SEPARATING CLASSES OF COMPOSITION OPERATORS VIA SUBNORMAL CONDITION

IL BONG JUNG, MI RYEONG LEE, AND SANG SOO PARK

(Communicated by Joseph A. Ball)

ABSTRACT. Several classes have been considered to study the weak subnormalities of Hilbert space operators. One of them is n -hyponormality, which comes from the Bram-Halmos criterion for subnormal operators. In this note we consider $E(n)$ -hyponormality, which is the parallel version corresponding to the Embry characterization for subnormal operators. We characterize $E(n)$ -hyponormality of composition operators via k -th Radon-Nikodym derivatives and present some examples to distinguish the classes.

1. INTRODUCTION

Let $\mathcal{H}$ be an infinite-dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. Several bridges of operators have been developed to detect how far from being subnormal a given hyponormal operator is. In [4], R. Curto defined strongly n -hyponormal operators by using the Bram-Halmos criterion, which states that an operator $T \in \mathcal{L}(\mathcal{H})$ is subnormal if and only if $\sum_{i,j} \langle T^i f_j, T^j f_i \rangle \geq 0$ for all finite collections $\{f_i\}$ in $\mathcal{H}$. Recall that T is *strongly n -hyponormal* if $\sum_{i,j=0}^n \langle T^i f_j, T^j f_i \rangle \geq 0$ for any elements $f_0, f_1, \dots, f_n$ in $\mathcal{H}$ ([4]). This notion has been studied to find gaps between subnormality and hyponormality of operators when their main models are weighted shifts (cf. [5], [6], [7], [19], [21], [22], [24]). Also those techniques in weighted shifts were useful to study k -contractive and k -hypercontractive operators (cf. [1], [13], [14]). In [10], they proved that every 2-hyponormal trigonometric Toeplitz operator is necessarily subnormal. An abstract solution to the lifting problem for commuting subnormal operators is obtained in [9] via k -hyponormality of multivariable weighted shifts. In [17] and [18] they show that strong k -hyponormality of a cyclic subnormal operator can be applied to a truncated complex moment problem by studying the flatness of moment matrices that was introduced in [8].

An alternative characterization of subnormality is due to Embry ([11]), which is that an operator T is subnormal if and only if $\sum_{i,j} \langle T^{i+j} f_i, T^{i+j} f_j \rangle \geq 0$ for all finite collections $\{f_i\}$ in $\mathcal{H}$. The following is defined as parallel with strong n -hyponormality.

Definition 1.1. For a positive integer n , an operator $T \in \mathcal{L}(\mathcal{H})$ is $E(n)$ -hyponormal if $\sum_{i,j=0}^n \langle T^{i+j} f_i, T^{i+j} f_j \rangle \geq 0$ for any $f_0, f_1, \dots, f_n$ in $\mathcal{H}$.

Received by the editors June 14, 2006 and, in revised form, November 7, 2006.

2000 *Mathematics Subject Classification.* Primary 47B20, 47B33; Secondary 47A63.

Key words and phrases. Composition operator, subnormal operator.

It follows from [22] that for the case of weighted shifts, n -hyponormality and $E(n)$ -hyponormality are equivalent, but the two notions are different in general. However, some computations show that if T has dense range, then T is $E(n)$ -hyponormal if and only if T is strongly n -hyponormal. Since a (2×2) -operator matrix $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ (with A invertible) is positive if and only if $A \geq 0$, $C \geq 0$, and $B^*A^{-1}B \leq C$, it is easy to see that an operator $T \in \mathcal{L}(\mathcal{H})$ is $E(1)$ -hyponormal if and only if $|T^2|^2 \geq |T|^4$. By the Löwner-Heinz inequality, every $E(1)$ -hyponormal operator is always in the class A (i.e., $|T^2| \geq |T|^2$) (cf. [15]). Note that every operator in the class A is paranormal (cf. [16]). So $E(n)$ -hyponormality provides a bridge between subnormal and paranormal operators in $\mathcal{L}(\mathcal{H})$.

Let $X := (X, \mathcal{F}, \mu)$ be a σ -finite measure space and T be a measurable transformation from X into X , i.e., $T^{-1}\mathcal{F} \subset \mathcal{F}$. Let C_T be a composition operator on $L^2 := L^2(X)$ defined by $C_T f = f \circ T$. In [3], they studied gaps between quasinormality and p -hyponormality via composition operators on L^2 for $p > 0$. (Recall that T is p -hyponormal if $(T^*T)^p \geq (TT^*)^p$ ($p > 0$) and see the literature in [15] for further study). However, the information about the bridge of $E(n)$ -hyponormal operators is not so well known, and so it is worthwhile to study $E(n)$ -hyponormal composition operators on L^2 . We will see that those two notions are different in Section 4.2.

This note consists of the following. In Section 2 we characterize the $E(n)$ -hyponormal composition operators on L^2 as a positive matrix consisting of the k -th Radon-Nikodym derivatives generated by the absolutely continuous property $\mu \circ T^{-k} \ll \mu$ ($k \leq 2n$). In Section 3 we obtain a formula separating the classes of $E(n)$ -hyponormal composition operators on L^2 . In Section 4 we provide a useful related example.

