STABILITY OF THE EQUATOR MAP FOR THE HESSIAN ENERGY

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ABSTRACT. In this paper we show that the equator map is a minimizer of the Hessian energy $H(u) = \int_{\Omega} |\Delta u|^2 dx$ in $H^2(\Omega; S^n)$ for $n \geq 10$ and is unstable for $5 \leq n \leq 9$.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and set

$$H^2(\Omega; S^n) = \{u \in H^2(\Omega; \mathbb{R}^{n+1}) : |u| = 1 \text{ a.e.}\}.$$

The Hessian energy of a map $u \in H^2(\Omega; S^n)$ is given by

$$(1.1) \quad \mathbb{H}(u) = \int_{\Omega} |\Delta u|^2 dx.$$

A map u is called a weakly biharmonic map if it is a critical point of $\mathbb{H}(\cdot)$ in $H^2(\Omega; S^n)$; that is, $u \in H^2(\Omega; S^n)$ satisfies

$$(1.2) \quad -\Delta^2 u = u (|\Delta u|^2 + \Delta |\nabla u|^2 + 2\nabla u \cdot \nabla \Delta u)$$

in the distribution sense.

A map $u \in H^2(\Omega; S^n)$ is said to be a minimizing biharmonic map if it is a local minimizer of the functional $\mathbb{H}$ in $H^2(\Omega; S^n)$; that is,

$$(1.3) \quad \mathbb{H}(u) \leq \mathbb{H}(v)$$

for all $v \in H^2(\Omega; S^n)$ with $u - v \in H_0^2(\Omega; \mathbb{R}^{n+1})$.

The study of biharmonic maps was initiated by Hardt-Mou in an unpublished paper. Chang-Wang-Yang [CWY] established the partial regularity of stationary biharmonic maps into spheres (see also [A]). Wang in [W1]-[W3] generalized the result of [CWY] to the case of stationary biharmonic maps into any compact smooth Riemannian submanifold N of Euclidean space. In [HW], Hong and Wang proved that the Hausdorff dimension of the singular set of minimizing biharmonic maps from Ω into S^k is $n - 5$.

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On the other hand, for the case of harmonic maps, Jäger and Kaul [JK2] proved that the equator map u^* is a minimizer of the Dirichlet energy $E(u) = \int_{\Omega} |\nabla u|^2 dx$ in H^1 for $n \geq 7$ and is unstable for $E(u)$ in H^1 for $3 \leq n \leq 6$. The result of Jäger and Kaul was generalized by the first author in [H] for the p -energy.

We say that a map u is stable for the Hessian functional in H^2 if the second variation of the Hessian functional at u is non-negative (see section 2 for details).

In this paper we establish the analogue of the Jäger and Kaul results for biharmonic maps. In particular, we prove the following result.

Theorem A. *Let $u^* = (\frac{x}{|x|}, 0)$ be the equator map from Ω into S^n with $0 \in \Omega$.*

- (i) *When $n \geq 10$, u^* is a minimizer of $\mathbb{H}$ in $H^2(\Omega; S^n)$.*
- (ii) *For $5 \leq n \leq 9$, u^* is unstable for $\mathbb{H}$ in $H^2(\Omega; S^n)$.*

2. SECOND VARIATION OF THE HESSIAN ENERGY

Let $u = (u_1, \dots, u_n, 0)$ be a map to the equator $(S^{n-1}, 0)$ of S^n . Set

$$u_t = \frac{u + t\phi}{(1 + t^2\phi^2)^{1/2}},$$

where $\phi = (0, \dots, 0, \eta)$ is an arbitrary smooth function $\eta \in C_0^\infty(\Omega, \mathbb{R})$ and $\phi^2 = |\phi|^2 = \eta^2$.

Then

$$\begin{aligned} \nabla u_t &= \frac{\nabla u + t\nabla\phi}{(1 + t^2\phi^2)^{1/2}} - \frac{1}{2} \frac{t^2(u + t\phi)\nabla\phi^2}{(1 + t^2\phi^2)^{3/2}} \text{ and} \\ \Delta u_t &= \frac{\Delta u + t\Delta\phi}{(1 + t^2\phi^2)^{1/2}} - \frac{t^2\nabla(u + t\phi) \cdot \nabla\phi^2}{(1 + t^2\phi^2)^{3/2}} \\ &\quad - \frac{1}{2} t^2(u + t\phi) \left[\frac{\Delta\phi^2}{(1 + t^2\phi^2)^{3/2}} - \frac{3t^2}{2} \frac{|\nabla\phi^2|^2}{(1 + t^2\phi^2)^{5/2}} \right]. \end{aligned}$$

