ON THE LOCAL HÖLDER CONTINUITY OF THE INVERSE OF THE p -LAPLACE OPERATOR

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ABSTRACT. We prove an interpolation type inequality between C^α , L^∞ and L^p spaces and use it to establish the local Hölder continuity of the inverse of the p -Laplace operator: $\|(-\Delta_p)^{-1}(f) - (-\Delta_p)^{-1}(g)\|_{C^1(\bar{\Omega})} \leq C\|f - g\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}$, for any f and g in a bounded set in $L^\infty(\Omega)$.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^N$ with $C^{1,\gamma}$ boundary and $1 < p < \infty$. Consider the Dirichlet problem

$$(1.1) \quad \begin{cases} -\Delta_p u &= f, \text{ in } \Omega, \\ u &= 0, \text{ on } \partial\Omega, \end{cases}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplace operator. The solution u to (1.1) is unique for a given f in $L^\infty(\Omega)$. It follows from regularity results in [2] and [3] that the inverse of the p -Laplace operator

$$(1.2) \quad L := (-\Delta_p)^{-1} : L^\infty(\Omega) \rightarrow C^{1,\alpha}(\bar{\Omega})$$

is a continuous and compact mapping for some $0 < \alpha < 1$. We write $u = Lf$ if u solves (1.1) in the weak sense (see (2.1) for the definition of weak solutions).

In this paper we study the Hölder continuity of the operator L . As we will see in section 2, one obtains $\|Lf - Lg\|_\infty \leq C\|f - g\|_\infty^{\frac{1}{p-1}}$, for p large, by using Sobolev embedding theorems.

However, if we want to estimate the $C^1(\bar{\Omega})$ norm of $(Lf - Lg)$ in terms of $\|f - g\|_\infty$, then Sobolev embeddings do not give the result. It is so because $W^{1,p}(\Omega)$ is not a subset of $C^1(\bar{\Omega})$.

Our idea is to establish an interpolation inequality between $C^{1,\alpha}(\bar{\Omega})$, $C^1(\bar{\Omega})$ and $W^{1,p}(\Omega)$. Such an inequality together with the compactness of the operator L gives us an estimate for $\|Lf - Lg\|_{C^1}$ and thus shows that L is locally Hölder continuous.

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We use the following notation:

$$\begin{aligned}
 \alpha, \beta, \gamma &: \text{positive constants in } (0, 1). \\
 U &: \text{an open set in } \mathbb{R}^N. \\
 \Omega &: \text{an open bounded domain in } \mathbb{R}^N \text{ with } C^{1,\gamma} \text{ boundary.} \\
 c, r, \theta &: \text{positive constants depending only on the universal} \\
 &\quad \text{parameters such as } N, p, \Omega. \\
 c(M) &: \text{a constant depending on } M \text{ and possibly on} \\
 &\quad \text{the universal parameters.} \\
 C^\alpha(\bar{U}), C^\alpha(\mathbb{R}^N) &: \{u : |u|_\alpha := \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha} < \infty\}. \\
 \|u\|_{C^{0,\alpha}} = \|u\|_{0,\alpha} &= \|u\|_\infty + |u|_\alpha. \\
 C^1(\bar{U}), C^1(\mathbb{R}^N) &: \{u : \nabla u \text{ is continuous}\}. \\
 \|u\|_{C^1} = \|u\|_{1,0} &= \|\nabla u\|_\infty + \|u\|_\infty.
 \end{aligned}$$

We denote by $\|\cdot\|_{1,\alpha}$ and $\|\cdot\|_q$ the norms in $C^{1,\alpha}$ and L^q , $1 \leq q \leq \infty$, respectively.

The spaces $W_0^{1,p}$ and $W^{1,p}$, $1 < p < \infty$, are used in the standard way.

Our first result is about an interpolation inequality in $\mathbb{R}^N$.

