CONVERGENCE OF ALMOST-ORBITS OF NONEXPANSIVE SEMIGROUPS IN BANACH SPACES

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ABSTRACT. The purpose of this paper is to show that the study of mean ergodic theorems for almost-orbits of semigroups of nonexpansive mappings on closed convex subsets of a Banach space can be reduced to the study of orbits for semigroups of nonexpansive mappings. This provides a unified approach to various mean ergodic theorems for almost-orbits in the literature and new applications.

1. INTRODUCTION

Let C be a nonempty closed convex subset of a Banach space E . Then, a mapping T of C into itself is said to be *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for every $x, y \in C$. Let S be a semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a representation of the semigroup S as nonexpansive mappings from C into itself, i.e., for each $t \in S$, $T(t)$ is a nonexpansive mapping of C into itself and $T(st) = T(s)T(t)$ for every $s, t \in S$. Then, a function $u : S \rightarrow C$ is said to be an *almost-orbit* of $\mathcal{S}$ if

$$\inf_{w \in S} \sup_{s, t \in S} \|u(swt) - T(s)u(wt)\| = 0.$$

In this paper, we prove that mean ergodic theorems for almost-orbits of nonexpansive semigroups can be reduced to those for nonexpansive semigroups.

The notion of almost-orbit was first introduced by Kiuchi and Takahashi in [7]. Our approach unifies various mean ergodic theorems for almost-orbits already in the literature (see [7], [9], [10] and [13]) and gives new applications to results in [1], [2], [3], [5] and [6]. Baillon [4] proved the first nonlinear ergodic theorem for nonexpansive mappings in the framework of a Hilbert space: Let C be a closed convex subset of a Hilbert space and let T be a nonexpansive mapping of C into itself. If the set $F(T)$ of fixed points of T is nonempty, then for each $x \in C$, the Cesàro means $S_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$ converge weakly to some $y \in F(T)$. The notion of an almost-orbit for a nonexpansive mapping was introduced by Bruck in [5]. It was extended by Miyadera and Kobayasi to the case of a one-parameter nonexpansive semigroup in [9] where they established the weak and strong almost

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convergence of such an almost-orbit in Banach spaces. These results were then extended to various forms by many authors; for example [6, 8, 10, 11, 13].

Our paper is organized as follows: in Section 2 we define some terminologies that we use; in Section 3 we prove our main results; in Section 4 we provide the applications of our results in Section 3.

2. PRELIMINARIES

Let E be a Banach space and let S be a semigroup. We denote by E^* the dual space of E and by $\langle x, x^* \rangle$ the value of $x^* \in E^*$ at $x \in E$. We also denote by $B(S)$ the Banach space of all bounded real-valued functions on S with supremum norm. For each $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ of $B(S)$ by $(l_s f)(t) = f(st)$ and $(r_s f)(t) = f(ts)$ for all $t \in S$. We sometimes use $\mu_t(f(t))$ instead of $\mu(f)$ for $\mu \in B(S)^*$ and $f \in B(S)$.

Let X be a subspace of $B(S)$ such that $1 \in X$ and X is l_s - and r_s -invariant, i.e., $l_s(X) \subset X$ and $r_s(X) \subset X$ for every $s \in S$. Then, a net $\{\mu_\alpha\}_{\alpha \in I}$ of continuous linear functionals on X is said to be *strongly regular* [6] if it satisfies the following conditions:

- (1) $\sup_\alpha \|\mu_\alpha\| < \infty$;
- (2) $\lim_\alpha \mu_\alpha(1) = 1$;
- (3) $\lim_\alpha \|\mu_\alpha - l_s^* \mu_\alpha\| = 0$ and $\lim_\alpha \|\mu_\alpha - r_s^* \mu_\alpha\| = 0$ for every $s \in S$.