It is easy to see that

$$\begin{aligned} \frac{d}{dt}(1 + t^2\phi^2)^{-1/2} &= -t \frac{\phi^2}{(1 + t^2\phi^2)^{3/2}}, \quad \frac{d^2}{dt^2}(1 + t^2\phi^2)^{-1/2} \Big|_{t=0} = -\phi^2, \\ \frac{d}{dt}(\Delta u_t) \Big|_{t=0} &= \Delta\phi, \quad \text{and} \\ \frac{d^2}{dt^2}(\Delta u_t) \Big|_{t=0} &= -\Delta u\phi^2 - 2\nabla u \cdot \nabla\phi^2 - u\Delta\phi^2, \end{aligned}$$

where $\phi = (0, \dots, 0, \eta)$ is a vector function with $\eta \in C_0^\infty(\Omega, \mathbb{R})$.

If u is stable for the Hessian functional in $H^2(\Omega, S^n)$, then

$$\begin{aligned} \frac{d^2}{dt^2} \Big|_{t=0} \int |\Delta u_t|^2 dx &= \int \left[\frac{d}{dt} \Delta u_t \cdot \frac{d}{dt} \Delta u_t + \Delta u_t \cdot \frac{d^2}{dt^2} \Delta u_t \right] dx \Big|_{t=0} \\ &= \int [|\Delta\eta|^2 - \Delta u \cdot (\Delta u\eta^2 + 2\nabla u \cdot \nabla\eta^2 + u\Delta\eta^2)] dx \\ &= \int [|\Delta\eta|^2 - |\Delta u|^2\eta^2 - 2\Delta u \cdot (\nabla u \cdot \nabla\eta^2) - \Delta u \cdot (u\Delta\eta^2)] dx \geq 0 \end{aligned}$$

for an arbitrary smooth function $\eta \in C_0^\infty(\Omega, \mathbb{R})$. Substituting $\eta = \phi_i$ in the above and summing over i from 1 to $n+1$, if u is stable it follows that

$$(2.1) \quad \int_{\Omega} |\Delta \phi|^2 - |\Delta u|^2 |\phi|^2 - 2 \Delta u \cdot (\nabla u \cdot \nabla |\phi|^2) - \Delta u \cdot (u \Delta |\phi|^2) dx \geq 0$$

for every vector function $\phi = (\phi_1, \dots, \phi_n, \phi_{n+1}) \in C_0^\infty(\Omega, \mathbb{R}^{n+1})$.

Proposition 1. *If u satisfies the stability inequality (2.1), then u is a minimizer of $\mathbb{H}$.*

Proof. Let $v \in H^2(\Omega, S^n)$ satisfy $u - v \in H_0^2(\Omega, \mathbb{R}^{n+1})$. Thus $|\nabla u|, |\nabla v| \in L^4(\Omega)$.

Since u is a stable biharmonic map into the equator, it follows from setting $\phi = u - v$ in (2.1) that

$$(2.2) \quad \begin{aligned} & \int_{\Omega} |\Delta(u-v)|^2 dx - 2 \int_{\Omega} |\Delta u|^2 (1 - u \cdot v) dx \\ & - 2 \int_{\Omega} \Delta u \cdot (\nabla u \cdot \nabla(|u-v|^2)) dx - \int_{\Omega} \Delta u \cdot (u \Delta(|u-v|^2)) dx \geq 0. \end{aligned}$$

Note

$$\int_{\Omega} |\Delta(u-v)|^2 dx = \int_{\Omega} [|\Delta u|^2 - 2 \Delta u \cdot \Delta v + |\Delta v|^2] dx.$$

Using the fact that $|u|^2 = |v|^2 = 1$, we have

$$\begin{aligned} \int_{\Omega} |\Delta u|^2 |u-v|^2 dx &= 2 \int_{\Omega} |\Delta u|^2 (1 - u \cdot v) dx, \\ \int_{\Omega} \Delta u \cdot (\nabla u \cdot \nabla(|u-v|^2)) dx &= -2 \int_{\Omega} \Delta u \cdot (\nabla u \cdot \nabla(u \cdot v)) dx \end{aligned}$$

and

$$\int_{\Omega} \Delta u \cdot u \Delta |u-v|^2 dx = -2 \int_{\Omega} \Delta u \cdot u \Delta(u \cdot v) dx.$$

