹Here and below, the notation $c(n)$ is used for constants depending only on the dimension n ; their particular values may vary from one occurrence of this symbol to another.

with $\alpha_1, \dots, \alpha_n \in (0, 1)$. Since $\rho(x, z) < t \leq \rho(x, y)$, we have

$$(2.18) \quad \begin{aligned} \rho(x, u) &= \max_i |x_i - \alpha_i z_i - (1 - \alpha_i) y_i| \\ &\leq \max_i (\alpha_i |x_i - z_i| + (1 - \alpha_i) |x_i - y_i|) < \rho(x, y). \end{aligned}$$

Next, since $\Omega \cap R_i \neq \emptyset$ and $y \in \bar{\Omega}$, by the convexity of Ω we see that in $\Omega \cap R_i$ there are points u arbitrarily close to y . Then $\rho(x, u) < \rho(x, y)$ by (2.18), contrary to the choice of y .

The contradiction proves the claim.

3. PROOF OF THEOREM 1 IN THE CASE WHEN Ω IS AN n -DIMENSIONAL RECTANGLE

We saw in §2 that it suffices to prove the inequality

$$(3.1) \quad \varphi(t) = \sup_{x \in \mathbb{R}^n \setminus \Omega_t} F_\Omega(x) \leq c(n) \frac{|\Omega|}{|\Omega_t \setminus \Omega|},$$

where

$$(3.2) \quad \Omega = (-a_1, a_1) \times \cdots \times (-a_n, a_n),$$

$$(3.3) \quad \Omega_t = \{x \in \mathbb{R}^n \mid \rho(x, \Omega) < t\},$$

and

$$(3.4) \quad \rho(x, y) = \max |x_i - y_i|, \text{ where } x = (x_1, \dots, x_n) \text{ and } y = (y_1, \dots, y_n).$$

In this section we verify (3.1). The proof is based on two lemmas. Below we assume for definiteness that

$$(3.5) \quad 0 < a_1 \leq \cdots \leq a_n.$$

Lemma 1. *a) If $0 < t < a_1$, then*

$$(3.6) \quad |\Omega_t \setminus \Omega| \leq c(n) \frac{t}{a_1} |\Omega|,$$

b) If $a_k \leq t \leq a_{k+1}$ ($k = 1, \dots, n-1$) or $t \geq a_n$ ($k = n$), then

$$(3.7) \quad |\Omega_t \setminus \Omega| \leq c(n) \frac{t}{a_1} \cdots \frac{t}{a_k} |\Omega|.$$

Proof. Inequality (3.6) is a consequence of the estimates

$$\begin{aligned} |\Omega_t \setminus \Omega| &\leq (2a_1 + 2t) \cdots (2a_n + 2t) - 2a_1 \cdots 2a_n \\ &= 2a_1 \cdots 2a_n \left[\left(1 + \frac{t}{a_1}\right) \cdots \left(1 + \frac{t}{a_n}\right) - 1 \right] \\ &\leq |\Omega| \left[\left(1 + \frac{t}{a_1}\right)^n - 1 \right] \leq n \cdot 2^{n-1} \cdot \frac{t}{a_1} |\Omega|, \end{aligned}$$

where at the last step we have used the formula $f(b) - f(a) = f'(c) \cdot (b - a)$ for the function $f(x) = (1 + x)^n$ and $a = 0$, $b = \frac{t}{a_1}$.

Inequality (3.7) is also elementary:

$$\begin{aligned} |\Omega_t \setminus \Omega| &\leq (2a_1 + 2t) \cdots (2a_n + 2t) \leq (4t)^k (4a_{k+1}) \cdots (4a_n) \\ &= 2^n \frac{t}{a_1} \cdots \frac{t}{a_k} |\Omega|. \end{aligned}$$

□

In the proof of another lemma, we shall use the fact that it suffices to estimate $\sup_{x \in \mathbb{R}^n \setminus \Omega_t} F_\Omega(x)$ for x belonging to the union of coordinate axes. Moreover, the following statement is true.

Proposition 1. For $\Omega = (-a_1, a_1) \times \cdots \times (-a_n, a_n)$ we have

$$(3.8) \quad \sup_{x \in \mathbb{R}^n \setminus \Omega_t} F_\Omega(x) \leq c(n) \max_{i=1, \dots, n} F_\Omega((a_i + t)e_i),$$

where $\{e_1, \dots, e_n\}$ is the standard basis in $\mathbb{R}^n$.

