EXAMPLES OF FANO VARIETIES OF INDEX ONE THAT ARE NOT BIRATIONALLY RIGID

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ABSTRACT. A conjecture of Pukhlikov states that a smooth Fano variety of dimension at least 4 and index one is birationally rigid. We show that a general member of the linear system given by the ample generator of the Picard group of the moduli space of stable, rank 2 bundles with fixed determinant of odd degree on a curve of genus at least 3 is not birationally rigid.

1. PUKHLIKOV'S CONJECTURE

The following is a conjecture of Pukhlikov:

Conjecture 1.1 ([P4, Conjecture 5.1]). *Let V be a smooth Fano variety of dimension $\dim V \geq 4$ with Picard group $\text{Pic}(V)$ generated by the canonical class K_V . Then V is birationally rigid. If $\dim V \geq 5$, then V is superrigid.*

Conjecture 1.1 was proved for a large class of Fano complete intersections [P1], [P3], [P4], [dF], [dFEM]. More examples are given by complete intersections in weighted projective spaces [P2]. Conjecture 1.1 is a generalization of the famous theorem of Iskovskikh and Manin [IM] that states that a smooth quartic threefold is birationally rigid, and therefore not rational.

In this note we give counterexamples (in arbitrarily large dimension) to Conjecture 1.1 using moduli spaces of bundles on curves. I am grateful to Professor János Kollár for telling me about this question and suggesting that I look at sections of the theta divisor on moduli spaces of bundles on curves. I thank Jenia Tevelev and Sean Keel for reading this note and for helpful suggestions.

We recall the basic definitions about birational rigidity from [P4]. Let V be a uniruled $\mathbb{Q}$ -Gorenstein variety with terminal singularities. For an effective divisor $D \neq 0$ on V , one defines the *canonical threshold of canonical adjunction* $c(D)$ of the divisor D as follows:

$$c(D) = \sup \{ b/a \mid b, a \in \mathbb{Z}_+ \setminus \{0\}, |aD + bK_X| \neq \emptyset \}.$$

(If there are no $a, b \in \mathbb{Z}_+ \setminus \{0\}$ such that $|aD + bK_X| \neq \emptyset$, then we set $c(D) = 0$.)

Note, for an effective divisor $D \neq 0$ on a Fano variety V with $\text{Pic}(V) = \mathbb{Z}K_V$, the canonical threshold $c(D)$ is the number $-m$, if $D = mK_V$ in $\text{Pic}(V)$. Clearly, in this case $c(D) > 0$.

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In what follows all varieties are assumed to be $\mathbb{Q}$ -factorial with terminal singularities. For simplicity, we work over an algebraically closed field of characteristic zero.

Definition 1.2 ([P4, Def. 5.1]). A variety V is called *birationally rigid*, if for any V' , any birational map $\phi : V \dashrightarrow V'$ and any moving linear system Σ' on V' , there exists a birational self-map $\alpha : V \dashrightarrow V$ such that if Σ is the birational transform of Σ' via the composition $\phi \circ \alpha$ (i.e., the linear system induced on V when composing with $\phi \circ \alpha$), one has $c(\Sigma) \leq c(\Sigma')$. The variety V is called *birationally superrigid*, if one may always take $\alpha = \text{id}$.

The following proposition is an immediate consequence of the above definitions:

Proposition 1.3 ([P4, Prop. 5.1]). *Let V be a smooth Fano variety with $\text{Pic}(V) = \mathbb{Z}K_V$. If V is birationally rigid, then it is impossible to have a birational map $V \dashrightarrow V'$, with $V' \rightarrow S'$ a morphism with uniruled general fiber and S' a projective variety of dimension $\dim S' \geq 1$.*

Proof. Assume there is a birational map $V \dashrightarrow V'$, with $\pi : V' \rightarrow S'$ a morphism with uniruled general fiber and $\dim S' \geq 1$. Let D' be an effective Cartier divisor on S' and let $\Sigma' = \pi^*D'$. By Definition 1.2, there is a birational map $\alpha : V \dashrightarrow V$ such that if Σ is the birational transform of Σ' via the composition $\phi \circ \alpha$, then $c(\Sigma) \leq c(\Sigma')$. Note that since V is a Fano variety with $\text{Pic}(V) = \mathbb{Z}K_V$, one has $c(\Sigma) > 0$.

We claim that $c(\Sigma') = 0$: if there are $a, b > 0$ such that $a\pi^*D' + bK_{V'}$ is effective on V' , then for a general fiber F of π (choose F uniruled, in the smooth locus of π and such that it is not contained in the divisor $a\pi^*D' + bK_{V'}$), one has that the divisor

$$(a\pi^*D' + bK_{V'})|_F = (bK_{V'})|_F = bK_F$$

is effective on F . This is a contradiction, since F uniruled implies that $H^0(bK_F) = 0$ for all $b > 0$ [K]. $\square$

Note that since the condition of being uniruled is a closed condition [K], one may drop the word “general” from the statement of Proposition 1.3. Moreover, since the condition of being uniruled is a birational property, one may as well replace the condition of having a morphism $V' \rightarrow S'$ with having a birational map $V' \dashrightarrow S'$ with the same properties.