Substituting the above identities into (2.2) yields

$$(2.3) \quad \begin{aligned} & \int_{\Omega} |\Delta v|^2 dx - \int_{\Omega} |\Delta u|^2 dx + 2 \int_{\Omega} [|\Delta u|^2 (u \cdot v) - \Delta u \cdot \Delta v] dx \\ & + 4 \int_{\Omega} \Delta u \cdot (\nabla u \cdot \nabla(u \cdot v)) dx + 2 \int_{\Omega} \Delta u \cdot u \Delta(u \cdot v) dx \geq 0. \end{aligned}$$

Since u is a weakly biharmonic map, we have

$$(2.4) \quad - \int_{\Omega} \Delta^2 u \cdot (u-v) dx = \int_{\Omega} (1 - u \cdot v) [|\Delta u|^2 + \Delta |\nabla u|^2 + 2 \nabla u \cdot \nabla \Delta u] dx.$$

Integration by parts yields

$$(2.5) \quad - \int_{\Omega} \Delta^2 u \cdot (u-v) dx = - \int_{\Omega} |\Delta u|^2 dx + \int_{\Omega} \Delta u \cdot \Delta v dx$$

and

$$(2.6) \quad \int_{\Omega} (1 - u \cdot v) \Delta |\nabla u|^2 dx = - \int_{\Omega} |\nabla u|^2 \Delta(u \cdot v) dx.$$

Integrating by parts again, we have

$$(2.7) \quad \begin{aligned} & \int_{\Omega} (1 - u \cdot v) \nabla u \cdot \nabla \Delta u \, dx \\ &= - \int_{\Omega} (1 - u \cdot v) |\Delta u|^2 \, dx + \int_{\Omega} (\nabla u \cdot \nabla(u \cdot v)) \cdot \Delta u \, dx. \end{aligned}$$

Substituting (2.5)-(2.7) into (2.4), we obtain

$$\begin{aligned} & - \int_{\Omega} |\Delta u|^2 \, dx + \int_{\Omega} \Delta u \cdot \Delta v \, dx \\ &= \int_{\Omega} (1 - u \cdot v) |\Delta u|^2 \, dx - \int_{\Omega} |\nabla u|^2 \Delta(u \cdot v) \, dx \\ & \quad - 2 \int_{\Omega} (1 - u \cdot v) |\Delta u|^2 \, dx + 2 \int_{\Omega} (\nabla u \cdot \nabla(u \cdot v)) \cdot \Delta u \, dx. \end{aligned}$$

It follows that

$$(2.8) \quad \int_{\Omega} \Delta u \cdot \Delta v = \int_{\Omega} |\Delta u|^2 (u \cdot v) \, dx - \int_{\Omega} |\nabla u|^2 \Delta(u \cdot v) \, dx + 2 \int_{\Omega} (\nabla u \cdot \nabla(u \cdot v)) \cdot \Delta u \, dx.$$

From (2.8) and (2.3) we obtain

$$\int_{\Omega} |\Delta v|^2 \, dx - \int_{\Omega} |\Delta u|^2 \, dx \geq 0$$

for all maps v in $H^2(\Omega, S^n)$ with $u - v \in H_0^2(\Omega, \mathbb{R}^{n+1})$, and hence u is a minimizer. $\square$

3. PROOF OF THEOREM A

In this section, without loss of generality we consider $\Omega = B^n$ and take $u^* = (\frac{x}{|x|}, 0)$. Note that

$$|\nabla \frac{x}{|x|}|^2 = \frac{n-1}{r^2}, \quad -\Delta u^* = u^* |\nabla u^*|^2, \quad \text{and} \quad |\nabla \phi|^2 = |\frac{d\phi}{dr}|^2.$$

This implies that

$$\Delta u^* \cdot u^* = -\frac{n-1}{r^2} \quad \text{and} \quad \Delta u^* \cdot \nabla u^* = -|\nabla u^*|^2 u^* \cdot \nabla u^* = 0.$$

Take $\phi(x) = \phi(r) \in C_0^\infty(B^n, S^n)$ where $r = |x|$. If $u^* = (\frac{x}{|x|}, 0)$ is stable, then

$$(3.1) \quad \int_{B^n} [|\Delta \phi|^2 - \frac{(n-1)^2}{r^4} \phi^2 + \frac{(n-1)}{r^2} \Delta \phi^2] \, dx \geq 0.$$

Using polar coordinates, we have

$$\Delta \phi = \frac{d^2 \phi}{dr^2} + \frac{n-1}{r} \frac{d\phi}{dr} \quad \text{and} \quad \Delta \phi^2 = 2|\nabla \phi|^2 + 2\phi \Delta \phi.$$