Proof. Suppose $x \in \mathbb{R}^n \setminus \Omega_t$, i.e., $\rho(x, \Omega) \geq t$. The symmetry of Ω allows us to assume that all coordinates of x are nonnegative. Then

$$\rho(x, \Omega) = \max_i (x_i - a_i) \geq t.$$

There is an index i_0 with $x_{i_0} - a_{i_0} = \rho(x, \Omega)$. We replace by a_i all coordinates x_i of x such that $i \neq i_0$ and $x_i - a_i > 0$, obtaining a new vector $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$ with the property

$$|\tilde{x} - y| \leq |x - y|, \text{ for any } y \in \Omega.$$

Then the relation $\rho(\tilde{x}, \Omega) = \rho(x, \Omega)$ and (1.1) show that $F_\Omega(\tilde{x}) \geq F_\Omega(x)$. Thus, when estimating the left-hand side of (3.8), we may assume that exactly one coordinate x_{i_0} of x is greater than a_{i_0} .

Next, consider the translation

$$\begin{aligned} \tilde{y}_i &= y_i - x_i, \text{ for all } i \neq i_0, \\ \tilde{y}_{i_0} &= y_{i_0}, \end{aligned}$$

which takes x to $x_{i_0}e_{i_0}$, and Ω to some rectangle $\tilde{\Omega}$. Clearly, $F_\Omega(x) = F_{\tilde{\Omega}}(x_{i_0}e_{i_0})$. Furthermore, the set

$$\bar{\Omega} = (-2a_1, 2a_1) \times \cdots \times (-a_{i_0}, a_{i_0}) \times \cdots \times (-2a_n, 2a_n)$$

(we multiply all a_i except a_{i_0} by 2) contains $\tilde{\Omega}$, whence it follows that $F_\Omega(x) \leq F_{\bar{\Omega}}(x_{i_0}e_{i_0})$. The change of variables

$$\begin{aligned} \bar{y}_i &= \frac{1}{2}\tilde{y}_i, \text{ for all } i \neq i_0, \\ \bar{y}_{i_0} &= y_{i_0}, \end{aligned}$$

maps $\bar{\Omega}$ into Ω and fixes $x_{i_0}e_{i_0}$. After simple calculations this yields

$$F_{\bar{\Omega}}(x_{i_0}e_{i_0}) \leq 2^{n-1}F_\Omega(x_{i_0}e_{i_0}).$$

Therefore, $F_\Omega \leq 2^{n-1}F_\Omega(x_{i_0}e_{i_0})$. It remains to observe that $F_\Omega(x_{i_0}e_{i_0})$ increases as x_{i_0} decreases, and so, under the condition $\rho(x_{i_0}e_{i_0}, \Omega) \geq t$, its minimal value is attained for $x_{i_0} = a_{i_0} + t$. $\square$

We shall use the following elementary estimates:

$$(3.9) \quad \int_t^{t+a} \frac{1}{s^m} ds \leq \int_t^{t+a} \frac{1}{t^m} ds = \frac{a}{t^m}, \text{ where } t, a > 0 \text{ and } m \geq 0;$$

$$(3.10) \quad \int_t^{t+a} \frac{1}{s^m} ds \leq \int_t^\infty \frac{1}{s^m} ds = \frac{1}{m-1} \cdot \frac{1}{t^{m-1}}, \text{ where } t, a > 0 \text{ and } m > 1.$$

Also, we need the fact that the integrals

$$(3.11) \quad c_{m,n} = \int_{\mathbb{R}^m} \frac{1}{\left(1 + \sum_{i=1}^m u_i^2\right)^{\frac{n}{2}}} du_1 \cdots du_m < \infty$$

are finite for $m < n$.

Lemma 2. *a) If $0 < t < a_1$, then*

$$(3.12) \quad \sup_{x \in \mathbb{R}^n \setminus \Omega_t} F_{\Omega}(x) \leq c(n) \frac{a_1}{t}.$$

b) If $a_k \leq t \leq a_{k+1}$ ($k = 1, \dots, n-1$) or $t \geq a_n$ ($k = n$), then

$$(3.13) \quad \sup_{x \in \mathbb{R}^n \setminus \Omega_t} F_{\Omega}(x) \leq c(n) \frac{a_1}{t} \cdots \frac{a_k}{t},$$

where $c(n)$ is a constant depending on the dimension n only.