Theorem 1.1. *Given $0 < \alpha < 1$ and $1 \leq q < \infty$, there exist constants c and $0 < \theta < 1$ (depending only on N, p and α) such that for any u in $C^\alpha(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$,*

$$(1.3) \quad \|u\|_\infty \leq c \|u\|_{0,\alpha}^{1-\theta} \|u\|_q^\theta.$$

Next, we establish the inequality (1.3) for an open set U in $\mathbb{R}^N$. It is possible if U satisfies the cone property (see Definition 3.1).

Theorem 1.2. *Let U be an open set in $\mathbb{R}^N$ such that U satisfies the cone property. Given $0 < \alpha < 1$ and $1 \leq q < \infty$, there exist constants c and $0 < \theta < 1$ such that for any u in $C^\alpha(\bar{U}) \cap L^q(U)$*

$$(1.4) \quad \|u\|_\infty \leq c \|u\|_{0,\alpha}^{1-\theta} \|u\|_q^\theta.$$

An immediate consequence of Theorem 1.2 is the interplay between $C^{1,\alpha}(\bar{U})$, $C^1(\bar{U})$ and $W^{1,p}(U)$.

Corollary 1.3. *Let U be an open set in $\mathbb{R}^N$ such that U satisfies the cone property. There exist constants $c > 0$ and $0 < \theta < 1$ such that for any $u \in C^{1,\alpha}(\bar{U}) \cap W^{1,p}(U)$,*

$$(1.5) \quad \|u\|_{1,0} \leq c \|u\|_{1,\alpha}^{1-\theta} \|u\|_{W^{1,p}}^\theta.$$

Inequality (1.5) is the key result which is used to show the Hölder continuity of the operator L defined by (1.2). Let B_M^∞ be a ball in $L^\infty(\Omega)$ with radius $M > 0$. By compactness, the image $L(B_M^\infty)$ is a bounded set in $C^{1,\alpha}(\bar{\Omega})$.

Theorem 1.4. *Given $M > 0$, then there exist positive constants $c(M)$ and r such that*

$$(1.6) \quad \|Lf - Lg\|_{1,0} \leq c(M) \|f - g\|_\infty^r,$$

for any f and g in B_M^∞ (i.e. $0 < \|f\|_\infty, \|g\|_\infty < M$).

Theorem 1.4 shows that the operator L , as a mapping from $L^\infty(\Omega)$ to $C^1(\bar{\Omega})$, is locally Hölder continuous if $0 < r < 1$ and is locally Lipschitz continuous if $r \geq 1$. The rest of the paper is organized as follows. In section 2 we recall preliminary results and study the $\|Lf - Lg\|_\infty$ estimate of problem (1.1). In section 3, we give the proofs of Theorem 1.1 and Theorem 1.2. Finally in section 4, we establish the Hölder continuity of the operator L by proving Theorem 1.4.

2. PRELIMINARY RESULTS

Let us recall the definition of weak solutions to problem (1.1). Given a measurable function f , we say a function u in $W_0^{1,p}(\Omega)$ solves (1.1) in the weak sense provided that

$$(2.1) \quad \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx = \int_{\Omega} f v dx, \quad \forall v \in W_0^{1,p}(\Omega).$$

Evidently, if f is in $L^q(\Omega)$, for any $q \in [p, \infty]$, then the left hand side of (2.1) is finite for any v in $W_0^{1,p}(\Omega)$.

In fact if f is a bounded function, the corresponding solution u is unique and differentiable. The following proposition is a summary of regularity theory of the p -Laplace operator.

Proposition 2.1. *Let f be in $L^q(\Omega)$ for some $q \in [p, \infty]$. Then there exists a unique solution u to problem (1.1), i.e. the inverse p -Laplace operator L is well defined. Moreover,*

- (i) *If $q > \frac{Np}{p-1}$, then u is in $C^{1,\beta}(\Omega)$ locally, i.e. u is of class $C^{1,\beta}$ on any compact subset of Ω .*
- (ii) *If $q = \infty$, then u is in $C^{1,\alpha}(\bar{\Omega})$. Furthermore, for any number M positive there exists a constant $c = c(M)$ such that*

$$\|u\|_{1,\alpha} = \|Lf\|_{1,\alpha} \leq c(M),$$

whenever $0 \leq \|f\|_\infty \leq M$.

