POINCARÉ SERIES ON BOUNDED SYMMETRIC DOMAINS

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(Communicated by Mei-Chi Shaw)

ABSTRACT. We show that any holomorphic automorphic form of sufficiently large weight on an irreducible bounded symmetric domain in $\mathbb{C}^n$, $n > 1$, is the Poincaré series of a polynomial in $z_1, \dots, z_n$ and give an upper bound for the degree of this polynomial. We also give an explicit construction of a basis in the space of holomorphic automorphic forms.

1. INTRODUCTION

Let $D = G/K$ be an irreducible bounded symmetric domain in $\mathbb{C}^n$, $n > 1$; i.e., G is one of the following Lie groups: $SU(p, q)$, $SO^*(2p)$, $Sp(p, \mathbb{R})$, $SO_o(p, 2)$, $E_6^{(-14)}$, $E_7^{(-25)}$ [1], and K is a maximal compact subgroup of G . Denote by 0 the fixed point of K in D . Consider a compact smooth quotient $X = \Gamma \backslash D$, where Γ is a discrete subgroup of G . Denote by L the canonical line bundle on X . Let k be a positive integer, and let χ be a normalized character on Γ . Denote by $S_k(\Gamma, \chi)$ the space of holomorphic (Γ, χ) -automorphic forms of weight k on D . It is a finite-dimensional complex vector space. If k is sufficiently large (one can take $k \geq (n+2)^{n+7}$, but this is not the best bound), then $L^{\otimes k}$ is very ample, and the Poincaré map is surjective (Theorem 16.5 [12]); i.e., every element of $S_k(\Gamma, \chi)$ is the Poincaré series of a holomorphic function $f : D \rightarrow \mathbb{C}$. In this paper we show that the function f can always be chosen as a polynomial in $z_1, \dots, z_n$ of degree not higher than $N_0 = \frac{k^n n!}{(2\pi)^n} \text{vol}(X) = k^n \int_X c_1(L)^n$ or as a linear combination of weighted Bergman kernels at $\dim S_k(\Gamma, \chi)$ points in D which project to $\dim S_k(\Gamma, \chi)$ distinct points in general position on X under the covering projection $\pi : D \rightarrow X$. The main results of this paper are Theorems 2.1, 2.2. The general approach is as follows. Denote $N = \dim S_k(\Gamma, \chi) - 1$. Choose points $p_0, \dots, p_N$ in D such that $\pi(p_0), \dots, \pi(p_N)$ are $N+1$ points in general position on X . To each of these points, p_j , associate the automorphic form $\varphi_{k,p_j,\chi}$ with the property $(f, \varphi_{k,p_j,\chi}) = f(p_j)$ for any $f \in S_k(\Gamma, \chi)$. Here $(\cdot, \cdot)$ denotes the inner product in $S_k(\Gamma, \chi)$. The $\dim S_k(\Gamma, \chi)$ automorphic forms we obtain this way form a basis in $S_k(\Gamma, \chi)$.

Now let us change the approach a little. For $l = (l_1, \dots, l_n)$, where l_j , $j = 1, \dots, n$, are nonnegative integers, denote by $\varphi_{k,l,0,\chi} \in S_k(\Gamma, \chi)$ the Poincaré series of the monomial $z_1^{l_1} \cdots z_n^{l_n}$. We show that if $(f, \varphi_{k,l,0,\chi}) = 0$ for all l such that $l_1 + \cdots + l_n \leq N_0$, then all partial derivatives $\frac{\partial^{l_1+\cdots+l_n} f}{\partial z_1^{l_1} \cdots \partial z_n^{l_n}}(0)$, $l_1 + \cdots + l_n \leq N_0$, are

Received by the editors October 6, 2005 and, in revised form, July 25, 2006.

2000 *Mathematics Subject Classification.* Primary 32N10; Secondary 32N05, 32N15.

The research of this author was supported in part by NSF grant DMS-0204154.

zero. This implies that the set $\{\varphi_{k,l,0,\chi} | l_1 + \dots + l_n \leq N_0\}$ spans $S_k(\Gamma, \chi)$. This spanning set for $S_k(\Gamma, \chi)$ is, in a sense, associated to the point $\pi(0) \in X$.