Proof. We start with case a). By Proposition 1, we may assume that $x = (t + a_{i_0}) e_{i_0}$. We consider two cases: $i_0 = 1$ and $i_0 \neq 1$. In the first case

$$F_{\Omega}(x) = \int_{\Omega} \frac{1}{\left((t + a_1 - y_1)^2 + \sum_{i \neq 1} y_i^2\right)^{\frac{n}{2}}} dy.$$

We substitute $s = t + a_1 - y_1$ and extend integration in $y_2, \dots, y_n$ over $(-\infty, +\infty)$, obtaining

$$F_{\Omega}(x) \leq \int_t^{t+2a_1} \left(\int_{\mathbb{R}^{n-1}} \frac{1}{\left(s^2 + \sum_{i \neq 1} y_i^2\right)^{\frac{n}{2}}} dy_2 \cdots dy_n \right) ds.$$

Factoring out s^n in the denominator and changing the variables $u_2 = \frac{y_2}{s}, \dots, u_n = \frac{y_n}{s}$, we arrive at

$$F_{\Omega}(x) \leq \int_t^{t+2a_1} \frac{1}{s} \left(\int_{\mathbb{R}^{n-1}} \frac{1}{\left(1 + \sum_{i \neq 1} u_i^2\right)^{\frac{n}{2}}} du_2 \cdots du_n \right) ds.$$

Therefore, (3.9) and (3.11) imply

$$(3.14) \quad F_{\Omega}(x) \leq c(n) \frac{a_1}{t}.$$

But if $i_0 \neq 1$, we have

$$F_{\Omega}(x) = \int_{\Omega} \frac{1}{\left((a_{i_0} + t - y_{i_0})^2 + \sum_{i \neq i_0} y_i^2\right)^{\frac{n}{2}}} dy.$$

Replacing y_1 by 0 in the denominator, we see that

$$F_{\Omega}(x) \leq 2a_1 \int_{\Omega'} \frac{1}{\left((a_{i_0} + t - y_{i_0})^2 + \sum' y_i^2\right)^{\frac{n}{2}}} dy_2 \cdots dy_n,$$

where Ω' is the intersection of Ω with the coordinate hyperplane $y_1 = 0$, and $\sum'$ means summation over $i \neq i_0, i \neq 1$. Proceeding as before, we substitute $s = a_{i_0} + t - y_{i_0}, u_i = \frac{y_i}{s}$ ($i \neq i_0, i \neq 1$) and extend integration with respect to u_i over $(-\infty, +\infty)$. This leads to

$$(3.15) \quad F_{\Omega}(x) \leq 2a_1 \int_t^{t+2a_{i_0}} \frac{1}{s^2} \left(\int_{\mathbb{R}^{n-2}} \frac{1}{(1 + \sum' u_i^2)^{\frac{n}{2}}} du \right) ds \leq c(n) \frac{a_1}{t};$$

we have used (3.10) and (3.11). Inequalities (3.14) and (3.15) prove a).

Case b). Three possibilities may occur: $k < i_0$, $k = i_0$, and $k > i_0$. We treat them separately.

If $k < i_0$, we replace $y_1, \dots, y_k$ by 0 and extend integration with respect to y_i ($i > k, i \neq i_0$) over $(-\infty, +\infty)$. Arguing much as in case a), we arrive at

$$F_{\Omega}(x) \leq 2a_1 \cdots 2a_k \int_t^{t+2a_{i_0}} \frac{1}{s^{k+1}} \left(\int_{\mathbb{R}^{n-k-1}} \frac{1}{(1 + \sum_i u_i^2)^{\frac{n}{2}}} du \right) ds.$$