Then

$$\int_{B^n} \frac{1}{r^2} \Delta \phi^2 \, dx = \int_{B^n} \phi^2 \Delta \frac{1}{r^2} \, dx = \int_{B^n} \frac{8-2n}{r^4} \phi^2 \, dx.$$

Then (3.1) becomes

$$\int_0^1 \left[\left(\frac{d^2 \phi}{dr^2} \right)^2 + \frac{2(n-1)}{r} \frac{d\phi}{dr} \frac{d^2 \phi}{dr^2} + \frac{(n-1)^2}{r^2} \left(\frac{d\phi}{dr} \right)^2 - \frac{3(n-1)(n-3)}{r^4} \phi^2 \right] r^{n-1} dr \geq 0.$$

Integrating by parts yields

$$\int_0^1 r^{n-2} \frac{d\phi}{dr} \frac{d^2 \phi}{dr^2} dr = \frac{1}{2} \int_0^1 r^{n-2} \frac{d}{dr} \left(\left| \frac{d\phi}{dr} \right|^2 \right) dr = -\frac{n-2}{2} \int_0^1 \left| \frac{d\phi}{dr} \right|^2 r^{n-3} dr.$$

If u^* is stable, then

$$(3.2) \quad \int_0^1 \left[\left(\frac{d^2 \phi}{dr^2} \right)^2 + \frac{n-1}{r^2} \left(\frac{d\phi}{dr} \right)^2 - \frac{3(n-1)(n-3)}{r^4} \phi^2 \right] r^{n-1} dr \geq 0.$$

For simplicity, we assume that $\phi(r) \in C_0^\infty([0, 1])$. Estimating the first term of the above inequality, integrating by parts yields

$$\begin{aligned} \int_0^1 \left(\frac{d^2 \phi}{dr^2} \right)^2 r^{n-1} dr &= - \int_0^1 \frac{d\phi}{dr} \frac{d}{dr} \left(r^{n-1} \frac{d^2 \phi}{dr^2} \right) dr \\ &= - \int_0^1 \frac{d\phi}{dr} \left[(n-1) r^{n-2} \frac{d^2 \phi}{dr^2} + r^{n-1} \frac{d^3 \phi}{dr^3} \right] dr \\ (3.3) \quad &= \int_0^1 \left[\frac{(n-1)(n-2)}{r^2} \frac{d^2 \phi}{dr^2} + \frac{2(n-1)}{r} \frac{d^3 \phi}{dr^3} + \frac{d^4 \phi}{dr^4} \right] \phi r^{n-1} dr. \end{aligned}$$

To estimate the second term of the righthand side in (3.2), integrating by parts we obtain

$$\begin{aligned} \int_0^1 \frac{1}{r^2} \left(\frac{d\phi}{dr} \right)^2 r^{n-1} dr &= - \int_0^1 \phi \frac{d}{dr} \left(r^{n-3} \frac{d\phi}{dr} \right) dr \\ (3.4) \quad &= - \int_0^1 \left(\frac{(n-3)}{r^3} \frac{d\phi}{dr} + \frac{1}{r^2} \frac{d^2 \phi}{dr^2} \right) \phi r^{n-1} dr. \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^1 \left[\left(\frac{d^2 \phi}{dr^2} \right)^2 + \frac{n-1}{r^2} \left(\frac{d\phi}{dr} \right)^2 - \frac{3(n-1)(n-3)}{r^4} \phi^2 \right] r^{n-1} dr \\ = \int_0^1 \left(\frac{d^4 \phi}{dr^4} + \frac{2(n-1)}{r} \frac{d^3 \phi}{dr^3} + \frac{(n-1)(n-3)}{r^2} \frac{d^2 \phi}{dr^2} \right. \\ (3.5) \quad \left. - \frac{(n-1)(n-3)}{r^2} \frac{d\phi}{dr} - \frac{3(n-1)(n-3)}{r^4} \phi \right) \phi r^{n-1} dr. \end{aligned}$$

In order to establish instability we consider the differential equation

$$\begin{aligned} \frac{d^4 \phi}{dr^4} + \frac{2(n-1)}{r} \frac{d^3 \phi}{dr^3} + \frac{(n-1)(n-3)}{r^2} \frac{d^2 \phi}{dr^2} \\ (3.6) \quad - \frac{(n-1)(n-3)}{r^3} \frac{d\phi}{dr} - \frac{3(n-1)(n-3) - \varepsilon}{r^4} \phi = 0, \end{aligned}$$

where $\varepsilon > 0$ is chosen below.