Similar statements for $n = 1$, $D = SU(1, 1)/U(1) \simeq SL(2, \mathbb{R})/SO(2)$, are well known and are due to Petersson [20], [7]. They are formulated and proved in terms of standard notions of the theory of Riemann surfaces such as holomorphic k -differentials and k -Weierstrass points. In particular, every automorphic form in $S_k(\Gamma, \chi)$ is the Poincaré series of a polynomial in z of degree not higher than $k(2g - 2)$, where g is the genus of the hyperbolic Riemann surface $\Gamma \backslash D$. If $0 \in D$ is not a k -Weierstrass point on the uniformized Riemann surface X , then every automorphic form of weight k is the Poincaré series of a polynomial in z of degree not greater than $k(2g - 2) - g$.

This circle of ideas for $n = 1$ is conveyed, in particular, by the following references: [3], [7], [8], [13], [14], [15], [16]. For $n \geq 1$ see [24], where the general case of flag domains is studied (bounded symmetric domains are a small part of this class of homogeneous complex manifolds).

2. AUTOMORPHIC FORMS ON BOUNDED SYMMETRIC DOMAINS

2.1. Preliminaries. Let $D = G/K$ be a bounded symmetric domain in $\mathbb{C}^n$, $n > 1$:

type I: $G = SU(p, q)$, $K = S(U(p) \times U(q))$,

type II: $G = SO^*(2p)$, $K = U(p)$,

type III: $G = Sp(p, \mathbb{R})$, $K = U(p)$,

type IV: $G = SO_o(p, 2)$, $K = SO(p) \times SO(2)$,

type V: $G = E_6^{(-14)}$, $K = SO(10) \cdot SO(2)$,

type VI: $G = E_7^{(-25)}$, $K = E_6 \cdot SO(2)$

(we follow the notation of [1, Table 1]).

Denote by $d\mu_L$ the standard Lebesgue measure on $D \subset \mathbb{C}^n$. Recall (see e.g. [11], [17]) that the Bergman kernel $K(z, w)$ for D is the kernel of the orthogonal projection from $L^2(D, d\mu_L)$ to the closed subspace $(L^2 \cap H)(D)$ which consists of holomorphic functions, $K(z, w) = \overline{K(w, z)}$, $K(z, z) > 0$ on D , for $\gamma \in G$, $K(\gamma z, \gamma w)J(\gamma, z)\overline{J(\gamma, w)} = K(z, w)$, where $J(\gamma, z)$ is the determinant of the Jacobi matrix of $\gamma : D \rightarrow D$ at z , $K(z, w)$ is holomorphic in z , antiholomorphic in w , and reproducing on holomorphic functions; i.e.,

$$\int_D K(z, w)f(w)d\mu_L(w) = f(z)$$

for any holomorphic function $f \in L^2(D, d\mu_L)$.

Denote $\omega = -\frac{i}{2\pi}\partial\bar{\partial}\ln K(z, z)$; it is a G -invariant Kähler form on D . Consider the Hermitian metric on D given by $ds^2 = 2\sum_{j,m} h_{jm}dz_j \otimes d\bar{z}_m$, where $h_{jm} = \frac{\partial^2 \ln K(z, z)}{\partial z_j \partial \bar{z}_m}$. The Bergman metric on D is $Re(ds^2) = 2\sum_{j,m} h_{jm}dz_j d\bar{z}_m$, the associated $(1, 1)$ form is $-\frac{1}{2}Im(ds^2) = 2\pi\omega$, and the volume form is $\frac{1}{n!}(2\pi\omega)^n$.

Define $d\mu_r = K(z, z)^r d\mu_L$, where $d\mu_L$ is the standard Lebesgue measure on $D \subset \mathbb{C}^n$ and $r \leq 0$. Also denote $d\mu(z) = K(z, z)d\mu_L(z)$. Consider the weighted Bergman space $\mathcal{H}_r^2(D)$ which consists of $\mathbb{C}$ -valued functions which are holomorphic in D and belong to $L^2(D, d\mu_r)$. The reproducing kernel for $\mathcal{H}_r^2(D)$ is $c(D, r)K(z, w)^{-r+1}$, where $c(D, r)$ is a positive constant given by the following equality [22]:

$$\frac{1}{c(D, r)} = \int_D \left(\frac{K(z, w)K(w, z)}{K(z, z)K(w, w)} \right)^{-r+1} d\mu(w).$$

2.2. Γ -automorphic forms. In this subsection we shall recall necessary facts regarding holomorphic automorphic forms with the canonical automorphy factor.