2. $E(n)$ -HYPONORMAL COMPOSITION OPERATORS

Let $X := (X, \mathcal{F}, \mu)$ be a σ -finite measure space and let T be a non-singular transformation on X (i.e., $\mu \circ T^{-1} \ll \mu$) throughout this paper. Since μ is σ -finite, there exists a measurable non-negative extended real-valued function (Radon-Nikodym derivative) h_1 such that $\mu \circ T^{-1}(A) = \int_A h_1 d\mu$, for all A in $\mathcal{F}$. More generally, since $\mu \circ T^{-k} \ll \mu$ ($k \geq 2$), we have the k -th Radon-Nikodym derivative h_k defined by $\mu \circ T^{-k}(A) = \int_A h_k d\mu$, for all A in $\mathcal{F}$, i.e., $h_k := \frac{d\mu \circ T^{-k}}{d\mu}$ (cf. [3]). For a convention we write $h_0(x) = 1$. We now begin our work with the following lemma.

Lemma 2.1. *Let $\{\phi_{ij}\}_{i,j=0}^n$ be a subset of functions in L^∞ such that $\phi_{ij}(x) = \overline{\phi_{ji}(x)}$. Assume that, for any $f_i \in L^2$, $0 \leq i \leq n$,*

$$\sum_{i,j=0}^n \int_X \phi_{ji}(x) \cdot f_i(x) \cdot \overline{f_j(x)} d\mu \geq 0.$$

Then, for any $f_i \in L^2$, $1 \leq i \leq n$,

$$\sum_{i,j=1}^n \int_X (\phi_{00}(x)\phi_{ji}(x) - \phi_{i0}(x)\phi_{0j}(x)) \cdot f_i(x) \cdot \overline{f_j(x)} d\mu \geq 0.$$

Proof. Assuming the contrary, we suppose there exist $f_j \in L^2$, $1 \leq j \leq n$, such that

$$\sum_{i,j=1}^n \int_X (\phi_{00}(x)\phi_{ji}(x) - \phi_{i0}(x)\phi_{0j}(x)) \cdot f_i(x) \cdot \overline{f_j(x)} d\mu < 0.$$

Let $A_n := \{x \in X | \phi_{00}(x) \geq 1/n\}$ and let χ_{A_n} be the characteristic function on A_n . Since $\phi_{00}(x) \geq 0$ a.e. x in X , by the Lebesgue dominated convergence theorem, there exists a number $n_0 \in \mathbb{N}$ satisfying

$$\sum_{i,j=0}^n \int_X (\phi_{00}(x)\phi_{ji}(x) - \phi_{i0}(x)\phi_{0j}(x)) \cdot f_i(x) \cdot \overline{f_j(x)} \chi_{A_{n_0}} d\mu < 0.$$

Now let $g_j = \chi_{A_{n_0}} f_j$, $1 \leq j \leq n$, and define g_0 by

$$g_0(x) = \begin{cases} -\frac{1}{\phi_{00}(x)} (\phi_{01}(x)g_1(x) + \cdots + \phi_{0n}(x)g_n(x)), & \text{if } x \in A_{n_0}, \\ 0, & \text{if } x \notin A_{n_0}. \end{cases}$$

Then, $g_0 \in L^2$ and $\phi_{00}(x)g_0(x) + \phi_{01}(x)g_1(x) + \cdots + \phi_{0n}(x)g_n(x) = 0$, for all $x \in X$. Hence we have $\sqrt{\phi_{00}}g_j \in L^2$, $0 \leq j \leq n$, but

$$\begin{aligned} & \sum_{i,j=0}^n \int_X \phi_{ji}(x) \left(\sqrt{\phi_{00}}g_i \right)(x) \cdot \left(\sqrt{\phi_{00}}g_j \right)(x) d\mu \\ &= \int_X |\phi_{00}(x)g_0(x) + \phi_{01}(x)g_1(x) + \cdots + \phi_{0n}(x)g_n(x)|^2 d\mu \\ &+ \sum_{i,j=1}^n \int_X (\phi_{00}(x)\phi_{ji}(x) - \phi_{i0}(x)\phi_{0j}(x)) \cdot g_i(x) \cdot \overline{g_j(x)} d\mu \\ &= \sum_{i,j=1}^n \int_X (\phi_{00}(x)\phi_{ji}(x) - \phi_{i0}(x)\phi_{0j}(x)) \cdot f_i(x) \overline{f_j(x)} \chi_{A_{n_0}}(x) d\mu < 0, \end{aligned}$$

which leads to a contradiction. $\square$

Lemma 2.2. Let $\phi_{ij} \in L^\infty$ such that $\phi_{ij}(x) = \overline{\phi_{ji}(x)}$ for $0 \leq i, j \leq n$. Then the following assertions are equivalent:

(i) for any $f_i \in L^2$, $0 \leq i \leq n$,

$$(2.1) \quad \sum_{i,j=0}^n \int_X \phi_{ji}(x) \cdot f_i(x) \cdot \overline{f_j(x)} d\mu \geq 0;$$

(ii) the $(n+1) \times (n+1)$ complex matrix $(\phi_{ij}(x))_{i,j=0}^n$ is (pointwise) positive for a.e. x in X w.r.t. μ .