We say that $\{\mu_\alpha\}_{\alpha \in I}$ is *weak* regular* if (3) is replaced by the weaker condition:

- (4) $\text{weak}^* - \lim_\alpha (\mu_\alpha - l_s^* \mu_\alpha) = 0$ and
 $\text{weak}^* - \lim_\alpha (\mu_\alpha - r_s^* \mu_\alpha) = 0$ for every $s \in S$.

Let f be a function of S into E such that $\{f(t) : t \in S\}$ is relatively weakly compact and let X be a subspace of $B(S)$ such that for each $x^* \in E^*$, the function $t \mapsto \langle f(t), x^* \rangle$ is an element of X . Then, for any $\mu \in X^*$ there exists a unique element f_μ in E such that

$$\mu_t \langle f(t), x^* \rangle = \langle f_\mu, x^* \rangle$$

for every $x^* \in E^*$; see [12, 6]. Following [6], we write such f_μ by $\int f(t) d\mu(t)$.

Let C be a nonempty closed convex subset of E . Let $\mathcal{S} = \{T(t) : t \in S\}$ be a representation of the semigroup S as nonexpansive mappings from C into itself and let X be a subspace of $B(S)$. Then, a function $u : S \rightarrow C$ is said to be a *nearly almost-orbit* of $\mathcal{S}$ if

$$\inf_{s \in S} \sup_{t \in S} \|u(ts) - T(t)u(s)\| = 0.$$

Clearly, every almost-orbit of $\mathcal{S}$ is a nearly almost-orbit. We denote by $NAO(\mathcal{S}, X)$ the set of all nearly almost-orbits u of $\mathcal{S}$ such that $\{u(t) : t \in S\}$ is relatively weakly compact and for each $x^* \in E^*$, the function $t \mapsto \langle u(t), x^* \rangle$ is an element of X .

3. MAIN RESULTS

First we show the following lemma.

Lemma 3.1. *Let E be a Banach space and let S be a semigroup. Let f be a function of S into E such that $\{f(t) : t \in S\}$ is relatively weakly compact and let X be a subspace of $B(S)$ such that $1 \in X$, for each $x^* \in E^*$, the function $t \mapsto \langle f(t), x^* \rangle$ is an element of X and X is l_s - and r_s -invariant for every $s \in S$. Let $\{\mu_\alpha\}_{\alpha \in I}$ be a weak* regular net of continuous linear functionals on X . If for each $h \in S$,*

$\int f(ht)d\mu_\alpha(t)$ converges weakly to some $y_h \in E$, then there exists $\mu \in X^*$ such that $\|\mu\| \leq K$,

$$\mu(l_sg) = \mu(r_sg) = \mu(g),$$

for every $g \in X$ and $s \in S$, and

$$y_h = \int f(t)d\mu(t)$$

for every $h \in S$, where $K = \sup_{\alpha \in I} \|\mu_\alpha\|$.

Proof. Assume that for each $h \in S$, $\int f(ht)d\mu_\alpha(t)$ converges weakly to some $y_h \in E$. Let $B = \{\mu \in X^* : \|\mu\| \leq K\}$. Since B is compact in the weak* topology, there exists a subnet $\{\mu_{\alpha_\beta}\}_{\beta \in J}$ of $\{\mu_\alpha\}_{\alpha \in I}$ such that μ_{α_β} converges to some $\mu \in B$ in the weak* topology. Let $\varepsilon > 0$, $g \in X$ and $s \in S$. Then, there exists $\alpha_0 \in I$ such that $|\mu_\alpha(l_sg) - \mu_\alpha(g)| < \varepsilon$ for every $\alpha \geq \alpha_0$. Since μ_{α_β} converges to $\mu \in B$ in the weak* topology, we can choose $\alpha_1 \geq \alpha_0$ such that $|\mu_{\alpha_1}(g) - \mu(g)| < \varepsilon$ and $|\mu_{\alpha_1}(l_sg) - \mu(l_sg)| < \varepsilon$. So, we have