Let Γ be a discrete cocompact subgroup of G which acts freely on D . Let k be a positive integer. By definition a holomorphic Γ -automorphic form of weight k on D is a holomorphic function $f : D \rightarrow \mathbb{C}$ such that

$$f(\gamma z)[J(\gamma, z)]^k = f(z)$$

for any $\gamma \in \Gamma$, $z \in D$. Denote by $S_k(\Gamma)$ the space of holomorphic Γ -automorphic forms of weight k on D . We have $S_k(\Gamma) \simeq H^0(X, L^{\otimes k})$, and $\pi^*s = f(z)(dz_1 \wedge \cdots \wedge dz_n)^{\otimes k}$, where $X = \Gamma \backslash D$, $L = \Lambda^n T^{*'} X$ is the canonical line bundle on X , $s \in H^0(X, L^{\otimes k})$, $f \in S_k(\Gamma)$, and $\pi : D \rightarrow X$ is the covering map.

We note that $\pi^*c_1(L) = [\omega]$. For a sufficiently large k there is a well-defined embedding $X \hookrightarrow \mathbb{CP}^N \simeq \mathbb{P}(H^0(X, L^{\otimes k})^*)$, $p \mapsto [s_0(p), \dots, s_N(p)]$, where $N = \dim H^0(X, L^{\otimes k}) - 1$ and $s_0, \dots, s_N$ is a basis in $H^0(X, L^{\otimes k})$.

Recall that, by definition, the Poincaré series of a holomorphic function $f \in L^1(D, d\mu_{1-\frac{k}{2}})$ is $\sum_{\gamma \in \Gamma} f(\gamma z)[J(\gamma, z)]^k \in S_k(\Gamma)$. For sufficiently large k the series converges absolutely and uniformly on compact sets ([4, Theorem 9.1]) and the Poincaré map

$$(L^1 \cap H)(D, (\Lambda^n T^{*'} D)^{\otimes k}) \rightarrow H^0(X, L^{\otimes k}),$$

$$s \mapsto \sum_{\gamma \in \Gamma} s(\gamma z),$$

is surjective (see [12, Chapter 7] and references therein).

2.3. (Γ, χ) -automorphic forms. In this subsection we shall discuss holomorphic automorphic forms with the canonical automorphy factor twisted by the character χ^k (see e.g. [19, section 10], [22], also [6, 1.2.1]).

Let $\chi : \Gamma \rightarrow U(1)$ be a normalized character on Γ . Denote by $Pic^\Gamma(D)$ the set of isomorphism classes of holomorphic line bundles on D equipped with a holomorphic action of Γ which is compatible with the action of Γ on D and is linear on the fibers. We note that there is a well-defined map $Pic(X) \rightarrow Pic^\Gamma(D)$, $[E] \mapsto [\pi^*E]$. Denote by $Pic^0(X)$ the subgroup of $Pic(X)$ formed by holomorphic line bundles with $c_1 = 0$. Denote by $S_k(\Gamma, \chi)$ the space of holomorphic functions $f : D \rightarrow \mathbb{C}$ satisfying the automorphy law

$$f(\gamma z)[J(\gamma, z)\chi(\gamma)]^k = f(z)$$

for any $z \in D$, $\gamma \in \Gamma$. The space $S_k(\Gamma, \chi)$ is isomorphic to $H^0(X, (L \otimes E)^{\otimes k})$, where E is a holomorphic line bundle on X which is isomorphic to $D \times_{\chi^{-1}\mathbb{C}} \mathbb{C} = D \times \mathbb{C} / \sim$, $(z, \xi) \sim (\gamma z, \chi(\gamma)\xi)$ for $z \in D$, $\xi \in \mathbb{C}$, $\gamma \in \Gamma$.