Proof. Since (ii) $\Rightarrow$ (i) is obvious, it is sufficient to show that (i) $\Rightarrow$ (ii). First we claim that $\phi_{00}(x) \geq 0$ a.e. To the contrary, we suppose that there exists $\epsilon > 0$ such that $\phi_{00}(x) < -\epsilon$ on a positive measurable set E . Let $f_0(x) = \chi_E(x)$. Then we have

$$\int_X \phi_{00}(x) |f_0(x)|^2 d\mu < -\epsilon \cdot \mu(E) < 0,$$

which is a contradiction of (2.1). Note that this is the case $n = 0$ in the mathematical induction. We assume that the implication holds for the case $n = k - 1$. If (i)

holds in the case of $n = k$, i.e.,

$$(2.2) \quad \sum_{i,j=0}^k \int_X \phi_{ji}(x) \cdot f_i(x) \cdot \overline{f_j(x)} \, d\mu \geq 0,$$

then it follows from Lemma 2.1 that for any $f_i \in L^2$, $1 \leq i \leq k$,

$$(2.3) \quad 0 \leq \sum_{i,j=1}^k \int_X (\phi_{00}(x)\phi_{ji}(x) - \phi_{i0}(x)\phi_{0j}(x)) \cdot f_i(x) \cdot \overline{f_j(x)} \, d\mu.$$

Then the hypothesis of mathematical induction shows that

$$(2.4) \quad (\phi_{00}(x)\phi_{ji}(x) - \phi_{j0}(x)\phi_{0i}(x))_{i,j=1}^k \geq 0.$$

If x satisfies $\phi_{00}(x) \neq 0$, then obviously we have

$$\left(\phi_{ji}(x) - \frac{\phi_{j0}(x)\phi_{0i}(x)}{\phi_{00}(x)} \right)_{i,j=1}^k \geq 0,$$

which is equivalent to $(\phi_{ji}(x))_{i,j=0}^k \geq 0$.

In the case that $N := \{x : \phi_{00}(x) = 0\}$ is a non-zero measurable set, it follows immediately from (2.3) that $\phi_{j0}(x) = 0$ for $x \in N$ a.e., $1 \leq j \leq k$. Moreover, applying $f_0(x) \equiv 0$ to (2.2), we have that for any $f_j \in L^2$, $1 \leq j \leq k$,

$$\sum_{i,j=1}^k \int_X \phi_{ji}(x) f_i(x) \overline{f_j(x)} \, d\mu \geq 0.$$

By the hypothesis of mathematical induction, it follows that $(\phi_{ij}(x))_{i,j=1}^k \geq 0$ a.e. Hence the condition $\phi_{i0}(x) = 0$ ($0 \leq i \leq k$) yields that $(\phi_{ij}(x))_{i,j=0}^k \geq 0$, which completes the proof. $\square$

Theorem 2.3. Suppose $n \in \mathbb{N}$. Let C_T be a composition operator on L^2 . Then C_T is $E(n)$ -hyponormal if and only if the $(n+1) \times (n+1)$ complex matrix $\Delta_n(x) \geq 0$ for a.e. x w.r.t. μ , where

$$(2.5) \quad \Delta_n(x) := \begin{pmatrix} h_0(x) & h_1(x) & \cdots & h_n(x) \\ h_1(x) & h_2(x) & \cdots & h_{n+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ h_n(x) & h_{n+1}(x) & \cdots & h_{2n}(x) \end{pmatrix}.$$

Proof. Since $(C_T^{m*} C_T^m) f = h_n f$ for all $n \in \mathbb{N}$ and any f in L^2 , we have that

$$\langle C_T^{*i+j} C_T^{i+j} f_i, f_j \rangle = \int_X h_{i+j}(x) f_i(x) \overline{f_j(x)} \, d\mu \quad (f_i \in L^2, 0 \leq i \leq n).$$

Hence by Lemma 2.2, we have this theorem. $\square$

According to Theorem 2.3, we have that C_T is subnormal if and only if $\Delta_n(x) \geq 0$ a.e. x and for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ if and only if $\{h_n(x)\}_{n=0}^\infty$ is a moment sequence a.e. x . The following is an immediate corollary of Theorem 2.3.

Corollary 2.4. Let C_T be a composition operator on L^2 . Then C_T is $E(1)$ -hyponormal if and only if $h_2(x) \geq h_1(x)^2$ for a.e. x in X if and only if C_T is paranormal.

Proof. Let $E(f) := E(f|T^{-1}\mathcal{F})$ be the conditional expectation of f in L^2 with respect to $T^{-1}\mathcal{F}$ (cf. [3], [23]). Recall that $E(h_1) \geq h_1 \circ T$ if and only if C_T is paranormal ([2]). Since $h_2 = h_1 E(h_1) \circ T^{-1}$ (cf. [20]), a simple computation proves that if C_T is paranormal, then $h_2(x) \geq h_1(x)^2$. $\square$

The following is a generalization of recursively generated weighted shifts (whose definition will appear in the next section) corresponding to composition operators.

Corollary 2.5. *Let C_T be a composition operator on L^2 and let $k \in \mathbb{N}$. Assume that there exist $\psi_j \in L^\infty$ for $j = 0, \dots, k-1$, such that*

$$(2.6k) \quad h_{k+i}(x) = \psi_{k-1}(x)h_{k-1+i}(x) + \psi_{k-2+i}(x)h_{k-2+i}(x) + \dots + \psi_0(x)h_i(x)$$

for a.e. x in X and $i = 0, 1, 2, \dots$. Let r be the smallest positive integer satisfying (2.6r). Then C_T is $E(r)$ -hyponormal if and only if C_T is subnormal.