Then (3.10) and (3.11) imply

$$(3.16) \quad F_{\Omega}(x) \leq c(n) \frac{a_1}{t} \cdots \frac{a_k}{t}.$$

If $k = i_0$, we replace $y_1, \dots, y_{k-1}$ by 0 and extend integration in the variables $y_{k+1}, \dots, y_n$ over $(-\infty, +\infty)$. After similar substitution we obtain

$$F_{\Omega}(x) \leq 2a_1 \cdots 2a_{k-1} \int_t^{t+2a_k} \frac{1}{s^k} \left(\int_{\mathbb{R}^{n-k}} \frac{1}{(1 + \sum_i u_i^2)^{\frac{n}{2}}} du \right) ds.$$

So, (3.9) and (3.11) imply

$$(3.17) \quad F_{\Omega}(x) \leq c(n) \frac{a_1}{t} \cdots \frac{a_k}{t}.$$

Finally, if $k > i_0$, we replace $y_1, \dots, y_{i_0-1}, y_{i_0+1}, \dots, y_k$ by 0 and extend integration in the variables $y_{k+1}, \dots, y_n$ over $(-\infty, +\infty)$. Similar changes of variables yield

$$(3.18) \quad F_{\Omega}(x) \leq 2a_1 \cdots 2a_{i_0-1} \cdot 2a_{i_0+1} \cdots 2a_k \cdot \int_t^{t+2a_{i_0}} \frac{1}{s^k} \left(\int_{\mathbb{R}^{n-k}} \frac{1}{(1 + \sum u_i^2)^{\frac{n}{2}}} du \right) ds \leq c(n) \frac{a_1}{t} \cdots \frac{a_k}{t}.$$

We have used (3.9) and (3.11) at the last step. Now, b) follows from (3.16)–(3.18). $\square$

Theorem 1 is now proved by an immediate application of Lemmas 1 and 2.

4. COUNTEREXAMPLE

Examples given below show that the convexity condition in Theorem 1 is essential. We begin with the one-dimensional case; then we show how to use it for constructing an example on the plane with connected Ω .

We will need the following simple observation.

Proposition 2. *Let Ω be a bounded open subset of $\mathbb{R}^n$ such that the statement of Theorem 1 is correct. Then*

$$(4.1) \quad \|F_\Omega(x)\|_{L_{1,\infty}(\mathbb{R}^n \setminus \Omega)} \leq c |\Omega|,$$

where $c = c(n) > 0$ is a constant depending only on the dimension n .

Proof. Clearly,

$$\varphi(t) = \sup_{x \in \mathbb{R}^n \setminus \Omega_t} F_\Omega(x)$$

is a monotone nonincreasing function of t . Since Ω is bounded, the definition (1.1) of F_Ω implies that $\varphi(t) \rightarrow 0$ as $t \rightarrow \infty$. Since Ω is open, we have $\varphi(t) \rightarrow \infty$ as $t \rightarrow 0$. Next, the continuity of F_Ω shows that φ has no jumps and, therefore, is continuous. So, φ is a continuous monotone nondecreasing function taking all values between 0 and ∞ . Now, take a number $c > 0$ and put

$$t_c = \inf \{t > 0 \mid \varphi(t) \leq c\}.$$

Then $\varphi(t_c) = c$, and if $F_\Omega(x) > c$, then $x \in \Omega_{t_c} \setminus \Omega$. Since we suppose that the statement of Theorem 1 is correct, we have

$$\varphi(t_c) \leq c(n) \frac{|\Omega|}{|\Omega_{t_c} \setminus \Omega|},$$

and therefore

$$(4.2) \quad c \cdot |\{x \mid F_\Omega(x) > c\}| \leq \varphi(t_c) \cdot |\Omega_{t_c} \setminus \Omega| \leq c(n) \cdot |\Omega|. \quad \square$$

Remark 1. The inequality $\|F_\Omega(x)\|_{L_{1,\infty}(\mathbb{R}^n \setminus \Omega)} \leq c |\Omega|$ for a convex Ω with the constant dependent only on the dimension n can also be obtained from the proof of Theorem 1 in [8] and some rather simple considerations connected with the convexity. Authors are very thankful to P. Sjögren for this observation.