- (iii) *The operator L is continuous and compact from L^∞ to $C^1(\bar{\Omega})$.*

We recall that a mapping is said to be compact if it maps bounded sets to precompact sets.

Proof. The existence of a solution u can be established by minimizing in $W_0^{1,p}(\Omega)$ the functional

$$J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p dx - \int_{\Omega} f u dx.$$

The uniqueness of u is a direct consequence of Lemma 2.2. Assertion (i) is a special case of the results proved in [2] or in [6] and assertion (ii) is shown in [3]. Finally, due to (ii) the operator L maps bounded sets in $L^\infty(\Omega)$ to bounded sets in $C^{1,\alpha}(\bar{\Omega})$. Consequently, we obtain (iii) by the compact embedding from $C^{1,\alpha}(\bar{\Omega})$ into $C^1(\bar{\Omega})$.

In fact, L is a continuous and compact mapping from $L^\infty(\Omega)$ to $C^{1,\alpha'}(\bar{\Omega})$, for any $0 \leq \alpha' < \alpha$. $\square$

Let A be the differential mapping of the functional $\frac{1}{p} \int_{\Omega} |\nabla u|^p dx$,

$$(2.2) \quad \begin{aligned} A &: W_0^{1,p}(\Omega) \rightarrow W_0^{1,p}(\Omega)^*, \\ \langle Au, v \rangle &= \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v dx, \quad \forall u, v \in W_0^{1,p}(\Omega). \end{aligned}$$

Clearly, $u = Lf$ is equivalent to $\langle Au, \cdot \rangle = \langle f, \cdot \rangle_{L^2}$.

In the proof of Theorem 1.4 we use some properties of the operator A . In [5], Lindqvist shows that there exists a constant $c = c(p)$ such that for any vector x and y in $\mathbb{R}^N$, one has

$$\begin{aligned} |y|^p &\geq |x|^p + p|x|^{p-2}x \cdot (y-x) + c|x-y|^p, \quad \text{if } p \geq 2, \\ |y|^p &\geq |x|^p + p|x|^{p-2}x \cdot (y-x) + c \frac{|x-y|^2}{(|x|+|y|)^{2-p}}, \quad \text{if } 1 < p < 2. \end{aligned}$$

Direct calculations yield the following:

Lemma 2.2. *For any u and v in $W_0^{1,p}(\Omega)$, we have*

$$(2.3) \quad \langle Au - Av, u - v \rangle \geq c(p) \int_{\Omega} |\nabla u - \nabla v|^p dx, \quad \text{if } p \geq 2,$$

$$(2.4) \quad \langle Au - Av, u - v \rangle \geq c(p) \int_{\Omega} \frac{|\nabla u - \nabla v|^2}{(|\nabla u| + |\nabla v|)^{2-p}} dx, \quad \text{if } 1 < p < 2.$$

Equalities hold if and only if $u = v$ in Ω .

To motivate our work, let us estimate $\|Lf - Lg\|_{\infty}$. Using (2.3) in Lemma 2.2 one can show that for $p > N$,

$$(2.5) \quad \|Lf - Lg\|_{\infty} \leq c \|f - g\|_{\infty}^{\frac{1}{p-1}}.$$

To see this, let $u = Lf$ and $v = Lg$. It follows from (2.3) that

$$c \|u - v\|_{W_0^{1,p}(\Omega)}^p \leq \langle Au - Av, u - v \rangle = \int_{\Omega} (f - g)(u - v),$$

which, by Hölder inequality, implies

$$c \|u - v\|_{W_0^{1,p}(\Omega)}^p \leq \|f - g\|_{\infty} \|u - v\|_{W_0^{1,p}(\Omega)}.$$

As $p > N$, the space $W_0^{1,p}(\Omega)$ is compactly embedded into $L^{\infty}(\Omega)$. Thus, (2.5) follows.