Proof. Suppose C_T is $E(r)$ -hyponormal. Then, by the recursive relation (2.6k), $\text{rank } \Delta_m(x) = \text{rank } \Delta_{m+1}(x)$ a.e. x in X , for all $m \geq r$, and hence $\Delta_m(x) \geq 0$ a.e. x in X for all $m \geq r$ (cf. [6, Prop. 2.3]). So this corollary follows from Theorem 2.3. $\square$

The operator appearing in Corollary 2.5 is referred to as “a recursively generated composition operator with rank r ” (cf. [6]). An example satisfying the hypothesis of Corollary 2.5 will be introduced in Section 4.

Proposition 2.6. *Suppose that C_T has dense range and $h_1 > 0$. Let $n \in \mathbb{N}$. Then C_T^* is $E(n)$ -hyponormal if and only if the $(n+1) \times (n+1)$ complex matrix*

$$\Gamma_n(x) := \begin{pmatrix} h_0(x) & h_1 \circ T(x) & \dots & h_n \circ T^n(x) \\ h_1 \circ T(x) & h_2 \circ T^2(x) & \dots & h_{n+1} \circ T^{n+1}(x) \\ \vdots & \vdots & \ddots & \vdots \\ h_n \circ T^n(x) & h_{n+1} \circ T^{n+1}(x) & \dots & h_{2n} \circ T^{2n}(x) \end{pmatrix}$$

is positive a.e. x in X w.r.t. μ .

Proof. Let $E_n f = E(f|T^{-n}\mathcal{F})$ be the conditional expectation of f in L^2 with respect to $T^{-n}\mathcal{F}$ for $n \geq 1$. Then it follows from [12] that $C^n C^{*n} f = h_n \circ T^n E_n f$ ($f \in L^2$, $n \in \mathbb{N}$). Hence, we have that

$$\sum_{i,j=0}^n \langle C^{i+j} C^{*i+j} f_i, f_j \rangle \geq 0 \iff \sum_{i,j=0}^n \langle h_{i+j} \circ T^{i+j} E_{i+j} f_i, f_j \rangle \geq 0,$$

for any $f_i \in L^2$, $0 \leq i \leq n$. Since $T^{-1}\mathcal{F} = \mathcal{F}$, each of the above inequalities is equivalent to

$$\sum_{i,j=0}^n \langle h_{i+j} \circ T^{i+j} f_i, f_j \rangle \geq 0;$$

that is,

$$(2.7) \quad \sum_{i,j=0}^n \int_X h_{i+j} \circ T^{i+j}(x) f_i(x) \cdot \overline{f_j(x)} \, d\mu \geq 0.$$

By Lemma 2.2, (2.7) is equivalent to $\Gamma_n(x) \geq 0$ a.e. x in X . $\square$

We close this section with the following remark.

Remark 2.7. It is well known that $T^{-1}\mathcal{F} = \mathcal{F}$ if and only if C_T has dense range (cf. [26]). Thus, if $T^{-1}\mathcal{F} = \mathcal{F}$, then C_T is strongly n -hyponormal if and only if C_T is $E(n)$ -hyponormal. It is worthwhile to find characterizations of strong n -hyponormality [or $E(n)$ -hyponormality] of C_T [or C_T^* , resp.] without the hypothesis in Proposition 2.6. However we do not yet obtain a useful result about them.

3. SEPARATING CLASSES OF $E(n)$ -HYPONORMAL OPERATORS

Let $\gamma := \{\gamma_n\}_{n=0}^\infty$ be a set of positive real numbers. For brevity we consider $\gamma_0 = 1$ in this section. A sequence γ is said to be *recursively generated* if there exist a natural number r and a vector $\psi = (\psi_0, \dots, \psi_{r-1}) \in \mathbb{C}^r$ such that $\gamma_n = \psi_{r-1}\gamma_{n-1} + \dots + \psi_0\gamma_{n-r}$ ($n \geq r$). A sequence γ is *non-recursively generated* if it is not recursively generated. (Note that the weighted shift W_α with weight sequence $\alpha_n := \sqrt{\gamma_{n+1}/\gamma_n}$ is called *recursively generated*. The numbers $\gamma_0 := 1$ and $\gamma_n := \alpha_0^2 \cdots \alpha_{n-1}^2$ ($n \geq 1$) are called *moments* of W_α in [6, Prop. 2.6]).

Suppose $\{\gamma_k\}_{k=0}^\infty$ is a non-recursively generated sequence. Let

$$A(i, j) = \begin{pmatrix} \gamma_i & \gamma_{i+1} & \gamma_{i+2} & \cdots & \gamma_{i+j} \\ \gamma_{i+1} & \gamma_{i+2} & \gamma_{i+3} & \cdots & \gamma_{i+j+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{i+j} & \gamma_{i+j+1} & \gamma_{i+j+2} & \cdots & \gamma_{i+2j} \end{pmatrix}$$

for $i, j \in \mathbb{N}_0$. If $A(i, j) \geq 0$, the non-recursivity of $\{\gamma_k\}_{k=0}^\infty$ implies that $\det A(i, j) > 0$ ($i, j \in \mathbb{N}_0$) (cf. [6, Prop. 2.6], [19]). Let

$$(3.1) \quad \Psi_k(t) = \begin{pmatrix} t & \gamma_0 & \gamma_1 & \cdots & \gamma_{k-1} \\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma_{k-1} & \gamma_k & \gamma_{k+1} & \cdots & \gamma_{2k-1} \end{pmatrix},$$

where t is a positive real variable. Then $\det \Psi_k(t)$ is a polynomial in t of degree 1 and has the unique zero $\xi_k > 0$.