$$\begin{aligned} & |\mu(l_sg) - \mu(g)| \\ & \leq |\mu(l_sg) - \mu_{\alpha_1}(l_sg)| + |\mu_{\alpha_1}(l_sg) - \mu_{\alpha_1}(g)| + |\mu_{\alpha_1}(g) - \mu(g)| \\ & < \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\mu(l_sg) = \mu(g)$. Similarly, we have $\mu(r_sg) = \mu(g)$. Since μ_{α_β} converges to $\mu \in B$ in the weak* topology, it follows that for each $h \in S$, $\int f(ht)d\mu_{\alpha_\beta}(t)$ converges weakly to $\int f(ht)d\mu(t)$. Therefore, we obtain

$$y_h = \int f(ht)d\mu(t) = \int f(t)d\mu(t)$$

for every $h \in S$. □

Now we prove the main theorems of this paper.

Theorem 3.2. *Let C be a nonempty closed convex subset of a Banach space E . Let S be a semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a representation of the semigroup S as nonexpansive mappings from C into itself such that for each $x \in C$, $\{T(t)x : t \in S\}$ is relatively weakly compact. Let X be a subspace of $B(S)$ such that $1 \in X$, for each $x \in C$ and $x^* \in E^*$, the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of X and X is l_s - and r_s -invariant for every $s \in S$. Let $\{\mu_\alpha\}_{\alpha \in I}$ be a strongly regular net of continuous linear functionals on X . Then the following are equivalent:*

- (1) *For each $x \in C$, $\int T(ht)x d\mu_\alpha(t)$ weakly converges uniformly in $h \in S$.*
- (2) *For each $u \in NAO(\mathcal{S}, X)$, $\int u(ht)d\mu_\alpha(t)$ weakly converges uniformly in $h \in S$.*

Proof. Since (2) $\Rightarrow$ (1) clearly holds, it is sufficient to show (1) $\Rightarrow$ (2). Let $u \in NAO(\mathcal{S}, X)$ and $\varepsilon > 0$. Since u is a nearly almost-orbit $\mathcal{S}$, there exists $s_0 \in S$ such that

$$(3.1) \quad \sup_{t \in S} \|u(ts_0) - T(t)u(s_0)\| < \varepsilon.$$

Let $K = \sup_{\alpha \in I} \|\mu_\alpha\|$. From the hypothesis and Lemma 3.1, there exists $\mu \in X^*$ such that $\|\mu\| \leq K$, $\mu(l_sg) = \mu(r_sg) = \mu(g)$ for every $g \in X$ and $s \in S$ and $\int T(ht)u(s_0)d\mu_\alpha(t)$ converges weakly to $\int T(t)u(s_0)d\mu(t)$ uniformly in $h \in S$. We

show that $\int u(ht)d\mu_\alpha(t)$ converges weakly to $\int u(t)d\mu(t)$ uniformly in $h \in S$. By (3.1), we have

$$\begin{aligned} \left\| \int u(t)d\mu(t) - \int T(t)u(s_0)d\mu(t) \right\| &= \left\| \int u(ts_0)d\mu(t) - \int T(t)u(s_0)d\mu(t) \right\| \\ &\leq \|\mu\| \sup_{t \in S} \|u(ts_0) - T(t)u(s_0)\| \\ (3.2) \qquad \qquad \qquad &\leq K\varepsilon \end{aligned}$$

and for each α and $h \in S$,

$$\begin{aligned} \left\| \int u(hts_0)d\mu_\alpha(t) - \int T(ht)u(s_0)d\mu_\alpha(t) \right\| \\ \leq \|\mu_\alpha\| \sup_{t \in S} \|u(hts_0) - T(ht)u(s_0)\| \\ (3.3) \qquad \qquad \qquad \leq K\varepsilon. \end{aligned}$$

Since $\{\mu_\alpha\}_{\alpha \in I}$ is strongly regular, there exists α_0 such that for each $\alpha \geq \alpha_0$,

$$(3.4) \qquad \qquad \qquad \|\mu_\alpha - r_{s_0}^* \mu_\alpha\| < \varepsilon.$$

