ON THE CRITICAL POINTS OF THE E_k FUNCTIONALS IN KÄHLER GEOMETRY

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(Communicated by Richard A. Wentworth)

ABSTRACT. We prove that a Kähler metric in the anticanonical class, that is a critical point of the functional E_k and has nonnegative Ricci curvature, is necessarily Kähler-Einstein. This partially answers a question of X.X. Chen.

1. INTRODUCTION

One of the major problems in Kähler geometry is the study of extremal metrics, especially Kähler-Einstein metrics or constant scalar curvature Kähler metrics (cscK). Let (M, ω) be a compact Kähler manifold of complex dimension n , and assume the first Chern class of M to be negative or zero. Then by the celebrated work of Yau [13] (see also Aubin [1]), we know that M admits a Kähler-Einstein metric. The remaining case $c_1(M) > 0$ is more difficult and still largely open. There is a conjecture of Yau [14] that the existence of a Kähler-Einstein metric should be equivalent to the stability of the manifold in some algebro-geometric sense. The behaviour of the Mabuchi energy [6] plays a central role in this study [12]. X.X. Chen and G. Tian in [4] have introduced a family of functionals E_k ($0 \leq k \leq n$) that generalize the Mabuchi energy $\nu_\omega = E_0$, and they use E_1 in their study of the Kähler-Ricci flow. The functionals can also be defined in terms of the Deligne pairing [9], and they should possibly be related to stability. Bando and Mabuchi showed [2] that Kähler-Einstein metrics realize the minimum of the Mabuchi energy, and recently Song and Weinkove [10] showed that Kähler-Einstein metrics realize the minimum of the E_k , on the space of metrics with nonnegative Ricci curvature in the anticanonical class for $k \geq 2$, and on the whole anticanonical class for $k = 1$. Pali [8] also investigated the question of the lower boundedness of E_1 .

For the moment we drop the assumption $c_1(M) > 0$. Let $P(M, \omega)$ be the set of Kähler potentials for ω , i.e., the set of all smooth real functions ϕ such that

$$\omega_\phi := \omega + \sqrt{-1}\partial\bar{\partial}\phi$$

Received by the editors May 8, 2006 and, in revised form, October 2, 2006.

2000 *Mathematics Subject Classification*. Primary 32Q20, 58E11.

The author is supported by a Harvard Mathematics Department grant.

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is positive definite. For $\phi \in P(M, \omega)$, let ϕ_t be a smooth path in $P(M, \omega)$ with $\phi_0 = 0$, $\phi_1 = \phi$, and define

$$E_{k,\omega}(\phi) = \frac{k+1}{V} \int_0^1 \int_M (\Delta_{\phi_t} \dot{\phi}_t) \operatorname{Ric}(\omega_{\phi_t})^k \wedge \omega_{\phi_t}^{n-k} dt \\ - \frac{n-k}{V} \int_0^1 \int_M \dot{\phi}_t (\operatorname{Ric}(\omega_{\phi_t})^{k+1} - \mu_k \omega_{\phi_t}^{k+1}) \wedge \omega_{\phi_t}^{n-k-1} dt,$$

where $\dot{\phi}_t = \frac{\partial}{\partial t} \phi_t$, $V = \int_M \omega^n$ is the volume of M , and

$$\mu_k = \frac{1}{V} \int_M \operatorname{Ric}(\omega)^{k+1} \wedge \omega^{n-k-1} = (2\pi)^{k+1} \frac{[c_1(M)]^{k+1} \smile [\omega]^{n-k-1}}{[\omega]^n}$$

is a constant that depends only on the cohomology classes $[c_1(M)]$ and $[\omega]$. It can be checked that these functionals do not depend on the choice of the path. The functional $E_{0,\omega}$ is equal to the Mabuchi energy ν_ω . With a simple integration by parts we can rewrite these functionals as

$$E_{k,\omega}(\phi) \\ = \frac{1}{V} \int_0^1 \int_M \dot{\phi}_t \left[(k+1) \binom{n}{k}^{-1} (\Delta_{\phi_t}(\sigma_k(\omega_{\phi_t})) - \sigma_{k+1}(\omega_{\phi_t})) + (n-k)\mu_k \right] \omega_{\phi_t}^n dt,$$

where $\sigma_k(\omega)$ is given by

$$(\omega + t\operatorname{Ric}(\omega))^n = \left(\sum_{k=0}^n \sigma_k(\omega) t^k \right) \omega^n.$$

If at a point p we pick normal coordinates in which the Ricci tensor is diagonal with entries $\lambda_1, \dots, \lambda_n$, then $\sigma_k(\omega)$ at p is the k -th elementary symmetric function in the λ_i ,

$$\sigma_k(\omega) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_k}.$$

For example we have $\sigma_0(\omega) = 1$, $\sigma_1(\omega) = R$, where R is the scalar curvature of ω , and $\sigma_2(\omega) = \frac{1}{2}(R^2 - |\operatorname{Ric}|^2)$. From this we see that the critical points of $E_{k,\omega}$ are the functions ϕ such that ω_ϕ satisfies

$$(1) \quad \sigma_{k+1}(\omega_\phi) - \Delta_\phi \sigma_k(\omega_\phi) = \binom{n}{k+1} \mu_k.$$