To geometrically motivate Definition 1.2, we recall the other definition of birational rigidity from [Co]. Recall that a morphism $f : V \rightarrow S$ is called a *Mori fiber space* if it has relative Picard number 1, $-K_V$ is relatively ample for f and $\dim S < \dim V$. Note, Fano varieties with Picard number 1 are trivially Mori fiber spaces. A birational map $\phi : V \dashrightarrow V'$ to another Mori fiber space $f' : V' \rightarrow S'$ is *square* if there is a birational map $h : S \dashrightarrow S'$ such that $h \circ f = f' \circ \phi$ and the map ϕ induces an isomorphism on the general fibers.

Definition 1.4 ([Co, Def. 1.3]). A Mori fiber space $f : V \rightarrow S$ is called *birationally rigid* if for any birational map $\phi : V \dashrightarrow V'$ to another Mori fiber space $f' : V' \rightarrow S'$, there is a birational self-map $\alpha : V \dashrightarrow V$ such that $\phi \circ \alpha$ is square. If for any ϕ as above it follows that ϕ is square, then V is called *birationally superrigid*.

A Mori fiber space that satisfies Definition 1.2 also satisfies Definition 1.4 (by the Sarkisov program, based on the Noether-Fano-Iskovskikh inequalities; this is known

in dimension 3 and conjectured in higher dimensions). By the Mori program, any uniruled variety is birational to a Mori fiber space; hence, if in addition one assumes the Mori program, the two definitions are equivalent.

It follows from Definition 1.4 that if V is a smooth Fano variety with Picard number 1 and birationally rigid, then there is no birational map $\phi : V \dashrightarrow V'$, with $f' : V' \rightarrow S'$ another Mori fiber space with $\dim S' > 0$. From the Mori program for the relative case, one deduces Proposition 1.3. In particular, note that if V is a smooth Fano variety with Picard number 1 and birationally rigid, then for any V' another Fano variety of Picard number 1, if V' is birational to V , then $V \cong V'$.

2. COUNTEREXAMPLES USING MODULI SPACES OF BUNDLES ON CURVES

Let C be a smooth projective curve over $\mathbb{C}$ of genus $g \geq 3$. Fix ξ to be a degree 1 line bundle on C . Let M be the moduli space of stable, rank 2 vector bundles on C with determinant ξ . The moduli space M is a smooth, projective, variety of dimension $3g - 3$. The Picard group of M is $\mathbb{Z}[\text{DN}]$. Let Θ be the ample generator. In fact, Θ is very ample [BV]. Then $K_M \cong -2\Theta$ [R].

Let N be a non-singular element of the linear system $|\Theta|$. Let $\Theta' \in \text{Pic}(N)$ be the restriction of Θ to N . The canonical bundle of N is $-\Theta'$. Since $g \geq 3$, by Lefschetz's theorem, Θ' generates $\text{Pic}(N)$. Therefore, the variety N satisfies the conditions in Conjecture 1.1. We prove the following:

Proposition 2.1. *If N is a general element of the linear system $|\Theta|$, then N is a smooth Fano variety with $\text{Pic}(N) \cong \mathbb{Z}K_N$ that is not birationally rigid.*

Proof. We start with a general construction. Let $e \geq 0$ and let $\mathcal{L}$ be a line bundle of degree $-e$ on C . Denote by $V_{\mathcal{L}}$ the space of extensions $\text{Ext}^1(\mathcal{L}^{-1} \otimes \xi, \mathcal{L}) \cong H^1(C, \mathcal{L}^2 \otimes \xi^{-1})$. Then $V_{\mathcal{L}}$ parametrizes extensions of the form

$$(*) \quad 0 \rightarrow \mathcal{L} \rightarrow \mathcal{E} \rightarrow \mathcal{L}^{-1} \otimes \xi \rightarrow 0.$$

The dimension of $V_{\mathcal{L}}$ is $2e + g$. Clearly, any two non-zero elements in $V_{\mathcal{L}}$ which differ by a scalar define isomorphic vector bundles $\mathcal{E}$. Therefore, the isomorphism classes of non-trivial extensions as above are parametrized by the projective space $\mathbb{P}(V_{\mathcal{L}})$. The locus $Z_{\mathcal{L}} \subset \mathbb{P}(V_{\mathcal{L}})$ of extensions $(*)$ with $\mathcal{E}$ unstable is an irreducible subvariety of codimension at least g [Ca, Lemma 2.1]. There is a well-defined morphism

$$(2.1) \quad \kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \setminus Z_{\mathcal{L}} \rightarrow M$$

that associates to an extension $(*)$ the isomorphism class of $\mathcal{E}$. By [Ca, 2.3(1), or Lemma A.1]

$$\kappa_{\mathcal{L}}^* \Theta \cong \mathcal{O}(2e + 1).$$

Consider the case when $e = g - 1$. Then $\mathbb{P}(V_{\mathcal{L}}) \cong \mathbb{P}^{3g-3}$ and $\kappa_{\mathcal{L}}^* \Theta \cong \mathcal{O}(2g - 1)$. The following claim is a standard fact. For convenience, we include a short proof.

