ON THE ALGEBRAIC CLOSURE IN RINGS

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ABSTRACT. The “algebraic closure” of a subset $K \subseteq A$ of a ring is an algebraic analogue of topological closure.

Suppose A is a ring, or additive category, with identity 1 and invertible group A^{-1} . Then we make the following

Definition 1. The algebraic closure of a subset $K \subseteq A$ is the set

$$(1.1) \quad \text{cl}_{\text{alg}}(K) \equiv \text{cl}_{\text{alg}}^A(K) = \{a \in A : \forall b, c \in A \exists a' \in K, 1 - b(a - a')c \in A^{-1}\}.$$

Equivalently

$$(1.2) \quad a \in \text{cl}_{\text{alg}}(K) \iff \forall b \in A \exists a' \in K, 1 - b(a - a') \in A^{-1}$$

for if this holds, then for arbitrary $b, c \in A$ there is $a' \in A$ with

$$1 - cb(a - a') \in A^{-1} \iff 1 - b(a - a')c \in A^{-1}.$$

For example if A is a Banach algebra, then the norm closure is a subset of the algebraic closure: using the geometric series [10]

$$(1.3) \quad \|z\| \|a - a'\| \|w\| < 1 \implies 1 - z(a - a')w \in A^{-1}.$$

The reverse inclusion is not clear: for example

$$(1.4) \quad \text{cl}_{\text{alg}}\{a\} = a + \text{Rad}(A),$$

the algebraic closure of a singleton is the coset modulo the Jacobson radical.

The algebraic closure has almost all of the properties of topological closure, and is compatible with the algebraic operations:

Theorem 2. For arbitrary $K, H \subseteq A$ there is inclusion

$$(2.1) \quad K \subseteq \text{cl}_{\text{alg}}(K),$$

implication

$$(2.2) \quad K \subseteq H \implies \text{cl}_{\text{alg}}(K) \subseteq \text{cl}_{\text{alg}}(H),$$

and inclusion (necessarily equality)

$$(2.3) \quad \text{cl}_{\text{alg}}\text{cl}_{\text{alg}}(K) \subseteq \text{cl}_{\text{alg}}(K).$$

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There is also inclusion

$$(2.4) \quad \text{cl}_{\text{alg}}(K) + \text{cl}_{\text{alg}}(H) \subseteq \text{cl}_{\text{alg}}(K + H)$$

and

$$(2.5) \quad \text{cl}_{\text{alg}}(K) \cdot \text{cl}_{\text{alg}}(H) \subseteq \text{cl}_{\text{alg}}(K \cdot H) .$$

Proof. (2.1) and (2.2) are clear; towards (2.3), if $x \in \text{cl}_{\text{alg}}\text{cl}_{\text{alg}}(K)$, then for arbitrary $z, w \in A$ there is $x' \in \text{cl}_{\text{alg}}(K)$ for which $1 - z(x - x')w = c \in A^{-1}$, and then $x'' \in K$ for which $1 - c^{-1}z(x' - x'')w \in A^{-1}$. Now

$$1 - z(x - x'')w = 1 - z(x - x')w - z(x' - x'')w = c(1 - c^{-1}z(x' - x'')w) \in A^{-1} .$$

Towards (2.4) and (2.5), if $x \in \text{cl}_{\text{alg}}(K)$ and $y \in \text{cl}_{\text{alg}}(H)$ and if $z, w \in A$, then, with $1 - z(x - x')w = c \in A^{-1}$,

$$\begin{aligned} 1 - z(x + y - x' - y')w &= 1 - z(x - x')w - z(y - y')w \\ &= (1 - z(x - x')w)(1 - c^{-1}z(y - y')w) \in A^{-1}A^{-1} = A^{-1} \end{aligned}$$

provided $1 - c^{-1}z(y - y')w \in A^{-1}$ and, with $1 - z(x - x')yw = d \in A^{-1}$,

$$\begin{aligned} 1 - z(xy - x'y')w &= 1 - z(x - x')yw - zx'(y - y')w \\ &= (1 - z(x - x')yw)(1 - d^{-1}zx'(y - y')w) \in A^{-1} \end{aligned}$$

provided $1 - d^{-1}zx'(y - y')w \in A^{-1}$. $\square$

One of the Kuratowski closure axioms however seems to fail:

Example 3. For $K, H \subseteq A$, inclusion

$$(3.1) \quad \text{cl}_{\text{alg}}(K \cup H) \subseteq \text{cl}_{\text{alg}}(K) \cup \text{cl}_{\text{alg}}(H)$$

is liable to fail.

