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A SHARP VANISHING THEOREM FOR LINE BUNDLES ON K3 OR ENRIQUES SURFACES

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Abstract. Let L be a line bundle on a K3 or Enriques surface. We give a vanishing theorem for $H^1(L)$ that, unlike most vanishing theorems, gives necessary and sufficient geometrical conditions for the vanishing. This result is essential in our study of Brill-Noether theory of curves on Enriques surfaces (2006) and of Enriques-Fano threefolds (2006 preprint).

1. Introduction

Since Grothendieck's introduction of basic tools such as the cohomology of sheaves and the Grothendieck-Riemann-Roch theorem, vanishing theorems have proved to be essential in many studies in algebraic geometry.

Perhaps the most influential one, at least for line bundles, is the well-known Kawamata-Viehweg vanishing theorem ([K, V]) which, in its simplest form, asserts that $H^i(K_X+\mathcal{L}) = 0$ for $i > 0$ and any big and nef line bundle $\mathcal{L}$ on a smooth variety X. On the other hand, as most vanishing theorems (even for special surfaces [CD, Thm.1.5.1]), it gives only *sufficient conditions for the vanishing.* Practice shows though that, in many situations, it would be very useful to know that a certain vanishing is *equivalent* to some geometrical/numerical properties of $\mathcal{L}$.

In this short note we accomplish the above goal for line bundles on a K3 or Enriques surface, by proving that, when $L^2 > 0$, the vanishing of $H^1(L)$ is equivalent to the fact that the intersection of L with all effective divisors of self-intersection -2 is at least -1.

In the statement of the theorem we will employ the following

Definition 1.1. Let X be a smooth surface. We will denote by $\sim$ (respectively $\equiv$) the linear (respectively numerical) equivalence of divisors (or line bundles) on X. We will say that a line bundle L is *primitive* if $L \equiv kL'$ for some line bundle L' and some integer k implies $k = \pm 1$.

Theorem. *Let X be a K3 or an Enriques surface and let L be a line bundle on X such that $L > 0$ and $L^2 \geq 0$. Then $H^1(L) \neq 0$ if and only if one of the three following occurs:*

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(i) *$L \sim nE$ for $E > 0$ nef and primitive with $E^2 = 0$, $n \geq 2$ and $h^1(L) = n-1$ if X is a K3 surface, $h^1(L) = \lfloor \frac{n}{2} \rfloor$ if X is an Enriques surface;*
(ii) *$L \sim nE + K_X$ for $E > 0$ nef and primitive with $E^2 = 0$, X is an Enriques surface, $n \geq 3$ and $h^1(L) = \lfloor \frac{n-1}{2} \rfloor$;*
(iii) *there is a divisor $\Delta > 0$ such that $\Delta^2 = -2$ and $\Delta.L \leq -2$.*

Note that the hypothesis $L > 0$ is not restrictive since, if L is nontrivial, from $L^2 \geq 0$ we get by Riemann-Roch that either $L > 0$ or $K_X - L > 0$, and $h^1(L) = h^1(K_X - L)$ by Serre duality.

The theorem has of course many possible applications. For example, if L is base-point free and $|P|$ is an elliptic pencil on X, the knowledge of $h^0(L - nP)$ for $n \geq 1$ (which follows by Riemann-Roch if we know that $h^1(L-nP) = 0$) determines the type of scroll spanned by the divisors of $|P|$ in $\mathbb{P}H^0(L)$ and containing $\varphi_L(X)$ ([SD, KJ, Co]). Most importantly for us, this result proves crucial in our study of the Brill-Noether theory [KL1, KL2] and Gaussian maps [KL3] of curves lying on an Enriques surface, and especially in our proof of a genus bound for threefolds having an Enriques surface as a hyperplane section given in [KLM].

2. Proof of the Theorem

We first record the following simple but useful fact.

Lemma 2.1. *Let X be a smooth surface and let $A > 0$ and $B > 0$ be divisors on X such that $A^2 \geq 0$ and $B^2 \geq 0$. Then $A.B \geq 0$ with equality if and only if there exists a primitive divisor $F > 0$ and integers $a \geq 1, b \geq 1$ such that $F^2 = 0$ and $A \equiv aF, B \equiv bF$.*

Proof. The first assertion follows from the signature theorem [BPV, VIII.1]. If $A.B = 0$, then we cannot have $A^2 > 0$, otherwise the Hodge index theorem implies the contradiction $B \equiv 0$. Therefore $A^2 = B^2 = 0$. Now let H be an ample line bundle on X and set $\alpha = A.H, \beta = B.H$. We have $(\beta A - \alpha B)^2 = 0$ and $(\beta A - \alpha B).H = 0$, therefore $\beta A \equiv \alpha B$ by the Hodge index theorem. As there is no torsion in $\mathrm{Num}(X)$ we can find a divisor F as claimed. □

We now proceed with the theorem.

