ON POSITIVE UNIPOTENT OPERATORS ON BANACH LATTICES

ROMAN DRNOVŠEK

(Communicated by Joseph A. Ball)

ABSTRACT. Let T be a positive operator on a complex Banach lattice. We prove that T is greater than or equal to the identity operator I if

$$\lim_{n \rightarrow \infty} n \|(T - I)^n\|^{1/n} = 0.$$

1. INTRODUCTION

Throughout the paper, E denotes a complex Dedekind complete Banach lattice, i.e., $E = \operatorname{Re} E \oplus i \operatorname{Re} E$, where $\operatorname{Re} E$ is a real Dedekind complete Banach lattice. Denote by $\mathcal{L}_r(E)$ the space of all regular operators on E , i.e., those operators that can be expressed as linear combinations of positive operators. This space is a subspace of the space $\mathcal{L}(E)$ of all bounded linear operators on E , and it is also a complex Banach lattice algebra with respect to the regular norm that is defined by $\|T\|_r := \|(T - I)^n\|^{1/n}$. The center $\mathcal{Z}(E)$ defined by $\mathcal{Z}(E) := \{T \in \mathcal{L}_r(E) : |T| \leq \lambda I \text{ for some } \lambda \geq 0\}$ is a commutative full subalgebra of the space $\mathcal{L}(E)$. If $T \in \mathcal{Z}(E)$, then $\|T\| = \|T\|_r = \min\{\lambda \geq 0 : |T| \leq \lambda I\}$. Since $\mathcal{Z}(E)$ is also a band in $\mathcal{L}_r(E)$, we have the band decomposition $\mathcal{L}_r(E) = \mathcal{Z}(E) \oplus \mathcal{Z}(E)^d$. The associated band projection in $\mathcal{L}_r(E)$ onto $\mathcal{Z}(E)$ is denoted by $\mathcal{D}$, and it is called the diagonal map. By a result of Voigt [8], $\mathcal{D}$ is a contraction with respect to the operator norm, i.e., $\|\mathcal{D}(T)\| \leq \|T\|$ for all $T \in \mathcal{L}_r(E)$. It is not difficult to verify that $0 \leq \mathcal{D}(T) \leq r(T)I$ for any positive operator T on E , where $r(T)$ denotes the spectral radius of T . Other interesting properties of the diagonal map are explored in the paper [3]. For the terminology not explained in the text we refer e.g. to the books [9], [1], and [4].

An operator is said to be unipotent whenever its spectrum contains only the number 1. In this note we give a partial answer to the question posed by Huijsmans and de Pagter: Is it true that $T \geq I$ for every unipotent positive operator T on E ? This question was studied in the papers [11], [10] and [5]. In particular, Zhang [10] showed that the answer is affirmative under the additional hypothesis that there exist $\alpha \in (0, \frac{1}{2})$ and a constant $c \geq 0$ such that $\|T^{-n}\| = O(\exp(cn^\alpha))$ as $n \rightarrow \infty$. In this paper we use the same method to prove a similar result replacing

Received by the editors December 1, 2005 and, in revised form, August 23, 2006.

2000 *Mathematics Subject Classification*. Primary 47B65, 47A10.

Key words and phrases. Banach lattices, positive operators, spectrum.

This work was supported in part by the Ministry of Higher Education, Science and Technology of Slovenia.

©2007 American Mathematical Society
 Reverts to public domain 28 years from publication

the assumption on the growth of the negative powers of T by a nice assumption on the growth of the powers of the quasinilpotent operator $A = T - I$.

2. RESULTS

The main result of the paper is the following.

Theorem 1. *Let T be a positive operator on E , and let $A = T - I$. If*

$$\lim_{n \rightarrow \infty} n \|\mathcal{D}(A^n)\|^{1/n} = 0$$

and if $\mathcal{D}(T^n) \leq I$ for all $n \in \mathbb{N}$, then the operator A is positive.