Lemma 3.1 ([19, Th. 2.1]). *Under the same notation and hypothesis as above, it follows that*

$$(3.2) \quad \xi_{k+1} - \xi_k = \frac{(\det A(0, k))^2}{\det A(1, k-1) \cdot \det A(1, k)} > 0 \quad (k \geq 1).$$

Let $X := \mathbb{N}_0$ and let $\mathcal{F} := \mathcal{P}(X)$ be the power set of X . We define a measurable transformation T on X by $T(k) = k - 1$ ($k \geq 1$) and $T(0) = 0$. Let $m(k) = m_k \in \mathbb{R}^+$ be an arbitrary point mass measure on X with $m_0 = t$ as a positive real variable. Then the σ -algebra $T^{-1}\mathcal{F}$ is generated by atoms $\{0, 1\}$, $\{2\}$, $\{3\}$, $\dots$. By a computation, we obtain the Radon-Nikodym derivative $h_1 = \frac{d\mu \circ T^{-1}}{d\mu}$:

$$h_1(0) = \frac{\mu \circ T^{-1}(0)}{\mu(0)} = \frac{t + m_1}{t} \quad \text{and} \quad h_1(k) = \frac{m_{k+1}}{m_k} \quad (k \geq 1),$$

i.e., h_1 is a sequence of the form: $\frac{t+m_1}{t}, \frac{m_2}{m_1}, \frac{m_3}{m_2}, \frac{m_4}{m_3}, \dots$. By simple computations, we also have that for each $n \geq 1$,

$$(3.3) \quad h_n(0) = \frac{t + m_1 + \cdots + m_n}{t} \quad \text{and} \quad h_n(k) = \frac{\mu \circ T^{-n}(\{k\})}{\mu(\{k\})} = \frac{m_{n+k}}{m_k} \quad (k \geq 1).$$

Then we have the following lemma.

Lemma 3.2. Let $\{m_k\}_{k=0}^\infty$ be the sequence as above. Then we have $\Delta_n(0) = (h_{i+j}(0))_{i,j=0}^n = \frac{1}{t}(d_{ij})_{i,j=0}^n$, where $d_{ij} = \sum_{p=0}^{i+j} m_p$.

Lemma 3.3. Let $\{m_k\}_{k=0}^\infty$ be the sequence as above. For $n \in \mathbb{N}$, we let

$$\Theta_n(t) = \begin{pmatrix} t & \beta_0 & \beta_1 & \cdots & \beta_{n-1} \\ \beta_0 & \beta_1 & \beta_2 & \cdots & \beta_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n-1} & \beta_n & \beta_{n+1} & \cdots & \beta_{2n-1} \end{pmatrix}$$

with

$$(3.4) \quad \beta_k = \sum_{i=0}^k (-1)^i \binom{k}{i} m_{k+1-i}.$$

Then $\det(d_{ij})_{i,j=0}^n = \det \Theta_n(t)$.

Proof. According to the elementary row operations and column operations, without difficulty we may obtain the required equality. $\square$

With the same notation as in (2.5) and (3.3), we denote

$$RE(n, k) = \{t > 0 : \Delta_n(k, t) \geq 0\},$$

where $\Delta_n(k, t) \equiv \Delta_n(k)$ is as in (2.5), and

$$RD(n, k) = \{t > 0 : \det \Delta_i(k, t) > 0, 1 \leq i \leq n-1 \text{ and } \det \Delta_n(k, t) \geq 0\}$$

for $n = 1, 2, \dots$. Then we have the following lemma.

Lemma 3.4. Let $\{m_k\}_{k=0}^\infty$ be a sequence of positive real numbers with $m_0 = t$ as a positive real variable such that $\{\frac{m_{n+k}}{m_k}\}_{n=0}^\infty$ is a non-recursively generated sequence for any $k \in \mathbb{N}$. Let β_k be as in (3.4) and let ξ_n be the zero of $\det \Theta_n(t)$. Then we have the following assertions:

- (i) $RE(n, k) = RD(n, k)$ for all $n = 1, 2, \dots$ and $k \in \mathbb{N}_0$;
- (ii) $RE(n, k) \setminus RE(n+1, k) = [\xi_n, \xi_{n+1})$ ($n \geq 1, k \in \mathbb{N}_0$);
- (iii) $\bigcap_{n=1}^\infty RE(n, k) = [\xi_\infty, \infty)$ for any $k \in \mathbb{N}_0$, where $\xi_\infty = \lim \xi_n$.

Proof. By the Nested Determinants Test ([6]), obviously we have that $RE(n, k) \supset RD(n, k)$. To prove the reverse containment, let $t \in RE(n, k)$, i.e., $\Delta_n(k, t) \geq 0$. Since $\{\frac{m_{n+k}}{m_k}\}_{n=0}^\infty$ is non-recursively generated, $\det \left(\frac{m_{i+j+k}}{m_k} \right)_{i,j=0}^n > 0$ for any $k \in \mathbb{N}$. Hence $RE(n, k) \subset RD(n, k) = (0, \infty)$, $k \in \mathbb{N}$. So we only consider $k = 0$. For a contradiction, we suppose that there exists $n_0 \in \mathbb{N}$ with $1 \leq n_0 \leq n-1$ such that $\Delta_{n_0}(0, t) \geq 0$ and $\det \Psi_{n_0}(t) = 0$ for all $t > 0$. (Note that $\det \Theta_{n_0}(t) = \det \Delta_{n_0}(0, t)$ by Lemma 3.3.) Then it follows from [6] that

$$\det \Theta_{n_0}(t) = 0 = \det \Theta_{n_0+1}(t) = \cdots = \det \Theta_n(t) \quad (t > 0).$$