The line bundle $L \otimes E$ is ample. Therefore by [12, Theorem 16.5] for $k \geq (n+2)^{n+7}$, the bundle $(L \otimes E)^{\otimes k}$ is very ample and the Poincaré map

$$(L^1 \cap H)(D, \pi^*((L \otimes E)^{\otimes k})) \rightarrow H^0(X, (L \otimes E)^{\otimes k})$$

is surjective.

Denote by Γ' the commutator subgroup of Γ . $T = Pic^0(X)$ is a complex torus, $\dim T = b_1(X) = rk(\Gamma/\Gamma')$ which is necessarily zero if G is not $SU(n, 1)$ by [18, Theorem 7]. Ball quotients with nonzero first Betti number do exist; see e.g. [10] for explicit examples. [19, Theorem 4] gives the list of bounded symmetric domains

for which $\text{Pic}(X)$ is isomorphic to the direct product of $\mathbb{Z}$ and a finite abelian group $\text{Char}(\Gamma/\Gamma')$.

We note that $\pi^*c_1(L \otimes E) = \pi^*c_1(L) = [\omega]$. We shall later need an embedding $\iota : X \rightarrow \mathbb{CP}^N \simeq \mathbb{P}(H^0(X, (L \otimes E)^k)^*)$, $p \mapsto [s_0(p), \dots, s_N(p)]$, where $N = \dim H^0(X, (L \otimes E)^{\otimes k}) - 1$ and $s_0, \dots, s_N$ is a basis in $H^0(X, (L \otimes E)^{\otimes k})$.

2.4. Explicit constructions. For a point $p \in D$ define

$$(2.1) \quad \Theta_{k,0,p,\chi}(z) = \sum_{\gamma \in \Gamma} [K(\gamma z, p) J(\gamma, z) \chi(\gamma)]^k \in S_k(\Gamma, \chi)$$

and

$$(2.2) \quad \Theta_{k,l,0,\chi}(z) = \sum_{\gamma \in \Gamma} P_l(\gamma z) [J(\gamma, z) \chi(\gamma)]^k \in S_k(\Gamma, \chi),$$

where $P_l(z) = z_1^{l_1} \cdots z_n^{l_n}$, l_j are nonnegative integers, and $l = (l_1, \dots, l_n)$. Functions $K(z, p)$ and $P_l(z)$ are bounded continuous functions on D . Therefore series (2.1), (2.2) converge absolutely, uniformly on compact sets (by [12, Lemma 5.14]; see also [2, 5.2] or [4, Theorem 9.1] in the case $\chi \equiv 1$).

Theorem 2.1. *Let $p_0, \dots, p_N$ be points in D such that $\iota \circ \pi(p_0), \dots, \iota \circ \pi(p_N)$ are not on the same hyperplane in $\mathbb{CP}^N$. Then the Poincaré series $\Theta_{k,0,p_j,\chi}$, $j = 0, \dots, N$, form a basis in $S_k(\Gamma, \chi)$.*

Theorem 2.2. *The Poincaré series $\Theta_{k,l,0,\chi}$, $0 \leq l_1 + \dots + l_n \leq N_0$, where $N_0 = \frac{k^n n!}{(2\pi)^n} \text{vol}(X)$, span $S_k(\Gamma, \chi)$.*

Remarks. 1. The statement of Theorem 2.1 does not depend on the choice of the basis $\{s_j\}$ that was made to define the map ι .

2. In Theorem 2.1 we need $N+1$ points $\pi(p_j)$, $j = 0, \dots, N$, in general position on X , i.e., $N+1$ points whose images under ι are not on the same hyperplane in $\mathbb{CP}^N$. This is equivalent to requiring that $(v_0, \dots, v_N) \notin \lambda^{-1}(0)$, where $v_j \in V = \mathbb{C}^{N+1}$ are such that

$$\mu(v_j) = \pi(p_j), \mu : \mathbb{C}^{N+1} - \{0\} \rightarrow \mathbb{CP}^N$$

is the natural map, and

$$\lambda : V \times \dots \times V \rightarrow \bigwedge^{N+1} V \cong \mathbb{C}, (u_0, \dots, u_N) \mapsto u_0 \wedge \dots \wedge u_N.$$

3. Suppose $p' = \gamma' p$, where $\gamma' \in \Gamma$, $p, p' \in D$ and $\pi(p) = \pi(p') \in X$. Then

$$\Theta_{k,0,\gamma'p,\chi}(z) = \frac{\chi(\gamma')^k}{\bar{J}(\gamma', p)^k} \Theta_{k,0,p,\chi}(z).$$

Therefore the basis in Theorem 2.1 indeed depends not only on the choice of $N+1$ points in general position on X but also on the choice of points representing them on the universal cover. The Poincaré series (2.1) (and corresponding sections of $(L \otimes E)^{\otimes k}$ as well) associated to two different points in D which represent the same point in X differ by a nonzero constant multiplicative factor. It is also clear that in order to get a basis we do not have to choose the points in D in the same fundamental domain.