Let $M = \sup_{t \in S} \|u(t)\|$. By (3.4), we have for each $\alpha \geq \alpha_0$ and $h \in S$,

$$\begin{aligned} \left\| \int u(ht)d\mu_\alpha(t) - \int u(hts_0)d\mu_\alpha(t) \right\| \\ = \left\| \int u(ht)d\mu_\alpha(t) - \int u(ht)d(r(s_0)^* \mu_\alpha)(t) \right\| \\ \leq \|\mu_\alpha - r_{s_0}^* \mu_\alpha\| \sup_{t \in S} \|u(ht)\| \\ (3.5) \qquad \qquad \qquad \leq M\varepsilon. \end{aligned}$$

Let $x^* \in E^*$ with $\|x^*\| \leq 1$. In the case of weak convergence, there exists $\alpha_1 \geq \alpha_0$ such that for each $\alpha \geq \alpha_1$,

$$(3.6) \qquad \left| \left\langle \int T(ht)u(s_0)d\mu_\alpha(t) - \int T(t)u(s_0)d\mu(t), x^* \right\rangle \right| < \varepsilon.$$

From (3.2), (3.3), (3.5) and (3.6), we obtain for each $\alpha \geq \alpha_1$ and $h \in S$,

$$\begin{aligned} &\left| \left\langle \int u(ht)d\mu_\alpha(t) - \int u(t)d\mu(t), x^* \right\rangle \right| \\ &\leq \left| \left\langle \int u(ht)d\mu_\alpha(t) - \int u(hts_0)d\mu_\alpha(t), x^* \right\rangle \right| \\ &\quad + \left| \left\langle \int u(hts_0)d\mu_\alpha(t) - \int T(ht)u(s_0)d\mu_\alpha(t), x^* \right\rangle \right| \\ &\quad + \left| \left\langle \int T(ht)u(s_0)d\mu_\alpha(t) - \int T(t)u(s_0)d\mu(t), x^* \right\rangle \right| \\ &\quad + \left| \left\langle \int T(t)u(s_0)d\mu(t) - \int u(t)d\mu(t), x^* \right\rangle \right| \\ &\leq (M + 2K + 1)\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain the conclusion. $\square$

Remark 3.3. Theorem 3.2 also holds when “weakly” is replaced by “strongly”. Indeed, if for each $x \in C$, $\int T(ht)x d\mu_\alpha(t)$ strongly converges uniformly in $h \in S$, then there exists $\alpha_2 \geq \alpha_0$ such that for each $\alpha \geq \alpha_2$,

$$(3.7) \quad \left\| \int T(ht)u(s_0)d\mu_\alpha(t) - \int T(t)u(s_0)d\mu(t) \right\| < \varepsilon.$$

From (3.2), (3.3), (3.5) and (3.7), we obtain for each $\alpha \geq \alpha_2$ and $h \in S$,

$$\begin{aligned} & \left\| \int u(ht)d\mu_\alpha(t) - \int u(t)d\mu(t) \right\| \\ & \leq \left\| \int u(ht)d\mu_\alpha(t) - \int u(hts_0)d\mu_\alpha(t) \right\| \\ & \quad + \left\| \int u(hts_0)d\mu_\alpha(t) - \int T(ht)u(s_0)d\mu_\alpha(t) \right\| \\ & \quad + \left\| \int T(ht)u(s_0)d\mu_\alpha(t) - \int T(t)u(s_0)d\mu(t) \right\| \\ & \quad + \left\| \int T(t)u(s_0)d\mu(t) - \int u(t)d\mu(t) \right\| \\ & \leq (M + 2K + 1)\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this implies that $\int u(ht)d\mu_\alpha(t)$ converges strongly to $\int u(t)d\mu(t)$ uniformly in $h \in S$.

Concerning Theorems 3.2 and Lemma 3.1, we provide the following result.