Remark 2.3. (i) As we see from the argument, the constant c in (2.5) does not depend on f and g . (ii) Using the identity $L(kf) = k^{\frac{1}{p-1}} Lf$, for any $k > 0$, we obtain from Proposition 2.1(ii) that there exists a constant c independent of f such that

$$(2.6) \quad \|Lf\|_{1,\alpha} \leq c \|f\|_{\infty}^{\frac{1}{p-1}}.$$

3. INTERPOLATION INEQUALITIES

In this section we establish interpolation inequalities between C^{α} , L^{∞} and L^q in $\mathbb{R}^N$ and in its open subsets.

Proof of Theorem 1.1. If $\|u\|_\infty = \infty$, then $\|u\|_{0,\alpha} = \infty$ and (1.3) holds. Suppose $\|u\|_\infty < \infty$, it suffices to show

$$(3.1) \quad |u(0)| \leq c\|u\|_{0,\alpha}^{1-\theta}\|u\|_q^\theta.$$

Since for arbitrary y in $\mathbb{R}^N$, we let $v(x) = u(x+y)$ and use (3.1) to obtain

$$|u(y)| = |v(0)| \leq c\|u\|_{0,\alpha}^{1-\theta}\|u\|_q^\theta,$$

which implies (1.3).

To see (3.1) we first observe from the triangle inequality that

$$(3.2) \quad \begin{aligned} |u(0)| &\leq \frac{|u(x) - u(0)|}{|x|^\alpha} |x|^\alpha + |u(x)| \\ &\leq \|u\|_{0,\alpha} |x|^\alpha + |u(x)|, \quad \forall x \in \mathbb{R}^N. \end{aligned}$$

For any $r > 0$, we integrate both sides of (3.2) over the ball $B_r = B(0, r)$ to get

$$\begin{aligned} \omega r^N |u(0)| &\leq \|u\|_{0,\alpha} \int_{B_r} |x|^\alpha dx + \int_{B_r} |u(x)| dx \\ &\leq \frac{N}{N+\alpha} \omega r^{N+\alpha} \|u\|_{0,\alpha} + (\omega r^N)^{1/q'} \|u\|_{L^q(B_r)}. \end{aligned}$$

Here ω is the volume of the unit ball in $\mathbb{R}^N$ and $\frac{1}{q'} = 1 - \frac{1}{q}$ if $q > 1$, $\frac{1}{q'} = 0$ if $q = 1$. Thus,

$$\begin{aligned} |u(0)| &\leq \frac{N}{N+\alpha} r^\alpha \|u\|_{0,\alpha} + (\omega r^N)^{-1+1/q'} \|u\|_q \\ &\leq c(r^\alpha \|u\|_{0,\alpha} + \frac{1}{r^\beta} \|u\|_q), \end{aligned}$$

where $c = \max\{\frac{N}{N+\alpha}, \omega^{-1+1/q'}\}$ and $\beta = N(1 - 1/q') > 0$. We note that the above inequality holds for any $r > 0$. We use the fact

$$r^\alpha a + \frac{1}{r^\beta} b = a^{1-\theta} b^\theta, \quad \text{if } r^{\alpha+\beta} = \frac{b}{a} \text{ and } \theta = \frac{\alpha}{\alpha+\beta},$$

to conclude that

$$|u(0)| \leq c\|u\|_{0,\alpha}^{1-\theta}\|u\|_q^\theta,$$

with $\theta = \frac{\alpha}{\alpha+\beta}$.