4. The statement of Theorem 2.2 can be rephrased as follows: any $f \in S_k(\Gamma, \chi)$ is equal to the Poincaré series of a polynomial in $z_1, \dots, z_n$ of degree not greater than N_0 . This is, of course, the $n > 1$ generalization of the Petersson's observation for the

classical holomorphic automorphic forms. Note: if $n = 1$, then $\text{vol}(X) = 2\pi(2g-2)$, $N_0 = k(2g-2)$.

5. Heuristically (2.1) is the k -th component of the delta-function at $\pi(p)$ and (2.2) is a linear combination of the k -th components of the delta-function at $\pi(0)$ and its derivatives.

6. The number of elements in the finite spanning set we obtain is equal to the number of monomials in $z_1, \dots, z_n$ of degree not greater than N_0 . To see how close this is to $\dim H^0(X, (L \otimes E)^{\otimes k})$ asymptotically as $k \rightarrow \infty$ we note that $\dim H^0(X, (L \otimes E)^{\otimes k}) \sim \text{const}(X)k^n$ and the number of monomials in $z_1, \dots, z_n$ of degree not higher than N_0 (which is equal to the number of monomials in $n+1$ variables of degree N_0) is

$$\frac{(n+N_0)!}{n!N_0!} \sim \text{const}(X)k^{(n^2)}.$$

7. It is well known that for any points $z^{(1)}, \dots, z^{(m)}$ in D which represent m distinct points on X there is sufficiently large k and there exist Poincaré series $\theta_j(z)$, $j = 1, \dots, m$, of weight k such that the $m \times m$ matrix with entries $\theta_i(z^{(j)})$ has a nonzero determinant. The proof uses asymptotic representations as $k \rightarrow \infty$, with $m, z^{(1)}, \dots, z^{(m)}$ fixed. It follows that for such $z^{(1)}, \dots, z^{(m)}$ there exists a Poincaré series of sufficiently large weight with any preassigned values at these points. For this and other statements in this direction, see [21], [2], also [5].

Normalize the Hermitian inner product in $S_k(\Gamma, \chi)$ as follows:

$$(f, g) = c(D, 1-k) \int_{\Gamma \backslash D} f(z) \overline{g(z)} d\mu_{1-k}.$$

Let F be a Dirichlet fundamental domain for Γ .

Lemma 2.3. *For any $f \in S_k(\Gamma, \chi)$,*

$$(f, \Theta_{k,0,p,\chi}) = f(p).$$

Proof.

$$\begin{aligned} (f, \Theta_{k,0,p,\chi}) &= c(D, 1-k) \int_F f(z) \sum_{\gamma \in \Gamma} [K(\gamma p, z) \bar{J}(\gamma, z) \bar{\chi}(\gamma)]^k K(z, z)^{-k} d\mu(z) \\ &= c(D, 1-k) \sum_{\gamma \in \Gamma} \int_{\gamma F} f(w) \chi(\gamma)^k \left(\frac{K(p, w)}{K(w, w)} \right)^k \bar{\chi}(\gamma)^k d\mu(w) \\ &= c(D, 1-k) \int_D f(w) \left(\frac{K(p, w)}{K(w, w)} \right)^k d\mu(w) = f(p). \quad \square \end{aligned}$$

Lemma 2.4. *If $f \in S_k(\Gamma, \chi)$ and $(f, \Theta_{k,l,0,\chi}) = 0$ for all $l = (l_1, \dots, l_n)$ such that $l_1 + \dots + l_n < N'$, where N' is a positive integer, then all partial derivatives $\frac{\partial^{l_1+\dots+l_n} f}{\partial z_1^{l_1} \dots \partial z_n^{l_n}}(0)$, $l_1 + \dots + l_n < N'$, are equal to zero.*