Theorem 3.4. *Let C be a nonempty closed convex subset of a Banach space E . Let S be a semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a representation of the semigroup S as nonexpansive mappings from C into itself such that for each $x \in C$, $\{T(t)x : t \in S\}$ is relatively weakly compact. Let X be a subspace of $B(S)$ such that $1 \in X$, for each $x \in C$ and $x^* \in E^*$, the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of X and X is r_s -invariant for every $s \in S$. Let μ be an element of X^* such that $\mu(r(s)f) = \mu(f)$ for every $f \in X$ and $s \in S$. Then the following are equivalent:*

- (1) *For each $x \in C$, $\int T(t)x d\mu(t) \in F(\mathcal{S})$.*
- (2) *For each $u \in NAO(\mathcal{S}, X)$, $\int u(t)d\mu(t) \in F(\mathcal{S})$.*

Proof. Since (2) $\Rightarrow$ (1) clearly holds, it is sufficient to show (1) $\Rightarrow$ (2). Let $u \in NAO(\mathcal{S}, X)$ and $\varepsilon > 0$. As in the proof of Theorem 3.2, we have for some $s_0 \in S$,

$$\left\| \int u(t)d\mu(t) - \int T(t)u(s_0)d\mu(t) \right\| \leq \varepsilon.$$

So, we have for each $s \in S$,

$$\begin{aligned} & \left\| T(s) \int u(t)d\mu(t) - \int u(t)d\mu(t) \right\| \\ & \leq \left\| T(s) \int u(t)d\mu(t) - \int T(t)u(s_0)d\mu(t) \right\| + \left\| \int T(t)u(s_0)d\mu(t) - \int u(t)d\mu(t) \right\| \\ & \leq 2 \left\| \int u(t)d\mu(t) - \int T(t)u(s_0)d\mu(t) \right\| \\ & < 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we obtain $\int u(t)d\mu(t) \in F(\mathcal{S})$. □

4. APPLICATIONS

In this section, by using Theorems 3.2 and 3.4, we provide some colloraries. From [5] and [6], we obtain the following results of weak almost convergence.

Corollary 4.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E whose norm is Fréchet differentiable. Let T be a nonexpansive mapping of C into itself such that $F(T)$ is nonempty. Then for each nearly almost-orbit $\{x_n\}$ of T , $\frac{1}{n} \sum_{i=0}^{n-1} x_{i+h}$ converges weakly to some $y \in F(T)$ uniformly in $h = 0, 1, 2, \dots$*

Corollary 4.2. *Let C and E be as in Corollary 4.1. Let $S = \{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on C such that $F(S)$ is nonempty. Then for each continuous nearly almost-orbit u of S , $\frac{1}{t} \int_0^t u(h+s)ds$ converges weakly to some $y \in F(S)$ uniformly in $h \geq 0$.*

Corollary 4.3. *Let C and E be as in Corollary 4.1. Let S be a commutative semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a representation of the semigroup S as nonexpansive mappings from C into itself such that $F(\mathcal{S})$ is nonempty. Let X be a subspace of $B(S)$ such that $1 \in X$, for each $x \in C$ and $x^* \in E^*$, the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of X and X is r_s -invariant for every $s \in S$. Let $\{\mu_\alpha\}_{\alpha \in I}$ be a strongly regular net of continuous linear functionals on X . Then for each $u \in \text{NAO}(\mathcal{S}, X)$, $\int u(ht)d\mu_\alpha(t)$ converges weakly to some $y \in F(\mathcal{S})$ uniformly in $h \in S$.*

From [2], [3] and [1], we obtain the following results of strong almost convergence.