So we can find a root ξ_{n_0} satisfying $\det \Theta_{n_0}(\xi_{n_0}) = 0$. Since the slope of $\det \Theta_{n_0}(x)$ is positive and $\xi_{n_0} < \xi_{n_0+1}$, we have $\det \Theta_{n_0+1}(\xi_{n_0}) < 0$, which contradicts $\Delta_n(0, k) \geq 0$. Hence $RE(n, 0) \subset RD(n, 0)$. The remaining parts are routine. $\square$

By the above lemmas and Theorem 2.3, we have the following theorem.

Theorem 3.5. Consider a measure space $X := \mathbb{N}_0$ and the power set of $\mathbb{N}_0$ as a σ -algebra $\mathcal{F} := \mathcal{P}(X)$. Let $m(k) = m_k$ be an arbitrary point mass measure on $(X, \mathcal{F})$ with positive real variable $m_0 = t$ and let T be a measurable transformation on X defined by $T(k) = k - 1$ ($k \geq 1$) and $T(0) = 0$. Assume that $\{\frac{m_{n+k}}{m_k}\}_{n=0}^\infty$ is a non-recursive sequence for any $k \in \mathbb{N}$. Let β_k be as in (3.4) and let ξ_k be the zero of $\det \Theta_k(t)$ with $\beta_k = \sum_{i=0}^k (-1)^i \binom{k}{i} m_{k+1-i}$. Then

- (i) $\{\xi_k\}_{k=1}^\infty$ is a strictly increasing sequence of positive real numbers;
- (ii) $RE(n, 0) = \{t > 0 : C_T \text{ is } E(n)\text{-hyponormal}\} = [\xi_n, \infty)$;
- (iii) $RE(n, 0) \setminus RE(n+1, 0) = [\xi_n, \xi_{n+1})$ ($n \geq 1$);
- (iv) $\{t > 0 : C_T \text{ is subnormal}\} = [\xi_\infty, \infty)$.

We close this section with an example for Theorem 3.5.

Example 3.6. Let $(X, \mathcal{F})$ be as in Theorem 3.5 and let a measure $m(k)$ on X be defined by

$$m(k) \equiv m_k = \int_1^2 (\lambda^k - \lambda^{k-1}) d\lambda, \quad k \geq 1.$$

Let C_T be a composition operator defined as in Theorem 3.5. Since the moment measure $d\lambda$ has infinite support, $\{\frac{m_{n+k}}{m_k}\}_{n=0}^\infty$ is non-recursive for any $k \in \mathbb{N}$ (cf. [25, p.6]). By simple computations, we obtain $\xi_n = \frac{n(n+2)}{(n+1)^2}$. Therefore

$$C_T \text{ is } E(n)\text{-hyponormal} \iff \frac{n(n+2)}{(n+1)^2} \leq t < \infty.$$

It follows from [3, Prop. 2.4] that C_T is hyponormal if and only if $h_1 > 0$ and $E(1/h_1) \leq 1/(h_1 \circ T)$, where E is the usual conditional expectation, which is equivalent to $\frac{3}{4} \leq t < \infty$. Hence C_T is hyponormal if and only if it is $E(1)$ -hyponormal. Moreover, C_T is subnormal if and only if $1 \leq t < \infty$.

4. FURTHER EXAMPLES

In this section, we will discuss two examples.

4.1. Let r be any fixed non-negative integer and let $\eta = \sum_{i=0}^r \rho_i \delta_{\{p_i\}}$ be a positive measure on $X := \mathbb{N}_0$ for some densities $\rho_i > 0$ and $p_i \in \mathbb{N}_0$. Define $m_0 = x > 0$ and $m_k = \int t^{k+1} d\eta$ ($k \in \mathbb{N}$). Let T be a measurable transformation on $(X, \mathcal{P}(X))$ defined by $T(k) = 0$ if $k = 0, 1$, and $T(k) = k - 1$ otherwise. Let q_i ($0 \leq i \leq r$) be the coefficient of t^i in the polynomial $\prod_{j=0}^r (t - p_j)$. Since

$$\sum_{j=0}^r q_j m_{j+k} = \int \sum_{j=0}^r q_j t^{j+k+1} d\eta = \int t^{k+1} \prod_{j=0}^r (t - p_j) d\eta = 0, \quad \text{for } k \geq 1,$$

we have that for $k \geq 1$,

$$(4.1) \quad h_{r+n}(k) = \frac{m_{r+n+k}}{m_k} = -\frac{1}{m_k} \sum_{j=0}^{r-1} q_j m_{j+n+k} = -\sum_{j=0}^{r-1} q_j h_{j+n}(k) \quad (n \in \mathbb{N}_0).$$

We want to show that

$$\begin{aligned} h_{r+n+1}(0) &= (1 - q_{r-1})h_{r+n}(0) + (q_{r-1} - q_{r-2})h_{r+n-2}(0) \\ &\quad + \cdots + (q_1 - q_0)h_n(0) + q_0 h_{n-1}(0). \end{aligned}$$