The main idea in the previous proof is the integration over any ball B_r , with arbitrary radius $r > 0$, both sides of inequality (3.2). Given U an open set in $\mathbb{R}^N$, we want to establish (1.4) (in Theorem 1.2) by using such an idea. By translation, we can assume 0 is in $\bar{U}$, and thus we obtain (3.2) for any $x \in U$. However, U may not contain all the balls $B_r = B(0, r)$ with arbitrary radius, or U may not even contain any ball B_r at all if 0 is on the boundary of U . As we will see later, it turns out that we only need to integrate on a fixed portion of a ball, namely a cone, and up to a fixed radius r_0 . For this reason, we assume that U satisfies the cone property.

Definition 3.1. We say an open U satisfies the cone property if there is a cone $K(r_0)$ with radius $r_0 > 0$ such that for each point x in $\bar{U}$ we can place inside $\bar{U}$ a cone $K(x, r_0)$, center at x , which is congruent to the cone $K(r_0)$.

Proof of Theorem 1.2. It suffices to show, for any x in $\bar{U}$, that

$$(3.3) \quad |u(x)| \leq c \|u\|_{0,\alpha}^{1-\theta} \|u\|_q^\theta.$$

Following the argument made in the proof of Theorem 1.1 we obtain

$$(3.4) \quad |u(x)| \leq \|u\|_{0,\alpha} |y - x|^\alpha + |u(y)|,$$

for any y in $\bar{U}$. Let $K(x, r_0)$ be the cone center at x (by Definition 3.1). For any $0 < r < r_0$, integrating both sides of (3.4) over the cone $K(r) = K(x, r_0) \cap B(x, r)$ to get

$$\begin{aligned} \omega_K r^N |u(x)| &\leq \|u\|_{0,\alpha} \int_{K(r)} |y - x|^\alpha dy + \int_{K(r)} |u(y)| dy \\ &\leq \frac{N}{N + \alpha} \omega_K r^{N+\alpha} \|u\|_{0,\alpha} + (\omega_K r^N)^{1/q'} \|u\|_{L^q(B_r)}, \end{aligned}$$

here $\omega_K = |K(0, 1)|$, the volume of $\frac{1}{r_0^N} K(r_0)$. Thus there is a constant $c_1 > 0$ such that

$$|u(x)| \leq c_1 (r^\alpha \|u\|_{0,\alpha} + \frac{1}{r^\beta} \|u\|_q),$$

for any $0 < r < r_0$. Let $\theta = \frac{\alpha}{\alpha+\beta}$; we consider two cases.

(i) $\frac{\|u\|_q}{\|u\|_{0,\alpha}} < r_0^{\alpha+\beta}$. By taking $r^{\alpha+\beta} = \frac{\|u\|_q}{\|u\|_{0,\alpha}}$ and substituting into the above inequality we obtain

$$|u(x)| \leq c_1 \|u\|_{0,\alpha}^{1-\theta} \|u\|_p^\theta.$$

(ii) $\frac{\|u\|_q}{\|u\|_{0,\alpha}} \geq r_0^{\alpha+\beta}$. We note that

$$\|u\|_\infty \leq \|u\|_{0,\alpha} \leq \|u\|_{0,\alpha}^{1-\theta} \|u\|_{0,\alpha}^\theta \leq \|u\|_{0,\alpha}^{1-\theta} \|u\|_q^\theta r_0^\alpha.$$

Combining these two cases we conclude (3.3) with $c = \max\{c_1, r_0^\alpha\}$. $\square$

We leave the verification of Corollary 1.3 to the readers.

4. ON THE LOCAL HÖLDER CONTINUITY OF L

As mentioned in section 1, we assume Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,\gamma}$ boundary and $1 < p < \infty$. Evidently, Ω satisfies the cone property. Let us recall that $u = Lf$ if and only if $\langle Au, \cdot \rangle = \langle f, \cdot \rangle_{L^2}$ where the operators L and A are defined by (1.2) and by (2.2), respectively.