Corollary 4.4. *Let C be a nonempty closed convex subset of a strictly convex Banach space E . Let T be a nonexpansive mapping of C into itself such that $T(C)$ is relatively compact. Then for each nearly almost-orbit $\{x_n\}$ of T , $\frac{1}{n} \sum_{i=0}^{n-1} x_{i+h}$ converges strongly to some $y \in F(T)$ uniformly in $h = 0, 1, 2, \dots$*

Corollary 4.5. *Let E be as in Corollary 4.4 and let C be a nonempty compact convex subset of E . Let $S = \{T(t) : t \geq 0\}$ be a one-parameter nonexpansive semigroup on C . Then for each continuous nearly almost-orbit u of S , $\frac{1}{t} \int_0^t u(h+s)ds$ converges strongly to some $y \in F(S)$ uniformly in $h \geq 0$.*

Corollary 4.6. *Let C and E be as in Corollary 4.4. Let S be a commutative semigroup and let $\mathcal{S} = \{T(t) : t \in S\}$ be a representation of the semigroup S as nonexpansive mappings from C into itself such that $\bigcup_{t \in S} T(t)(C) \subset K$ for some compact subset K of C . Let X be a subspace of $B(S)$ such that $1 \in X$, for each $x \in C$ and $x^* \in E^*$, the function $t \mapsto \langle T(t)x, x^* \rangle$ is an element of X and X is r_s -invariant for every $s \in S$. Let $\{\mu_\alpha\}_{\alpha \in I}$ be a strongly regular net of continuous linear functionals on X . Then for each $u \in \text{NAO}(\mathcal{S}, X)$, $\int u(ht)d\mu_\alpha(t)$ converges strongly to some $y \in F(\mathcal{S})$ uniformly in $h \in S$.*

5. EXAMPLES AND REMARKS

In this section, we give examples and remarks concerning our results.

Remark 5.1. (1) If m is an invariant mean on $B(S)$, i.e., m is a translation invariant norm one positive linear functional on $B(S)$, then $\{m\}$ is strongly regular.

- (2) If m is an invariant mean on $B(S)$, then there exists a net of means with finite support, i.e., $\mu_\alpha = \sum_{i=1}^n \lambda_i \delta_{a_i}$ (convex combination of point evaluations) such that $\text{weak}^*\text{-}\lim_\alpha \mu_\alpha = m$. In particular, the net $\{\mu_\alpha\}$ is weak* regular. Consequently, weak* regular nets with finite supports always exist when S is commutative, or more generally when $B(S)$ has an invariant mean.

Example 5.2 ([11]). Let $\mathcal{S} = (\{0, 1, 2, \dots\}, +)$ and $\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \delta_k$. Then, $\{\mu_n\}$ is a

weak* regular sequence of finite means on $\mathcal{S}$. In particular, if T is any nonexpansive mapping from a nonempty convex subset C of a Hilbert space H into C , then the sequence of mappings associated to μ_n is given by $T_{\mu_n} = \frac{1}{n} \sum_{k=0}^{n-1} T^k$, where $T^0 = I$. Suppose the fixed point set for T in C is nonempty. Then by a result of Rodé in [11], the sequence $T_{\mu_n}x$ converges weakly to a fixed point for T . Corollary 4.1 is an improvement of Rodé's result. This also gives Corollary 4.4.

Example 5.3. Let $S = [0, \infty)$, $\mathcal{S} = \{S(t) : t \geq 0\}$ and let X be the Banach space $CB(S)$ of all bounded continuous functions on S with supremum norm. Define

$$\lambda_s(f) = \frac{1}{s} \int_0^s f(t) dt$$

for every $s > 0$ and $f \in X$. Then, we obtain

$$\begin{aligned} \|\lambda_s - r_k^* \lambda_s\| &= \sup_{\|f\| \leq 1} \left\| \frac{1}{s} \int_0^s f(t) dt - \frac{1}{s} \int_0^s f(t+k) dt \right\| \\ &= \frac{1}{s} \sup_{\|f\| \leq 1} \left\| \int_0^s f(t) dt - \int_k^{s+k} f(t) dt \right\| \\ &= \frac{1}{s} \sup_{\|f\| \leq 1} \left\| \int_0^k f(t) dt - \int_s^{s+k} f(t) dt \right\| \\ &\leq \frac{1}{s} \sup_{\|f\| \leq 1} \left(\int_0^k \|f(t)\| dt + \int_s^{s+k} \|f(t)\| dt \right) \\ &= \frac{2k}{s} \rightarrow 0, \end{aligned}$$

as $s \rightarrow \infty$. In particular, $\{\lambda_s; s > 0\}$ is a strongly regular net of means on X . This gives Corollary 4.2 and Corollary 4.5.