Proof. We have:

$$\begin{aligned}(f, \Theta_{k,l,0,\chi}) &= c(D, 1-k) \int_F f(z) \sum_{\gamma \in \Gamma} \overline{P_l(\gamma z)} [\bar{J}(\gamma, z) \bar{\chi}(\gamma)]^k K(z, z)^{-k} d\mu(z) \\ &= c(D, 1-k) \sum_{\gamma \in \Gamma} \int_{\gamma F} f(w) \chi(\gamma)^k \overline{P_l(w)} \bar{\chi}(\gamma)^k K(w, w)^{-k} d\mu(w) \\ &= c(D, 1-k) \int_D f(w) \overline{P_l(w)} (K(w, w))^{-k} d\mu(w).\end{aligned}$$

Now recall that monomials in $z_1, \dots, z_n$ form a complete system in $\mathcal{H}_r^2(D)$ (here we take $r = -k + 1$), monomials of different degrees are orthogonal to each other, but the inner product of two monomials of the same degree is not necessarily zero [9], [23]. Expand f into a series of homogeneous polynomials in $z_1, \dots, z_n$ ([23] 2.9): $f(z) = \sum_{j=0}^{+\infty} p_j(z)$, where $p_j(z)$ is of degree j . We would like to show that $p_j(z) = 0$ if $0 \leq j < N'$. We obtain:

$$(f, \Theta_{k,l,0,\chi}) = c(D, 1-k) \int_D p_j(z) \overline{P_l(w)} (K(w, w))^{-k} d\mu(w),$$

where $j = \sum_{m=1}^n l_m$. By assumption this is zero for all monomials $P_l(z)$ of degree j , $0 \leq j < N'$. Therefore $p_j = 0$ if $0 \leq j < N'$. $\square$

For domains of type I there is a very explicit relation between the inner product $(f, \Theta_{k,l,0,\chi})$ and the partial derivative $\frac{\partial^{l_1+\dots+l_n} f}{\partial z_1^{l_1} \dots \partial z_n^{l_n}}(0)$.

Lemma 2.5. *If D is of type I, then*

$$(f, \Theta_{k,l,0,\chi}) = c(D, 1-k) \frac{1}{\Pi_{j=1}^n l_j!} \frac{\partial^{l_1+\dots+l_n} f}{\partial z_1^{l_1} \dots \partial z_n^{l_n}}(0) \int_D |P_l(z)|^2 (K(z, z))^{-k} d\mu(z).$$

Proof.

$$\begin{aligned}(f, \Theta_{k,l,0,\chi}) &= c(D, 1-k) \int_F f(z) \sum_{\gamma \in \Gamma} \overline{P_l(\gamma z)} [\bar{J}(\gamma, z) \bar{\chi}(\gamma)]^k K(z, z)^{-k} d\mu(z) \\ &= c(D, 1-k) \sum_{\gamma \in \Gamma} \int_{\gamma F} f(w) \chi(\gamma)^k \overline{P_l(w)} \bar{\chi}(\gamma)^k K(w, w)^{-k} d\mu(w) \\ &= c(D, 1-k) \int_D f(w) \overline{P_l(w)} (K(w, w))^{-k} d\mu(w) \\ &= c(D, 1-k) \frac{1}{\Pi_{j=1}^n l_j!} \frac{\partial^{l_1+\dots+l_n} f}{\partial z_1^{l_1} \dots \partial z_n^{l_n}}(0) \int_D |P_l(z)|^2 (K(z, z))^{-k} d\mu(z).\end{aligned}$$

The last equality is proved as follows. A domain of type I has symmetries $z_j \mapsto e^{i\varphi} z_j$, $\varphi \in \mathbb{R}$. Therefore the equality is true for polynomials in $z_1, \dots, z_n$. Then we expand an arbitrary function in $\mathcal{H}_r^2(D)$ into a series of homogeneous polynomials (with compact convergence) and obtain the desired equality for any function in $\mathcal{H}_r^2(D)$. $\square$