Proof. In the ring $A = \mathbf{C}^2$ take

$$(3.2) \quad K = \{1\} \times \mathbf{C} , \quad H = \{-1\} \times \mathbf{C}$$

and claim

$$(3.3) \quad (2, 3) \in \text{cl}_{\text{alg}}(K \cup H) \setminus (\text{cl}_{\text{alg}}K \cup \text{cl}_{\text{alg}}H) .$$

To see that $(2, 3) \notin \text{cl}_{\text{alg}}K$ notice

$$(1, 1) - (1, 1)((2, 3) - (1, t)) = (0, t - 2) \notin A^{-1} .$$

To see that $(2, 3) \notin \text{cl}_{\text{alg}}H$ notice

$$(1, 1) - (1/3, 1)((2, 3) - (-1, t)) = (0, t - 2) \notin A^{-1} .$$

Finally to see that $(2, 3) \in \text{cl}_{\text{alg}}(K \cup H)$ suppose $(r, s) \in A$ is arbitrary. If $r \neq 1$, then for arbitrary $t \in \mathbf{C}$

$$(1, 1) - (r, s)((2, 3) - (1, t)) = (1 - r, 1 - s(t - 3)) \in A^{-1} \text{ if } 1 - s(3 - t) \neq 0 .$$

If $r = 1$, then for arbitrary $t \in \mathbf{C}$

$$(1, 1) - (r, s)((2, 3) - (-1, t)) = (-2, 1 - s(3 - t)) \in A^{-1} \text{ if } 1 - s(3 - t) \neq 0 .$$

$\square$

The algebraic closure of the invertibles participates in a curious minuet with the left and the right invertibles:

Theorem 4. *There is equality*

$$(4.1) \quad A_{\text{left} \cap \text{cl}_{\text{alg}}}^{-1} A^{-1} = A^{-1} = A_{\text{right} \cap \text{cl}_{\text{alg}}}^{-1} A^{-1} ,$$

and inclusion

$$(4.2) \quad \text{cl}_{\text{alg}} A^{-1} \subseteq A^{-1} + A^{-1} + A^{-1} .$$

Necessary and sufficient for

$$(4.3) \quad 0 \in \text{cl}_{\text{alg}} A^{-1}$$

is inclusion

$$(4.4) \quad A \subseteq A^{-1} + A^{-1} .$$

Proof. Towards (4.1) we prove more, and claim

$$(4.5) \quad A_{\text{left} \cap \text{cl}_{\text{alg}}}^{-1} A_{\text{right}}^{-1} = A^{-1} = A_{\text{right} \cap \text{cl}_{\text{alg}}}^{-1} A_{\text{left}}^{-1} .$$

Towards the first equality suppose $a \in A_{\text{left} \cap \text{cl}_{\text{alg}}}^{-1} A_{\text{right}}^{-1}$ so that there are $a', b, b' \in A$ for which $a'a = 1 = bb'$ with $1 - a'(a - b) \in A^{-1}$. But this means $a'a = 1$ with $a'b \in A^{-1}$; hence, $b \in A_{\text{left}}^{-1}$ and so $b \in A^{-1}$. The argument for the second equality in (4.5) is the same.

Towards (4.2) suppose $a \in \text{cl}_{\text{alg}} A^{-1}$. Then for arbitrary $b \in A^{-1}$ there is $c \in A^{-1}$ for which

$$1 - b(a - c) = d \in A^{-1} \implies a = b^{-1}(1 - d) + c \in A^{-1} + A^{-1} + A^{-1} .$$

Finally if (4.3) holds, then for arbitrary $b \in A$ there is $c \in A^{-1}$ for which $1 - b(-c) = d \in A^{-1}$, giving $b = (d - 1)c^{-1} \in A^{-1} + A^{-1}$, as in (4.4). Conversely for arbitrary $c, d \in A^{-1}$ we have

$$1 - (c + d)d^{-1} = -cd^{-1} \in A^{-1} ;$$

in the presence of (4.4) this is (4.3) □

The analogue of (4.1) for the topological closure is familiar in Banach algebras [8], [10]. From (4.1) it follows that everything in the algebraic closure of the invertibles is “consistent in regularity” [6], [7]:

$$(4.6) \quad \begin{aligned} \text{cl}_{\text{alg}} A^{-1} &\subseteq A^{-1} \cup (A \setminus (A_{\text{left} \cap \text{cl}_{\text{alg}}}^{-1} A_{\text{right}}^{-1})) \\ &= \{a \in A : ax \in A^{-1} \iff xa \in A^{-1}\} . \end{aligned}$$

The last part of Theorem 4 generalizes to radical elements and to idempotents:

Theorem 5. *If*

$$(5.1) \quad 1 + qA^{-1} \subseteq A^{-1} ,$$

then the condition (4.3) is necessary and sufficient for

$$(5.2) \quad q \in \text{cl}_{\text{alg}} A^{-1} .$$

If instead $p = p^2 \in A$ is idempotent, then

$$(5.3) \quad 0 \in \text{cl}_{\text{alg}}^A (pAp)^{-1} \iff 0 \in \text{cl}_{\text{alg}}^{pAp} (pAp)^{-1} \implies 1 - p \in \text{cl}_{\text{alg}} A^{-1} .$$

Proof. If $q \in A$ is in $\text{cl}_{\text{alg}} A^{-1}$, then for arbitrary $b \in A$ there is $c \in A^{-1}$ for which $1 - b(q - c) = d \in A^{-1}$ and hence

$$be = dc^{-1} - c^{-1} \text{ with } e = 1 - qc^{-1}.$$

Now if (5.1) holds, then $e \in A^{-1}$ giving $b = (d - 1)(ec)^{-1} \in A^{-1} + A^{-1}$, which is (4.3). Conversely for arbitrary $c, d \in A^{-1}$

$$1 - (c + d)(q + c^{-1}) = -(dc^{-1} + cq + dq) \in A^{-1} + A^{-1}q + A^{-1}q,$$

which is included in A^{-1} by (5.1). In the presence of (4.3) this applies to arbitrary $b = c + d \in A$. Towards (5.3), if 0 is the algebraic closure in pAp of the invertible group $(pAp)^{-1}$, then for arbitrary $b \in A$ there is $c = pcp \in (pAp)^{-1}$ for which $p(1 + bc) \in (pAp)^{-1}$, giving

$$1 + pbc = 1 - p + p(1 + bc) \in A^{-1},$$

which says $0 \in \text{cl}_{\text{alg}}^A(pAp)^{-1}$. Conversely if this holds, then for arbitrary $b \in A$, in particular $pbp \in pAp$, there is $c = cpc \in (pAp)^{-1}$ and $d \in A$ for which $(1 + pbc)d = 1 = d(1 + pbc)$, giving

$$(p + pbc)pdp = p = pd(p + pbc)$$

and hence $p(1 + bc) \in (pAp)^{-1}$. Also if for arbitrary $b \in A$ there is $c, d \in pAp$ with $1 - bc \in A^{-1}$ and $cd = p = dc$, then $a = 1 - p - c \in A^{-1}$ with $a^{-1} = 1 - p - d$ giving $1 - ba = 1 - bc \in A^{-1}$. $\square$

The algebraic closure intervenes in generalized inverse theory:

Lemma 6. *With*

$$(6.1) \quad \hat{A} = \{a \in A : a \in aAa\} \text{ and } \check{A} = \{a \in A : a \in aA^{-1}a\}$$

there is inclusion

$$(6.2) \quad \hat{A} \cap \text{cl}_{\text{alg}} A^{-1} \subseteq \check{A}.$$

Necessary and sufficient for equality in (6.2) is that

$$(6.3) \quad A^\bullet \equiv \{p \in A : p = p^2\} \subseteq \text{cl}_{\text{alg}} A^{-1}.$$

Proof. If $a = aa'a \in \hat{A}$, so that $a'a = p = p^2$ is idempotent, and if

$$b \in A^{-1} \text{ with } 1 + (b - a)a' = c^{-1} \in A^{-1},$$

then

$$a = (cb)p \in \check{A}.$$

For equality in (6.2), observe [9], [10]

$$\check{A} = A^{-1}A^\bullet = A^\bullet A^{-1}.$$

$\square$

In Banach algebras (6.3) is clear [9], [10]: generally

$$(6.4) \quad 0 \notin \text{int } \sigma(a) \implies a \in \text{cl}_{\text{alg}} A^{-1},$$

where $\sigma(a) = \{\lambda \in \mathbf{C} : a - \lambda \notin A^{-1}\}$ is the usual spectrum.

We can extend the algebraic closure to tuples:

Definition 7. For arbitrary $K \subseteq A^n$

$$(7.1) \quad \text{cl}_{\text{alg}} K = \{x \in A^n : \forall z, w \in A^n \exists x' \in K, 1 - \sum_{j=1}^n z_j(x_j - x'_j)w_j \in A^{-1}\}.$$

We can also define left and right invertible tuples:

$$(7.2) \quad A_{\text{left}}^{-n} = \{a \in A^n : 1 \in \sum_{j=1}^n Aa_j\} \text{ and } A_{\text{right}}^{-n} = \{a \in A^n : 1 \in \sum_{j=1}^n a_j A\} .$$