To solve (3.6), we use the transform $r = e^t$ for $-\infty < t \leq 0$ and set $\xi(t) = \phi(r) = \phi(e^t)$. Then

$$\begin{aligned}\xi'(t) &= \phi'(r)r, & \xi''(t) &= \phi''(r)r^2 + \phi'(r)r, & \xi''' &= \phi'''r^3 + 3r^2\phi'' + r\phi', \text{ and} \\ \xi^{(4)} &= r^4\phi^{(4)} + 6r^3\phi''' + 7r^2\phi'' + r\phi'.\end{aligned}$$

Solving the above equations, we have

$$\phi''r^2 = \xi'' - \xi' \text{ and } r^4\phi^{(4)} = \xi^{(4)} - 6\xi''' + 11\xi'' - 6\xi'.$$

Then $\xi(t)$ satisfies

$$\begin{aligned}(3.7) \quad & \frac{d^4\xi}{dt^4} + 2(n-4)\frac{d^3\xi}{dt^3} + (n^2 - 10n + 20)\frac{d^2\xi}{dt^2} \\ & - 2(n-2)(n-4)\frac{d\xi}{dt} - [3(n-1)(n-3) - \varepsilon]\xi = 0.\end{aligned}$$

The corresponding characteristic equation is

$$(3.8) \quad \lambda^4 + 2(n-4)\lambda^3 + (n^2 - 10n + 20)\lambda^2 - 2(n-2)(n-4)\lambda - [3(n-1)(n-3) - \varepsilon] = 0.$$

The roots of the quartic equation (3.8) can be found explicitly. We find two numbers a and b such that

$$\begin{aligned}(3.9) \quad & \lambda^4 + 2(n-4)\lambda^3 + (n^2 - 10n + 20)\lambda^2 - 2(n-2)(n-4)\lambda - [3(n-1)(n-3) - \varepsilon] \\ & = (\lambda^2 + (n-4)\lambda + a)(\lambda^2 + (n-4)\lambda + b).\end{aligned}$$

Note that (3.9) is equivalent to

$$\begin{aligned}(3.10) \quad & a + b = -2(n-2), \\ & a + b + (n-4)^2 = n^2 - 10n + 20, \\ & ab = -[3(n-1)(n-3) - \varepsilon].\end{aligned}$$

Solving equations (3.10), we have

$$a = -n + 2 \pm \sqrt{3(n-1)(n-3) + (n-2)^2 - \varepsilon}$$

and

$$b = -n + 2 \mp \sqrt{3(n-1)(n-3) + (n-2)^2 - \varepsilon}.$$

Therefore the four roots of the quartic equation (3.8) are the four roots of the following two quadratic equations:

$$(3.11) \quad \lambda^2 + (n-4)\lambda - n + 2 - \sqrt{3(n-1)(n-3) + (n-2)^2 - \varepsilon} = 0$$

and

$$(3.12) \quad \lambda^2 + (n-4)\lambda - n + 2 + \sqrt{3(n-1)(n-3) + (n-2)^2 - \varepsilon} = 0.$$

The quadratic equation (3.11) has two real roots

$$\lambda_1 = \frac{4 - n + \sqrt{(n-4)^2 + 4(n-2 + \sqrt{3(n-1)(n-3) + (n-2)^2 - \varepsilon})}}{2} > 0$$

and

$$\lambda_2 = \frac{4 - n - \sqrt{(n-4)^2 + 4(n-2 + \sqrt{3(n-1)(n-3) + (n-2)^2 - \varepsilon})}}{2} < 0.$$

For $5 \leq n \leq 9$, choosing ε sufficiently small yields

$$\delta = (n-4)^2 - 4(-n+2 + \sqrt{3(n-1)(n-3) + (n-2)^2 - \varepsilon}) < 0.$$

Therefore the quadratic equation (3.12) has two complex roots

$$\lambda_3 = \alpha + i\beta \text{ and } \lambda_4 = \alpha - i\beta$$

where $i = \sqrt{-1}$, $\alpha = \frac{4-n}{2}$ and $\beta = \frac{\sqrt{-\delta}}{2}$. Then $e^{\lambda_1 t}$, $e^{\lambda_2 t}$, $e^{\alpha t} \sin(\beta t)$ and $e^{\alpha t} \cos(\beta t)$ are fundamental solutions to differential equation (3.7). This means that differential equation (3.6) has fundamental solutions r^{λ_1} , r^{λ_2} , $r^\alpha \sin(\beta \ln r)$ and $r^\alpha \cos(\beta \ln r)$, where $\lambda_1 > 0$, $\lambda_2 < 0$, α and β are real numbers.