If we set $s_k := \sum_{j=0}^k m_j$, then we have, for $n \in \mathbb{N}_0$,

$$\begin{aligned} s_{r+n+1} &= s_{r+n} + m_{r+n+1} = s_{r+n} - \sum_{j=0}^{r-1} q_j m_{j+n+1} = s_{r+n} - \sum_{j=0}^{r-1} q_j (s_{j+n+1} - s_{j+n}) \\ &= 1 - q_{r-1} s_{r+n} + (q_{r-1} - q_{r-2}) s_{r+n-1} + \cdots + (q_1 - q_0) s_{n+1} + q_0 s_n. \end{aligned}$$

Using the fact that $h_{r+n+1}(0) = \frac{s_{r+n+1}}{m_0}$, it follows that

$$(4.2) \quad \begin{aligned} h_{r+n+1}(0) &= (1 - q_{r-1})h_{r+n}(0) + (q_{r-1} - q_{r-2})h_{r+n-1}(0) \\ &\quad + \cdots + (q_1 - q_0)h_{n+1}(0) + q_0 h_n(0). \end{aligned}$$

Thus, by (4.1) and (4.2), C_T is a recursively generated composition operator satisfying (2.6k) in Corollary 2.5 with rank less than or equal to $r+1$. Hence we obtain that C_T is $E(r+1)$ -hyponormal if and only if C_T is subnormal.

In particular, let us consider $\eta := \rho_0 \delta_{\{p_0\}} + \rho_1 \delta_{\{p_1\}}$ satisfying $\int t d\eta > \int d\eta$, $p_0 \neq 1$, and $p_1 \neq 1$. We denote $RE(n) := \{x > 0 : C_T \text{ is } E(n)\text{-hyponormal}\}$. Then,

$$RE(1) = \{x : x \geq \left(\int 1 d\eta\right)^2 \left(\int (t-1) d\eta\right)^{-1}\}.$$

It follows from the above fact that $RE(2) = RE(3) = \cdots = RE(\infty) := \bigcap_{n=0}^{\infty} RE(n)$. At first observe that

$$x + \int (1 + \cdots + t^n) d\eta = x - \int \frac{1}{t-1} d\eta + \int \frac{t^{n+1}}{t-1} d\eta.$$

Therefore we can easily verify that for any $a, b, c \in \mathbb{C}$,

$$\begin{aligned} \left\langle \Delta_2(0) \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle &= \left\langle \Omega \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle \\ &= \left(x - \int \frac{1}{t-1} d\eta \right) |a + b + c|^2 + \int \frac{|a + bt + bt^2|^2}{t-1} d\eta, \end{aligned}$$

where

$$\Omega := \begin{pmatrix} x & x + \int d\eta & x + \int (1+t) d\eta \\ x + \int d\eta & x + \int (1+t) d\eta & x + \int (1+t+t^2) d\eta \\ x + \int (1+t) d\eta & x + \int (1+t+t^2) d\eta & x + \int (1+t+t^2+t^3) d\eta \end{pmatrix}.$$

Hence C_T is $E(2)$ -hyponormal if and only if $x \geq \int \frac{1}{t-1} d\eta$, which also is equivalent to C_T being subnormal.

4.2. In this subsection we give examples showing that the notions of $E(n)$ -hyponormality and p -hyponormality are different as we promised in the introduction. Let $\alpha : 1, 1, 2, 2, \dots$ be a weight sequence and let W_α be its corresponding weighted shift. Then obviously W_α is p -hyponormal for $p > 0$. However W_α is $E(1)$ -hyponormal but not $E(k)$ -hyponormal for any $k \geq 2$. Now we consider $T = I_{\mathcal{H}} + S$, where S is the unilateral unweighted shift of multiplicity one. Then obviously T is subnormal, and thus T is $E(n)$ -hyponormal for all $n \in \mathbb{N}$. By some computations, we have that

$$|T|^{2(1+\varepsilon)} - |T^*|^{2(1+\varepsilon)} = \begin{pmatrix} t_1(\varepsilon) & t_2(\varepsilon) \\ t_2(\varepsilon) & t_3(\varepsilon) \end{pmatrix} \oplus 0_{\mathcal{H} \ominus \mathbb{C}^2} \quad (\varepsilon > 0),$$

where

$$\begin{aligned} t_1(\varepsilon) &= \frac{1+3^{1+\varepsilon}}{2} + \frac{-5+\sqrt{5}}{10} \left(\frac{3-\sqrt{5}}{2} \right)^\varepsilon - \frac{5+\sqrt{5}}{10} \left(\frac{3+\sqrt{5}}{2} \right)^\varepsilon, \\ t_2(\varepsilon) &= \frac{-1+3^{1+\varepsilon}}{2} + \frac{-5+3\sqrt{5}}{10} \left(\frac{3-\sqrt{5}}{2} \right)^\varepsilon - \frac{5+3\sqrt{5}}{10} \left(\frac{3+\sqrt{5}}{2} \right)^\varepsilon, \\ t_3(\varepsilon) &= \frac{1+3^{1+\varepsilon}}{2} - \frac{5-2\sqrt{5}}{5} \left(\frac{3-\sqrt{5}}{2} \right)^\varepsilon - \frac{5+2\sqrt{5}}{5} \left(\frac{3+\sqrt{5}}{2} \right)^\varepsilon. \end{aligned}$$

It can be easily verified that $\det \begin{pmatrix} t_1(\varepsilon) & t_2(\varepsilon) \\ t_2(\varepsilon) & t_3(\varepsilon) \end{pmatrix} < 0$ for all $\varepsilon > 0$, and hence T is not $(1+\varepsilon)$ -hyponormal for any $\varepsilon > 0$. Therefore “ $E(n)$ -hyponormality is independent from p -hyponormality for $p > 0$ ”.