For $k = 0$ we see that the critical metrics are precisely cscK metrics. The critical equation for $k = 1$ is

$$(2) \quad R^2 - |\operatorname{Ric}|^2 - 2\Delta R = n(n-1)\mu_1.$$

From now on assume that $c_1(M) > 0$ and that $[\omega] = 2\pi[c_1(M)]$ so that $\mu_k = 1$ for all k . Under these hypotheses, Chen and Tian [4] have shown that critical metrics for E_n with positive Ricci curvature must be necessarily Kähler-Einstein. In a recent paper [3], X.X. Chen asked the following:

Question (X.X. Chen). *Is a critical metric for E_1 in the anticanonical class necessarily Kähler-Einstein?*

Our result is a partial answer to this question, for all k , generalizing Chen and Tian:

Theorem 1. Fix $0 \leq k \leq n$ and assume ω is a Kähler metric in the anticanonical class that is critical for E_k and has nonnegative Ricci curvature. Then ω is Kähler-Einstein. If $k = 1$, then the same holds if ω satisfies only $R > -n$.

Remark. G. Maschler [7] has proved that a critical point of E_n in the anticanonical class is always Kähler-Einstein.

2. THE PROOF

In case $k = 0$ the critical metrics are cscK and it is well known that in the anticanonical class they are necessarily Kähler-Einstein, even if the Ricci curvature is not positive (see, for example, [11, Prop. 2.12]). Next we assume that $0 < k < n$, that $\text{Ric}(\omega) > 0$, and we use the minimum principle. Let $p \in M$ be a point where $\sigma_k(\omega)$ achieves its minimum, and pick normal coordinates around p in which the Ricci tensor is diagonal with *positive* entries $\lambda_1, \dots, \lambda_n$. Notice that all the $\sigma_k(\omega)$ are positive. Then using the critical equation (1) we have that

$$\sigma_{k+1}(\omega)(p) \geq \binom{n}{k+1}.$$

Define the normalized symmetric functions as

$$\Sigma_k(\omega) := \frac{\sigma_k(\omega)}{\binom{n}{k}},$$

so that at p we have $\Sigma_{k+1}(\omega) \geq 1$. Since the λ_i 's are positive, a classical inequality (see [5, Sect. 2.22, Thm. 51]) says that for every $0 \leq k \leq n-1$ we have

$$(3) \quad \Sigma_k^{k+1}(\omega) \geq \Sigma_{k+1}^k(\omega),$$

and this holds everywhere on M . Hence we have $\Sigma_k(\omega)(p) \geq 1$, and since p is the minimum point of $\Sigma_k(\omega)$, we have $\Sigma_k(\omega) \geq 1$ everywhere. Also using (3) repeatedly we have that

$$\Sigma_1(\omega) \geq \Sigma_2^{\frac{1}{2}}(\omega) \geq \dots \geq \Sigma_k^{\frac{1}{k}}(\omega) \geq 1$$

everywhere on M . Now by hypothesis $\omega \in 2\pi c_1(M)$. Therefore, there exists a smooth real function f such that $\text{Ric}(\omega) = \omega + \sqrt{-1}\partial\bar{\partial}f$, and taking the trace of this with respect to ω we get $R = n + \Delta f$, which can be rewritten as $\Sigma_1(\omega) = 1 + \frac{\Delta f}{n}$. Hence $\Delta f \geq 0$, from which it follows easily that f is constant, and so $\text{Ric}(\omega) = \omega$ and ω is Kähler-Einstein.

If now we only assume that $\text{Ric}(\omega) \geq 0$, we have that $\lambda_i \geq 0$. Using the minimum principle as before, we see that $\Sigma_{k+1}(\omega)(p) \geq 1$, so it is not zero. From this it follows that $\Sigma_j(\omega)(p) > 0$ for $0 \leq j \leq k$, and it is very easy to check that in this case the inequality (3) still holds and the reasoning proceeds as before.

If $k = n$, then the critical equation becomes $\Delta\sigma_n(\omega) = 0$. Hence $\sigma_n(\omega) = 1 = \Sigma_n(\omega)$, and, as before, $\Sigma_1(\omega) \geq \Sigma_n^{\frac{1}{n}}(\omega) = 1$, and so ω is Kähler-Einstein. G. Maschler called such metrics *central Kähler metrics*, and he proved [7] that the hypothesis $\text{Ric} \geq 0$ can be removed, using Demailly's holomorphic Morse inequalities.

If $k = 1$, then the critical equation is (2) with $\mu_1 = 1$. In this case the inequality (3) holds for arbitrary λ_i , and takes the form $|\text{Ric}|^2 \geq \frac{R^2}{n}$ (which follows from $|\text{Ric} - \frac{R}{n}g|^2 \geq 0$). With the same argument as above we get at the minimum p of R , $R^2(p) \geq n^2$. Since we are assuming $R > -n$, we must have $R(p) \geq n$, so $R \geq n$ everywhere and ω is Kähler-Einstein.

3. FURTHER REMARKS

Question. *Is Theorem 1 still true if we do not assume the Ricci curvature to be nonnegative (at least for $k = 1$)?*

Notice that this holds for $k = 0, n$.

Remark. Further evidence for the validity of Chen's question is provided by this recent result of Song and Weinkove [10]: assuming M has no nonzero holomorphic vector fields, the functional E_1 is proper in the sense of Tian [12] if and only if there exists a Kähler-Einstein metric. This leads one to expect that Kähler-Einstein metrics and critical points of E_1 coincide.

ACKNOWLEDGMENTS

The author thanks Jian Song and Ben Weinkove for many useful discussions and his advisor Professor S.-T. Yau for his constant support.

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