Example 1. Consider an interval $(a, b) \subset \mathbb{R}$. We split it into $4N$ subintervals of equal length and define Ω to be the union of even subintervals, that is, $\Omega = \bigcup_{k=1}^{2N} I_k$, where

$$I_k = \left(a + \frac{2k-1}{4N(b-a)}, a + \frac{2k}{4N(b-a)} \right).$$

Let $x \in (a, b) \setminus \Omega$. Then at least N intervals I_k lie to one side of x . Denote by Ω' the union of these intervals. Then

$$F_\Omega(x) \geq F_{\Omega'}(x) = \sum_{I_k \in \Omega'} F_{I_k}(x).$$

Therefore,

$$\begin{aligned}
 (4.3) \quad F_{\Omega}(x) &\geq \sum_{k=1}^N \int_{\frac{2k-1}{4N(b-a)}}^{\frac{2k}{4N(b-a)}} \frac{1}{t} dt \geq \sum_{k=1}^N \frac{4N(b-a)}{2k} \cdot \frac{1}{4N(b-a)} \\
 &= \sum_{k=1}^N \frac{1}{2k} \geq \frac{1}{2} \sum_{k=1}^N \frac{1}{k} \geq c_1 \ln(c_2 N),
 \end{aligned}$$

where $c_1, c_2 > 0$ are universal constants. Since the measure of $(a, b) \setminus \Omega$ is $\frac{b-a}{2}$, we have

$$(4.4) \quad |\{x \in \mathbb{R} \mid F_{\Omega}(x) > c_1 \ln(c_2 N)\}| \cdot c_1 \ln(c_2 N) \geq \frac{b-a}{2} \cdot c_1 \ln(c_2 N).$$

Remark 2. Since N in (4.4) is arbitrary, this example already shows that, without the convexity assumption, the constant c in Theorem 1 may become arbitrarily large.

Example 2. We continue the construction. Consider the union of the intervals $U_k = (\frac{1}{2^k}, \frac{1}{2^{k-1}})$, $k = 1, 2, \dots$, and split each U_k into $4N_k$ subintervals of equal length. We denote by Ω_k the union of $2N_k$ even intervals and put $\tilde{\Omega} = \bigcup \Omega_k$.

By (4.3), on the set $U_k \setminus \Omega_k$ we have

$$(4.5) \quad F_{\Omega}(x) \geq c_1 \ln(c_2 N_k).$$

Since $|U_k \setminus \Omega_k| = \frac{1}{2^{k+1}}$, we see that

$$|\{x \in \mathbb{R} \mid F_{\Omega}(x) > c_1 \ln(c_2 N_k)\}| \cdot c_1 \ln(c_2 N_k) \geq \frac{1}{2^{k+1}} \cdot c_1 \ln(c_2 N_k).$$

Therefore, if N_k are chosen in such a way that the right-hand side tends to infinity as $k \rightarrow \infty$, then the claim of Theorem 1 fails for $\tilde{\Omega}$, whatever $c > 0$.

The set $\tilde{\Omega} \subset \mathbb{R}$ constructed above is not connected; however, on the plane a connected set Ω with similar properties can be exhibited. For this, consider first the product $\tilde{\Omega} \times [0, 1]$, where $\tilde{\Omega}$ was constructed above. Routine but somewhat lengthy calculations show that

$$(4.6) \quad F_{\Omega}(x) \geq c_1 \ln(c_2 N_k)$$

for $x \in (U_k \setminus \Omega_k) \times (0, 1)$. Now, we add a “small cap” $[0, 1] \times [1, 1 + \varepsilon]$, and take the interior of the set $(\tilde{\Omega} \times [0, 1]) \cup ([0, 1] \times [1, 1 + \varepsilon])$ for Ω . By (4.6), the claim of Corollary 1 and therefore of Theorem 1 fails for Ω , whatever c . Clearly, Ω is open, connected and bounded.

The question now arises as to whether the claim of Theorem 1 is true for domains with C^∞ -smooth boundary. We show that in this case the constant c may happen to be arbitrarily large. For this, we take the segment $[0, 1]$, split it into $4N$ subintervals of equal length, and denote by $\tilde{\Omega}$ the union of even subintervals. As is easily seen, the above construction (adding a “small cap”) yields a set $\Omega \subset \mathbb{R}^2$ with rectifiable boundary, which can be regularized to become of class C^∞ . However, for this Ω the constant will be at least $c_1 \ln(c_2 N)$, where $c_1, c_2 > 0$ do not depend on N .

Thus, unlike the initial statement of Theorem 1, the constant can be made arbitrarily large.

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