Example 5.4 ([7]). Consider the initial value problem:

$$(5.1) \quad \begin{aligned} \frac{du(t)}{dt} + Au(t) &\ni f(t), \quad t > 0, \\ u(0) &= x, \end{aligned}$$

where A is an m -accretive operator in E , $f \in L^1(0, \infty; E)$ and $x \in \overline{D(A)}$. Then it is well known that (5.1) has a unique integral solution [7] and the integral solution $u(t)$ of (5.1) is an almost-orbit of a one-parameter nonexpansive semigroup on $\overline{D(A)}$ generated by $-A$.

Remark 5.5. Our main results (Theorem 3.2 and Theorem 3.4) are new even for a semigroup generated by a nonexpansive mapping (Corollary 4.1) and for a one-parameter semigroup (Corollary 4.2).

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REFERENCES

- [1] S. Atsushiba A. T. Lau and W. Takahashi, *Nonlinear strong ergodic theorems for commutative nonexpansive semigroups on strictly convex Banach spaces*, J. Nonlinear and Convex Anal., **1** (2000), 213–231. MR1777140 (2002d:47077)
- [2] S. Atsushiba and W. Takahashi, *A nonlinear strong ergodic theorem for nonexpansive mappings with compact domains*, Math. Japon., **52** (2000), 183–195. MR1793266 (2001i:47089)
- [3] S. Atsushiba and W. Takahashi, *Strong convergence theorems for one-parameter nonexpansive semigroups with compact domains*, Fixed Point Theory and Applications (Y. J. Cho, J. K. Kim and S. M. Kang, Eds.), Nova Science Publisher, Inc., Huntington New York (2002), 15–31. MR2081406 (2005i:47081)
- [4] J. B. Baillon, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Sér. A-B, **280** (1975), 1511–1514. MR0375009 (51:11205)
- [5] R. E. Bruck, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math., **32** (1979), 107–116. MR531254 (80j:47066)
- [6] N. Hirano, K. Kido and W. Takahashi, *Nonexpansive retractions and nonlinear ergodic theorems in Banach spaces*, Nonlinear Anal., **12** (1988), 1269–1281. MR969505 (90a:47138)
- [7] H. Kiuchi and W. Takahashi, *Asymptotic behavior of almost-orbits of nonexpansive semigroups without convexity*, Kodai Math. J., **15** (1992), 185–192. MR1185417 (93j:47092)
- [8] A. T. Lau, N. Shioji and W. Takahashi, *Existence of nonexpansive retractions for amenable semigroups of nonexpansive mappings and nonlinear ergodic theorems in Banach spaces*, J. Funct. Anal., **161** (1999), 62–75. MR1670206 (99j:47079)
- [9] I. Miyadera and K. Kobayasi, *On the asymptotic behaviour of almost-orbits of nonlinear contraction semigroups in Banach spaces*, Nonlinear Anal., **6** (1982), 349–365. MR654811 (84c:47064)
- [10] H. Oka, *Nonlinear ergodic theorems for commutative semigroups of asymptotically nonexpansive mappings*, Nonlinear Anal., **18** (1992), 619–635. MR1157563 (93d:47126)
- [11] G. Rodé, *An ergodic theorem for semigroups of nonexpansive mappings in a Hilbert space*, J. Math. Anal. Appl., **85** (1982), 172–178. MR647565 (83k:47038)
- [12] W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc., **81** (1981), 253–256. MR593468 (82f:47079)
- [13] W. Takahashi and J. Y. Park, *On the asymptotic behavior of almost-orbits of commutative semigroups in Banach spaces*, Nonlinear and Convex Analysis, Marcel Dekker, Inc., New York and Basel (1987), 271–293. MR892798 (88g:47131)

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