Proof of Theorem 2.1. The set $\{\Theta_{k,0,p_j,\chi}\}$ consists of exactly $N+1 = \dim S_k(\Gamma, \chi)$ elements. Assume that it is not a basis. Then there is $f \in S_k(\Gamma, \chi)$ which is not identically zero and such that $(f, \Theta_{k,0,p_j,\chi}) = 0$, $j = 0, \dots, N$. Then by Lemma 2.3 $f(p_0) = \dots = f(p_N) = 0$. Therefore the corresponding section

$s \in H^0(X, (L \otimes E)^{\otimes k})$ vanishes at $\pi(p_j)$, $j = 0, \dots, N$. But $s = \sum_{j=0}^N a_j s_j$, $a_j \in \mathbb{C}$, so the locus in $\iota(X)$ where $s = 0$ is the intersection of $\iota(X)$ and a hyperplane in $\mathbb{CP}^N$. Since the points $\pi(p_j)$ are, by assumption, not on the same hyperplane, s must be the zero section and hence f is identically zero. This is a contradiction; thus the given set is a basis. $\square$

Proof of Theorem 2.2. This will be a proof by contradiction. Assume that the given set does not span $S_k(\Gamma, \chi)$. Then there is $f \in S_k(\Gamma, \chi)$, not identically zero, such that $(f, \Theta_{k,l,0,\chi}) = 0$ for all $l = (l_1, \dots, l_n)$, $l_1 + \dots + l_n \leq N_0$. By Lemma 2.4 (with $N' = N_0 + 1$) all partial derivatives of f up to the order N_0 vanish at $z = 0$.

Denote by s the section of $(L \otimes E)^{\otimes k}$ that corresponds to f . Denote by V the locus in X where $s = 0$. We defined $N_0 = \frac{n!k^n}{(2\pi)^n} \text{vol}(X)$. With our normalization of the volume form (see subsection 2.1) $N_0 = \int_X (kc_1(L))^n$. Note that the value of this integral does not depend on the choice of a 2-form representing $c_1(L)$, and also note that $(L \otimes E)^{\otimes k} = \iota^* H$, where $H \rightarrow \mathbb{CP}^N$ is the hyperplane bundle. $\iota(V)$ is the intersection of $\iota(X)$ and a hyperplane in $\mathbb{CP}^N$. By Poincaré duality $N_0 = \text{ord}_V(s) \int_V (kc_1(L))^{n-1}$, and also we note that $a = \text{ord}_V(s)$ and $\int_V (kc_1(L))^{n-1}$ are positive integers. Hence $a \leq N_0$, i.e. the order of vanishing of s along V at $\pi(0)$ is not greater than N_0 , and this contradicts the statement that f is not identically zero and all its partial derivatives up to the order N_0 vanish at $z = 0$.

In more detail: the map $\iota \circ \pi$ gives a biholomorphic map from a small neighborhood of 0 in D (say, U) onto a small neighborhood of $\iota(\pi(0))$ in $\iota(X)$. Thus near 0 $f(z) = (h(z))^a g(z)$, where h, g are holomorphic in U , $h(0) = 0$, $g(0) \neq 0$. At least one of the first partial derivatives of h must be nonzero at $z = 0$ (because h is the pull-back of a linear function under a biholomorphic map), so $\frac{\partial h}{\partial z_m}(0) \neq 0$ for some m , $1 \leq m \leq n$. Also we observe that all partial derivatives of h^a up to the order $a - 1$ vanish at $z = 0$. Hence there is an a -th partial derivative of f which has a nonzero value at $z = 0$; indeed,

$$\frac{\partial^a f}{\partial z_m^a}(0) = a! \left(\frac{\partial h}{\partial z_m}(0) \right)^a g(0) + 0 \neq 0.$$

But $a \leq N_0$; hence this contradicts Lemma 2.4.

Therefore we indeed obtain a spanning set. $\square$

ACKNOWLEDGMENTS

I would like to thank Igor Dolgachev, Alejandro Uribe for discussions, David Borthwick, Ajneet Dhillon, Clifford Earle, Svetlana Katok, János Kollár, Irwin Kra, Peter Sarnak, and Osip Schwarzman - for their comments made by e-mail or in person on different stages of preparation of this paper. I am grateful to the Institut des Hautes Études Scientifiques for hospitality and support. I am thankful to the referee for helpful comments.

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