Proof. Let us consider the operator valued function f defined by

$$f(z) = \mathcal{D}(e^{zA}).$$

Since the diagonal map $\mathcal{D}$ is continuous with respect to the operator norm, f is an entire function, and we have

$$f(z) = \mathcal{D} \left(\sum_{n=0}^{\infty} \frac{z^n A^n}{n!} \right) = \sum_{n=0}^{\infty} c_n z^n,$$

where

$$c_n = \frac{\mathcal{D}(A^n)}{n!} \quad (n = 0, 1, 2, 3, \dots).$$

We claim that f is an entire function of minimal type with respect to the order $1/2$. Choose $\varepsilon > 0$. Then $\|\mathcal{D}(A^n)\| \leq \left(\frac{\varepsilon}{n}\right)^n$ for large $n \in \mathbb{N}$. Since $n! \geq \left(\frac{n}{e}\right)^n$ for all $n \in \mathbb{N}$, we have

$$\|c_n\| = \frac{\|\mathcal{D}(A^n)\|}{n!} \leq \frac{\varepsilon^n e^n}{n^{2n}}$$

for large $n \in \mathbb{N}$. Since the order ρ of f is determined by the formula (see [2])

$$\rho = \limsup_{n \rightarrow \infty} \frac{n \log n}{\log(1/\|c_n\|)},$$

we have

$$\rho \leq \limsup_{n \rightarrow \infty} \frac{n \log n}{2n \log n - n \log \varepsilon - n} = \frac{1}{2}.$$

If ρ equals $1/2$, then the following estimate for the type σ of f holds:

$$\sigma = \frac{1}{\rho e} \limsup_{n \rightarrow \infty} (n \|c_n\|^{\rho/n}) \leq \frac{2\sqrt{\varepsilon}}{\sqrt{e}},$$

which implies that $\sigma = 0$. This proves our claim.

Now, for any nonnegative number x we have

$$0 \leq f(x) = e^{-x} \mathcal{D}(e^{xT}) = e^{-x} \sum_{n=0}^{\infty} \mathcal{D}(T^n) \frac{x^n}{n!} \leq e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} I = I.$$

It follows that the operator valued entire function g defined by $g(z) = f(z^2)$ is of minimal type with respect to the order 1 and is bounded on the real axis. By the Phragmén-Lindelöf theorem (see [2, remark 3 on p. 39]), we conclude that g and f are constant functions. Since $f(0) = I$, we obtain that $f(x) = I$ for any nonnegative number x . This implies that $\mathcal{D}(T^n) = I$ for each n , so that $\mathcal{D}(A) = 0$, completing the proof of the theorem. $\square$

We now state and prove three corollaries. The first of them provides a partial answer to the question mentioned above.

Corollary 1. *Let T be a positive operator on E , and let $A = T - I$. If*

$$\lim_{n \rightarrow \infty} n \|A^n\|^{1/n} = 0,$$

then A is positive.

Proof. Since A is quasinilpotent, T and its powers are unipotent operators. It follows that $\mathcal{D}(T^n) \leq r(T^n)I = I$ for all $n \in \mathbb{N}$. Since the assumption implies that $\lim_{n \rightarrow \infty} n \|\mathcal{D}(A^n)\|^{1/n} = 0$, we can apply Theorem 1 to complete the proof. $\square$

Corollary 2. *Let T be a positive operator on E that is power bounded, i.e., $\sup\{\|T^n\| : n \in \mathbb{N}\} < \infty$. If the operator $A = T - I$ satisfies the condition $\lim_{n \rightarrow \infty} n \|\mathcal{D}(A^n)\|^{1/n} = 0$ (in particular, $\lim_{n \rightarrow \infty} n \|A^n\|^{1/n} = 0$), then $T = I$.*