Now we complete a proof of Theorem A.

Proof of Theorem A. For two constants c_1 and c_2 , set

$$\phi(r) = r^{\lambda_1} + c_1 r^\alpha \sin(\beta \ln r) + c_2 r^\alpha \cos(\beta \ln r).$$

Then

$$\phi'(r) = \lambda_1 r^{\lambda_1-1} + (\alpha c_1 - \beta c_2) r^{\alpha-1} \sin(\beta \ln r) + (c_2 \alpha + c_1 \beta) r^{\alpha-1} \cos(\beta \ln r).$$

To ensure that $\phi(x) = \phi(|x|)$ is in $H_0^2(B^n)$, we set $\phi(1) = \phi'(1) = 0$. This implies that

$$c_2 = -1 \text{ and } c_1 = \frac{\alpha - \lambda_1}{\beta}.$$

There are infinitely many small constants r_0 with $0 < r_0 < 1$ and $r_0 \rightarrow 0$ satisfying

$$(\alpha c_1 - \beta c_2) \sin(\beta \ln r_0) + (c_2 \alpha + c_1 \beta) \cos(\beta \ln r_0) = 0.$$

For such r_0 , we set

$$\eta(r) = \begin{cases} \phi, & \text{for } r_0 \leq r \leq 1, \\ r^{\lambda_1} + \phi(r_0) - r_0^{\lambda_1}, & \text{for } 0 \leq r \leq r_0 \end{cases}$$

for $5 \leq n \leq 9$. The function $\eta(x) = \eta(r)$ with $r = |x|$ is non-zero and belongs to $H_0^2(B^n)$. Actually, $\eta(r)$ can be chosen as a test function in (3.7). Integrating by parts, we have

$$\begin{aligned} I &= \int_0^1 \left[\frac{d^2 \phi}{dr^2} \frac{d^2 \eta}{dr^2} + \frac{n-1}{r^2} \frac{d\phi}{dr} \frac{d\eta}{dr} - \frac{3(n-1)(n-3)}{r^4} \phi \eta \right] r^{n-1} dr \\ &= - \int_0^1 \varepsilon \phi \eta r^{n-1} dr \end{aligned}$$

and

$$\begin{aligned} II &= \int_0^{r_0} \left[\frac{d^2 \eta}{dr^2} \left(\frac{d^2 \eta}{dr^2} - \frac{d^2 \phi}{dr^2} \right) \right. \\ &\quad \left. + \frac{n-1}{r^2} \frac{d\eta}{dr} \left(\frac{d\eta}{dr} - \frac{d\phi}{dr} \right) - \frac{3(n-1)(n-3)}{r^4} \eta(\eta - \phi) \right] r^{n-1} dr. \end{aligned}$$

Note that $\lim_{r_0 \rightarrow 0} II = 0$ and $\lim_{r_0 \rightarrow 0} I < 0$. Choosing a sufficiently small constant r_0 , we have

$$\begin{aligned} &\int_0^1 \left[\left(\frac{d^2 \eta}{dr^2} \right)^2 + \frac{n-1}{r^2} \left(\frac{d\eta}{dr} \right)^2 - \frac{3(n-1)(n-3)}{r^4} \eta^2 \right] r^{n-1} dr \\ &= I + II < 0 \end{aligned}$$

for $5 \leq n \leq 9$. This means that u^* is unstable for $5 \leq n \leq 9$.

When $n \geq 10$, it follows from [H] that

$$\int_{\Omega} |\nabla u^*|^4 dx \leq \int_{\Omega} |\nabla v|^4 dx$$

for all $v \in H^1(\Omega, S^n)$ and $v - u^* \in H_0^1(\Omega, \mathbb{R}^{n+1})$. On the other hand, $|\nabla u^*|^4 = |\Delta u^*|^2$ and $|\nabla v|^4 \leq |\Delta v|^2$. Thus for $n \geq 10$,

$$\int_{\Omega} |\Delta u^*|^2 dx \leq \int_{\Omega} |\Delta v|^2 dx$$

for all $v \in H^2(\Omega, S^n)$ and $v - u^* \in H_0^2(\Omega, \mathbb{R}^{n+1})$ and u^* is the minimizer, as required. $\square$

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