Proof of Theorem 1.4. Let f and g be in B_M^∞ , the ball with radius M in $L^\infty(\Omega)$. We will show that there exist positive constants $r > 0$ and $c = c(M)$ such that

$$(4.1) \quad \|u - v\|_{1,0} \leq c \|f - g\|_\infty^r,$$

where $u = Lf$ and $v = Lg$.

It is clear that

$$(4.2) \quad \langle Au - Av, u - v \rangle = \int_\Omega (f - g)(u - v) dx.$$

We first establish (4.1) for the case $p \geq 2$. By inequality (2.3) in Lemma 2.2,

$$c \int_\Omega |\nabla u - \nabla v|^p dx \leq \langle Au - Av, u - v \rangle.$$

Using Hölder's inequality, we can estimate from above the right hand side of (4.2)

$$\int_{\Omega} (f - g)(u - v) dx \leq c \|f - g\|_{\infty} \|u - v\|_{W_0^{1,p}}.$$

Thus,

$$(4.3) \quad \|u - v\|_{W_0^{1,p}}^{p-1} \leq c \|f - g\|_{\infty}.$$

On the other hand, as $u - v$ is in $C^{1,\alpha}(\bar{\Omega})$, it follows from (1.5) in Corollary 1.3 that

$$(4.4) \quad \|u - v\|_{1,0} \leq c \|u - v\|_{1,\alpha}^{1-\theta} \|u - v\|_{W_0^{1,p}(\Omega)}^{\theta}.$$

By the boundedness of L (Proposition 2.1(ii)),

$$(4.5) \quad \|u - v\|_{1,\alpha} \leq \|u\|_{1,\alpha} + \|v\|_{1,\alpha} \leq c(M).$$

Combining (4.3), (4.4) and (4.5) we conclude that

$$\|u - v\|_{1,0} \leq c(M) \|u - v\|_{W_0^{1,p}(\Omega)}^{\theta} \|f - g\|_{\infty}^{\frac{\theta}{p-1}},$$

which is (4.1) with $r = \frac{\theta}{p-1}$.

If $1 < p < 2$, we use inequality (2.4),

$$c(p) \int_{\Omega} \frac{|\nabla u - \nabla v|^2}{(|\nabla u| + |\nabla v|)^{2-p}} dx \leq \langle Au - Av, u - v \rangle,$$

(4.2) and the boundedness of L to obtain

$$\int_{\Omega} |\nabla u - \nabla v|^2 dx \leq c(M) \int_{\Omega} (f - g)(u - v) dx.$$

By Hölder inequality, the analog to (4.3) in this case is

$$(4.6) \quad \|u - v\|_{W_0^{1,2}} \leq c(M) \|f - g\|_{\infty}.$$

We carry the same argument made in the previous case with $p = 2$ to conclude (4.1).

We close the paper with some remarks.

Remark 4.1. The fact that the solution operator L is continuous (Proposition 2.1(iii)) is not new. In fact it is a direct consequence of Proposition 2.1(ii), which is proved in [3]. Yet the Hölder continuity of L says more about the stability of solutions with respect to data. Inequality (1.6) gives an estimate on the rate of convergence of solutions in C^1 spaces.

Remark 4.2. It is not clear to us that L is actually (*globally*) Hölder continuous from $L^{\infty}(\Omega)$ to $C^1(\bar{\Omega})$. In other words, it may be possible that the constant c in (1.6) and in (4.1) does not depend on M . This is the case if $p = 2$ or if $N = 1$ or if we consider radial solutions of problem (1.1) in the unit ball of $\mathbb{R}^N$. Also, as we note in Remark 2.3, inequalities (2.5) and (2.6) hold globally.

Remark 4.3. While reading the manuscript, the referee has noticed that condition $q > \frac{Np}{p-1}$ in Proposition 2.1(i) can be replaced by $q > N$. The idea is to use the arguments in Section 5 of [4] with the measure μ having Radon-Nikodym derivative equal to f . Also, as pointed out by the referee, Lemma 3.1 in [1] is very close to Theorem 1.1.

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