Proof. One immediately sees that $h^1(L)$ has the given values in (i) and (ii). In the case (iii) we first observe that $h^2(L-\Delta) = 0$. In fact $(K_X - L + \Delta)^2 > 0$, whence if $K_X - L + \Delta \geq 0$ the signature theorem [BPV, VIII.1] implies $0 \leq L.(K_X - L + \Delta) = -L^2 + L.\Delta \leq -2$, a contradiction. Therefore by Riemann-Roch we get

$$\frac{1}{2}L^2 + \chi(\mathcal{O}_X) < \frac{1}{2}L^2 - \Delta.L - 1 + \chi(\mathcal{O}_X) \leq h^0(L-\Delta) \leq h^0(L) = \frac{1}{2}L^2 + \chi(\mathcal{O}_X) + h^1(L)$$

whence $h^1(L) > 0$.

Now assume that $h^1(L) > 0$.

First we suppose that L is nef. By Riemann-Roch we have that $L + K_X > 0$. Since $h^1(-(L+K_X)) = h^1(L) > 0$, by [BPV, Lemma12.2], we deduce that $L + K_X$ is not 1-connected, whence that there exist $L' > 0$ and $L'' > 0$ such that $L + K_X \sim L' + L''$ and $L'.L'' \leq 0$. Now $(L')^2 \geq (L')^2 + L'.L'' = L'.L \geq 0$ and similarly $(L'')^2 \geq 0$, whence Lemma 2.1 implies that $L' \equiv aE$, $L'' \equiv bE$ for some $a, b \geq 1$ and for $E > 0$ nef and primitive with $E^2 = 0$. This gives us the two cases (i) and (ii).

Now assume that L is not nef, so that the set

$$\mathcal{A}_1(L) := \{\Delta > 0 \ : \ \Delta^2 = -2, \ \Delta.L \leq -1\}$$

is not empty. Similarly define the set

$$\mathcal{A}_2(L) = \{\Delta > 0 \ : \ \Delta^2 = -2, \ \Delta.L \leq -2\}.$$

If $\mathcal{A}_2(L) \neq \emptyset$ we are done. Assume therefore that $\mathcal{A}_2(L) = \emptyset$ and pick $\Gamma \in \mathcal{A}_1(L)$. Then $\Gamma.L = -1$, and we can clearly assume that Γ is irreducible. Hence if we set $L_1 = L - \Gamma$ we have that $L_1 > 0$, $L_1^2 = L^2$ and, since $h^0(L_1) = h^0(L)$, also that $h^1(L_1) = h^1(L) > 0$.

If L_1 is nef, by what we have just seen, we have $L_1 \equiv nE$, for $n \geq 2$, whence $L \equiv nE + \Gamma$ and $-1 = \Gamma.L = nE.\Gamma - 2$, a contradiction.

Therefore L_1 is not nef and $\mathcal{A}_1(L_1) \neq \emptyset$.

If $\mathcal{A}_2(L_1) \neq \emptyset$ we pick a $\Delta \in \mathcal{A}_2(L_1)$. We have $-2 \geq \Delta.L_1 = \Delta.(L - \Gamma) \geq -1 - \Delta.\Gamma$, whence $\Delta.\Gamma \geq 1$, $(\Delta + \Gamma)^2 \geq -2$ and $(\Delta + \Gamma).L_1 \leq -1$. Now Lemma 2.1 yields $(\Delta + \Gamma)^2 = -2$, so that $\Delta.\Gamma = 1$. Also $-1 \leq \Delta.L = \Delta.(L_1 + \Gamma) \leq -1$, whence $\Delta.L = -1$ and $(\Delta + \Gamma).L = -2$, contradicting $\mathcal{A}_2(L) = \emptyset$.

We have therefore shown that $\mathcal{A}_2(L_1) = \emptyset$.

This means that we can continue the process. But the process must eventually stop, since we always remove base components. This gives the desired contradiction. $\square$

Remark 2.2. A naive guess, to insure the vanishing of $H^1(L)$ for a line bundle $L > 0$ with $L^2 \geq 0$, could be that it is enough to add the hypothesis $L.R \geq -1$ for every irreducible rational curve R. However this is not true. Take, for example, a nef divisor B with $B^2 \geq 4$ and two irreducible rational curves R_1, R_2 such that $B.R_i = 0, R_1.R_2 = 1$. Then $L := B + R_1 + R_2$ satisfies the above requirements, but $L.(R_1 + R_2) = -2$, whence $H^1(L) \neq 0$ by the theorem.

Remark 2.3. It would be of interest to know if, in the statement of the theorem, it is possible to replace divisors $\Delta > 0$ such that $\Delta^2 = -2$ with chains of irreducible rational curves.

Definition 2.4. An effective line bundle L on a K3 or Enriques surface is said to be *quasi-nef* if $L^2 \geq 0$ and $L.\Delta \geq -1$ for every Δ such that $\Delta > 0$ and $\Delta^2 = -2$.

An immediate consequence of the theorem is

Corollary 2.5. *An effective line bundle L on a K3 or Enriques surface is quasi-nef if and only if $L^2 \geq 0$ and either $h^1(L) = 0$ or $L \equiv nE$ for some $n \geq 2$ and some primitive and nef divisor $E > 0$ with $E^2 = 0$.*

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