If we interpret A as an additive category, then in a sense we have already dealt with n -tuples, and more generally $n \times m$ matrices, over A , which just form another additive category $B = \text{Matrix}(A)$; thus most of Theorem 2 extends to n -tuples. For example the analogues of (2.1) and (2.2) for $K, H \subseteq A^n$ are immediate, while for the analogue of (2.3) we argue that if $x \in \text{cl}_{\text{alg}} \text{cl}_{\text{alg}}(K)$, then for arbitrary $z, w \in A^n$ there is $x' \in \text{cl}_{\text{alg}}(K)$ for which $1 - \sum_j z_j(x_j - x'_j)w_j = c \in A^{-1}$, and then $x'' \in K$ for which $1 - c^{-1} \sum_j z_j(x'_j - x''_j)w_j \in A^{-1}$. Now

$$\begin{aligned} 1 - \sum_j z_j(x_j - x''_j)w_j &= 1 - \sum_j (z_j(x_j - x'_j)w_j - z_j(x'_j - x''_j)w_j) \\ &= c(1 - c^{-1} \sum_j z_j(x'_j - x''_j)w_j) \in A^{-1} . \end{aligned}$$

Towards the extension of (2.4), if $x \in \text{cl}_{\text{alg}}(K)$ and $y \in \text{cl}_{\text{alg}}(H)$ and if $z, w \in A^n$ then, with $1 - \sum_j z_j(x_j - x'_j)w_j = c \in A^{-1}$,

$$\begin{aligned} 1 - \sum_j z_j(x_j + y_j - x'_j - y'_j)w_j \\ &= (\sum_j (1 - z_j(x_j - x'_j)w_j))(1 - c^{-1} \sum_j z_j(y_j - y'_j)w_j) \\ &\in A^{-1}A^{-1} = A^{-1} \text{ provided } 1 - c^{-1} \sum_j z_j(y_j - y'_j)w_j \in A^{-1} . \end{aligned}$$

Similarly, for an extended version of (2.5), with $1 - \sum_j z_j(x_j - x'_j)y_jw_j = d \in A^{-1}$,

$$\begin{aligned} 1 - \sum_j z_j(x_jy_j - x'_jy'_j)w_j \\ &= (1 - \sum_j z_j(x_j - x'_j)y_jw_j)(1 - d^{-1} \sum_j z_jx'_j(y_j - y'_j)w_j) \\ &\in A^{-1} \text{ provided } 1 - d^{-1} \sum_j z_jx'_j(y_j - y'_j)w_j \in A^{-1} . \end{aligned}$$

(2.5) would also extend, in a category A , to K, H and a more general bilinear image $K * H$.

Declare that a ring A has *left stable range* $\leq n$ provided

$$(7.3) \quad \forall (a, b) \in A^n \times A, (a, b) \in A_{\text{left}}^{-n-1} \implies \exists c \in A^n, a - cb \in A_{\text{left}}^{-n} .$$

Corach and Suarez [4], [5], and Blackadar [3], have considered this kind of situation when A is commutative, or a C^* -algebra.

Notice how the definition prefers the final element of an $n+1$ tuple; an alternative would say that if $d \in A^{n+1}$ was “left invertible”, then there would exist an index $1 \leq j \leq n+1$ for which an analogue of (7.3) held. In commutative Banach algebras the condition that A has stable range ≤ 1 reduces to the topological closure of the invertible group being the whole of A . We offer here a curious hybrid result:

Theorem 8. *If*

$$(8.1) \quad A \subseteq \text{cl}_{\text{alg}} A_{\text{left}}^{-1},$$

then there is implication

$$(8.2) \quad (a, b) \in A_{\text{left}}^{-2} \implies (a - Ab)_{\cap} A_{\text{right}}^{-1} \neq \emptyset.$$

Proof. Suppose

$$a'a + b'b = 1 \text{ with } a' \in \text{cl}_{\text{alg}} A_{\text{left}}^{-1},$$

so that there are a'' , a''' in A for which

$$b'b = 1 - a'a = d - a''a \text{ with } d \in A^{-1} \text{ and } a'''a'' = 1$$

giving

$$a - cb = a'''d \in A_{\text{right}}^{-1} \text{ with } c = -a'''b'.$$

□

When $b = 0$ this gives back (4.1). The same argument gives back a weaker version of (7.3): if $(a, b) \in A \times A^n$, then if (8.1) holds there is implication

$$(8.3) \quad (a, b) \in A_{\text{left}}^{-n-1} \implies (a - \sum_j Ab_j)_{\cap} A_{\text{right}}^{-1} \neq \emptyset.$$

We remark that the condition (8.2) says that the element $a \in A$ is in another kind [1], [2] of “closure” of the semigroup A_{right}^{-1} : for the moment we find the precise relationship between this condition and ours elusive.

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