Claim 2.2. If $\mathcal{L}$ is a line bundle of degree $1 - g$, the morphism $\kappa_{\mathcal{L}}$ in (2.1) is birational.

Proof of Claim 2.2. Note that for any $\mathcal{E} \in M$, by Riemann-Roch one has

$$\chi(\mathcal{E}^* \otimes \mathcal{L}^{-1} \otimes \xi) = 1.$$

Hence, for any $\mathcal{E} \in M$ one has

$$\text{Hom}(\mathcal{E}, \mathcal{L}^{-1} \otimes \xi) \cong H^0(\mathcal{E}^* \otimes \mathcal{L}^{-1} \otimes \xi) \neq 0.$$

If for general $\mathcal{E} \in M$, any non-zero morphism $\phi : \mathcal{E} \rightarrow \mathcal{L}^{-1} \otimes \xi$ is surjective, then we are done, as ϕ determines uniquely (up to scaling) the extension $(*)$, and by dimension considerations, one must have that

$$h^0(\mathcal{E}^* \otimes \mathcal{L}^{-1} \otimes \xi) = 1$$

(i.e., the fiber of κ at a general point contains a unique closed point).

We prove that for general $\mathcal{E}$, a non-zero morphism $\phi : \mathcal{E} \rightarrow \mathcal{L}^{-1} \otimes \xi$ must be surjective. Note that if ϕ is not surjective, then its image is $\mathcal{L}^{-1} \otimes \xi(-D)$, for some effective divisor D of degree $d > 0$, and there is an exact sequence:

$$(2.2) \quad 0 \rightarrow \mathcal{L}(D) \rightarrow \mathcal{E} \rightarrow \mathcal{L}^{-1} \otimes \xi(-D) \rightarrow 0.$$

It follows from the stability of $\mathcal{E}$ that $d < g$. For each $0 < d < g$ construct the total space $\mathbb{P}_d$ of extensions (2.2) by letting D vary in $\text{Sym}^d(C)$: the space $\mathbb{P}_d$ is a projective bundle over $\text{Sym}^d(C)$ with fiber at D isomorphic to $\mathbb{P}(V_{\mathcal{L}(D)})$. The dimension is:

$$\dim \mathbb{P}_d = d + \dim V_{\mathcal{L}(D)} - 1 = d + 2(g - 1 - d) + g - 1 = 3g - 3 - d.$$

The vector bundle which is the middle term of the universal extension over $\mathbb{P}_d$ induces a rational map $\rho : \mathbb{P}_d \dashrightarrow M$. Since $d > 0$, the map ρ is not dominant. Therefore, a general $\mathcal{E} \in M$ will not sit in an exact sequence (2.2).

From the previous discussion and since Θ is very ample, one has the following:

Claim 2.3. If N is a general element of the linear system $|\Theta|$, then N is birational to an irreducible hypersurface X_{2g-1} in $\mathbb{P}^{3g-3}$ of degree $2g - 1$.

Proposition 2.1 follows now from Claim 2.3 and Lemma 2.4. $\square$

Lemma 2.4. For any irreducible hypersurface $X_d \subset \mathbb{P}^n$ of degree $d < n$ (possibly singular) there is a rational map $\rho : X \dashrightarrow S$, $\dim S > 0$, with uniruled fibers.

Proof. Let S be a general pencil of hyperplanes $\{H_s\}_{s \in S}$ in $\mathbb{P}^n$. Then $X_s = X \cap H_s$ are hypersurfaces of degree d in $H_s \cong \mathbb{P}^{n-1}$. A smooth hypersurface of degree $d < n$ in $\mathbb{P}^{n-1}$ is Fano; hence, it is rationally connected [KMM]. It follows by deformation theory that any irreducible hypersurface of degree $d < n$ in $\mathbb{P}^{n-1}$ is uniruled. Hence, for all $s \in S$ such that X_s is irreducible, X_s is uniruled. Therefore, the induced rational map $\rho : X \dashrightarrow S$ has uniruled fibers. $\square$

Remark 2.5. The cohomology group $H^4(M; \mathbb{Q})$ has two independent generators [N]. Hence, by Lefschetz's theorem, if $g \geq 4$, the rank of $H^4(N; \mathbb{Q})$ is also 2. Since again by Lefschetz's theorem, the cohomology group H^4 of any smooth complete intersection of dimension ≥ 5 is of rank 1, it follows that N is not a complete intersection.