Proof. If we denote $c = \sup\{\|T^n\| : n \in \mathbb{N}\}$, then $\|\mathcal{D}(T^n)\| \leq \|T^n\| \leq c$ for all $n \in \mathbb{N}$, so that $\mathcal{D}(T^n) \leq cI$ for all $n \in \mathbb{N}$. Since $\mathcal{D}(T^n)^m \leq \mathcal{D}(T^{nm}) \leq cI$ for all $m, n \in \mathbb{N}$, we conclude that $\mathcal{D}(T^n) \leq I$ for all $n \in \mathbb{N}$. Now the operator A is positive by Theorem 1. Since $T^n = (I + A)^n \geq nA$ for all $n \in \mathbb{N}$, we have $n\|A\| \leq \|T^n\|$ for all $n \in \mathbb{N}$, so that $A = 0$. $\square$

Corollary 3. *Let T be a positive operator on a Banach lattice E that is of the form $\lambda I + K$, where $\lambda \geq 0$ and K is a compact operator. If $\lim_{n \rightarrow \infty} n \|K^n\|^{1/n} = 0$, then T has a non-trivial invariant closed ideal.*

Proof. An application of Theorem 1 gives that the operator K is positive. Since K is a quasinilpotent operator, K (and T as well) has a non-trivial invariant closed ideal by the well-known result of de Pagter (see [6] or [4, theorem 4.2.2]). $\square$

We remark that the compactness assumption in the last result cannot be omitted. Indeed, Schaefer (see [7] or [4, section 4.2]) constructed examples of quasinilpotent positive operators with no non-trivial invariant closed ideals. It is easy to see that among them there is an operator T satisfying the condition $\lim_{n \rightarrow \infty} n \|T^n\|^{1/n} = 0$.

REFERENCES

- [1] C.D. Aliprantis, O. Burkinshaw, *Positive operators*, Academic Press, Orlando 1985. MR809372 (87h:47086)
- [2] B. Ya. Levin, *Lectures on entire functions*, Translations of Mathematical Monographs 150, American Mathematical Society, Providence, 1996. MR1400006 (97j:30001)
- [3] W. A. J. Luxemburg, B. de Pagter, A. R. Schep, *Diagonals of the powers of an operator on a Banach lattice*, in: Operator theory in function spaces and Banach lattices, 223–273, Oper. Theory Adv. Appl. 75, Birkhäuser, Basel, 1995. MR1322506 (97i:47076)
- [4] P. Meyer-Nieberg, *Banach lattices*, Springer-Verlag, Berlin, 1991. MR1128093 (93f:46025)
- [5] S. Mouton, *A spectral problem in ordered Banach algebras*, Bull. Austral. Math. Soc. **67** (2003), no. 1, 131–144. MR1962967 (2004d:47075)
- [6] B. de Pagter, *Irreducible compact operators*, Math. Z. **192** (1986), 149–153. MR835399 (87d:47052)
- [7] H.H. Schaefer, *Topologische Nilpotenz irreduzibler Operatoren*, Math. Z. **117** (1970), 135–140. MR0276802 (43:2542)
- [8] J. Voigt, *The projection onto the center of operators in a Banach lattice*, Math. Z. **199** (1988), no. 1, 115–117. MR954756 (89f:47058)
- [9] A.C. Zaanen, *Riesz spaces II*, North Holland, Amsterdam, 1983. MR704021 (86b:46001)
- [10] X. D. Zhang, *On spectral properties of positive operators*, Indag. Math. (N.S.) **4** (1993), no. 1, 111–127. MR1213328 (94b:47047)

- [11] X. D. Zhang, *Some aspects of the spectral theory of positive operators*, Positive operators and semigroups on Banach lattices (Curaçao, 1990). Acta Appl. Math. **27** (1992), no. 1-2, 135–142. MR1184885 (93j:47056)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LJUBLJANA, JADRANSKA 19. SI-1000 LJUBLJANA, SLOVENIA

E-mail address: Roman.Drnovsek@fmf.uni-lj.si