3. DESCRIPTION OF THE HYPERSURFACE $X_{2g-1} \subset \mathbb{P}^{3g-3}$

Our construction of the map $\kappa_{\mathcal{L}}$ is a variant of the construction of Bertram [B]. The above results about the map $\kappa_{\mathcal{L}}$ ($\deg \mathcal{L} = 1 - g$) also follow from [B]. We chose to include the above considerations (which are enough for the purpose of this note) because of their simplicity and to avoid referring to the technical results in [B]. However, Bertram's powerful construction gives a precise description of the hypersurface $X_{2g-1} \subset \mathbb{P}^{3g-3}$. We describe this below.

Let $\mathcal{L}$ be a line bundle of degree $1 - g$ on C . Then C has a natural embedding $C \subset \mathbb{P}(V_{\mathcal{L}}) \cong \mathbb{P}^{3g-3}$ given by $\mathcal{L}^{-2} \otimes \xi \otimes K_C$, since by Serre duality one has:

$$V_{\mathcal{L}} \cong H^1(C, \mathcal{L}^2 \otimes \xi^{-1}) \cong H^0(C, \mathcal{L}^{-2} \otimes \xi \otimes K_C)^*.$$

Let $\text{Sec}^k(C)$ be the $(k+1)$ -secant variety of C (i.e., the closure in $\mathbb{P}^{3g-3}$ of the union of all the k -planes spanned by $k+1$ distinct points on C).

Theorem 3.1 ([B, Thm. 1]). *There is a sequence of blow-ups $\pi : \tilde{\mathbb{P}}_{\mathcal{L}} \rightarrow \mathbb{P}(V_{\mathcal{L}})$ with smooth centers (starting with the blow-up of $\mathbb{P}(V_{\mathcal{L}})$ along C) that resolves the rational map $\kappa_{\mathcal{L}} : \mathbb{P}(V_{\mathcal{L}}) \dashrightarrow M$ into a morphism $\tilde{\mathbb{P}}_{\mathcal{L}} \rightarrow M$. There are g exceptional divisors $E_0, E_1, \dots, E_{g-1}$, and E_k dominates the secant variety $\text{Sec}^k(C)$ for every k .*

Theorem 3.2 ([B, Thm. 2, Prop. 4.7]). *There is a natural identification*

$$(3.1) \quad H^0(M, \Theta) \cong H^0(\tilde{\mathbb{P}}_{\mathcal{L}}, (2g-1)H - (2g-3)E_0 - (2g-5)E_1 - \dots - E_{g-2}).$$

It follows from Theorem 3.2 that the proper transform $\tilde{X}$ in $\tilde{\mathbb{P}}_{\mathcal{L}}$ of the hypersurface $X_{2g-1} \subset \mathbb{P}^{3g-3}$ of Claim 2.3 is a general member of the linear system in (3.1). Hence, by Bertini, $\tilde{X}$ is smooth.

Proposition 3.3. *The singular locus of the hypersurface $X_{2g-1} \subset \mathbb{P}^{3g-3}$ has codimension $\geq g-2$. Hence, if $g \geq 4$, then X is normal and the canonical bundle K_X is Cartier. Moreover, in this case X has terminal singularities.*

Proof. Since the proper transform $\tilde{X}$ of X is smooth, it follows that X is smooth outside $\text{Sec}^{g-1}(C) \cap X$. Since a general X does not contain $\text{Sec}^{g-1}(C)$, it follows that the singular locus of X has dimension at most $\dim \text{Sec}^{g-1}(C) - 1 = 2g - 2$. It is well-known that hypersurfaces X whose singular locus has codimension at least 2 are normal and the canonical class K_X is Cartier. Hence, if $g \geq 4$, then X is normal and has the canonical class:

$$K_X = \mathcal{O}_X(1 - g).$$

Consider the resolution $\pi : \tilde{X} \rightarrow X$. The canonical class of $\tilde{\mathbb{P}}_{\mathcal{L}}$ is given by:

$$K_{\tilde{\mathbb{P}}_{\mathcal{L}}} = -(3g-2)H + (3g-5)E_0 + (3g-7)E_1 + \dots + (g-1)E_{g-2}.$$

The canonical class of $\tilde{X}$ is given by:

$$K_{\tilde{X}} = (K_{\tilde{\mathbb{P}}_{\mathcal{L}}} + \tilde{X})|_{\tilde{X}} = -(g-1)H + (g-2)E_0 + (g-2)E_1 + \dots + (g-2)E_{g-2}.$$

Hence, X has terminal singularities. $\square$

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