ACKNOWLEDGMENTS

The first author was supported by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD) (KRF-2006-312-C00027). The third author was supported by a grant, KRF-2004-050-C00003. We express our thanks to the referee for several excellent remarks that allowed us to correct some minor errors and to improve the exposition herein.

REFERENCES

- [1] J. Agler, *Hypercontractions and subnormality*, J. Operator Theory, **13**(1985), 203-217. MR775993 (86i:47028)
- [2] C. Burnap and I. Jung, *Composition operators with weak hyponormality*, J. Math. Anal. Appl., to appear.
- [3] C. Burnap, I. Jung and A. Lambert, *Separating partial normality classes with composition operators*, J. Operator Theory, **53**(2005), 381-397. MR2153155
- [4] R. Curto, *Quadratically hyponormal weighted shifts*, Integral Equation Operator Theory **13**(1990), 49-66. MR1025673 (90k:47061)
- [5] ———, *Joint hyponormality: A bridge between hyponormality and subnormality*, Proc. Sympos. Math. **51**(1990), 69-91. MR1025673 (90k:47061)
- [6] R. Curto and L. Fialkow, *Recursively generated weighted shifts and the subnormal completion problem*, Integral Equations Operator Theory, **17**(1993), 202-246. MR1233668 (94h:47050)
- [7] ———, *Recursively generated weighted shifts and the subnormal completion problem II*, Integral Equations Operator Theory, **18**(1994), 369-426. MR1233668 (94h:47050)
- [8] ———, *Solution of the truncated complex moment problems for flat data*, Memoirs Amer. Math. Soc. **568**(1996). MR1233668 (94h:47050)
- [9] R. Curto, S. Lee and J. Yoon, *k-Hyponormality of multivariable weighted shifts*, J. Funct. Anal., **229**(2005), 462-480. MR2183156
- [10] R. Curto and W. Lee, *Joint hyponormality of Toeplitz pairs*, Memoirs of Amer. Math. Soc., Vol. 150, No. 712 (2001). MR1810770 (2002c:47042)
- [11] M. Embry, *A generalization of the Halmos-Bram condition for subnormality*, Acta. Sci. Math.(Szeged) **35** (1973), 61-64. MR0328652 (48:6994)
- [12] M. Embry and A. Lambert, *Subnormality for the adjoint of a composition operator on L^2* , J. Operator Theory, **25**(1991), 309-318. MR1203036 (94f:47028)
- [13] G. Exner, *On n -contractive and n -hypercontractive operators*, Integral Equations Operator Theory, **56** (2006), 451-468.
- [14] G. Exner, I. Jung, and S. Park, *On n -hypercontractive operators*, II, submitted.
- [15] T. Furuta, *Invitation to linear operators*, Taylor & Francis Inc., 2001. MR1978629 (2004b:47001)

- [16] M. Ito and T. Yamazaki, *Relations between two inequalities $(B^{r/2}A^pB^{r/2})^{r/(p+r)} \geq B^r$ and $A^p \geq (A^{p/2}B^rA^{p/2})^{p/(p+r)}$ and their applications*, Integral Equations Operator Theory, **44**(2002), 442-450. MR1942034 (2003h:47032)
- [17] I. Jung, E. Ko, C. Li and S. Park, *Embry truncated complex moment problem*, Linear Algebra and Appl. **375**(2003), 95-114. MR2013458 (2004i:47030)
- [18] I. Jung, C. Li and S. Park, *Complex moment matrices via Halmos-Bram and Embry conditions*, J. Korean Math. Soc., to appear.
- [19] I. Jung and C. Li, *A formula for k -hyponormality of backstep extensions of subnormal weighted shifts*, Proc. Amer. Math. Soc. **129**(2000), 2243-2351. MR1823917 (2002b:47061)
- [20] A. Lambert, *Hyponormal composition operators*, Bull. London Math. Soc. **18**(1986), 395-400. MR838810 (87h:47059)
- [21] S. McCullough and V. I. Paulsen, *A note on joint hyponormality*, Proc. Amer. Math. Soc. **107**(1989), 187-195. MR972236 (90a:47062)
- [22] ———, *k -hyponormality of weighted shifts*, Proc. Amer. Math. Soc. **116**(1992), 165-169. MR1102858 (93e:47029)
- [23] M. Rao, *Conditional measures and applications*, Marcel Dekker, New York, 1993. MR1234936 (95d:28001)
- [24] J. Park and S. Park, *On k -hyponormal weighted translation semigroups*, Bull. Kor. Math. Soc. **39**(2002), No. 4, 527-534. MR1938992 (2003h:47042)
- [25] J. Shohat and J. Tamarkin, *The problem of moments*, Math. Surveys I, Amer. Math. Soc., Providence, 1943. MR0008438 (5:5c)
- [26] R. Singh and J. Manhas, *Composition operators on function spaces*, North-Holland Math. Stud. No. 179, Amsterdam, 1993. MR1246562 (95d:47036)

DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, KYUNGPOOK NATIONAL UNIVERSITY, DAEGU 702-702 KOREA

E-mail address: `ibjung@knu.ac.kr`

DEPARTMENT OF MATHEMATICS, COLLEGE OF NATURAL SCIENCES, KYUNGPOOK NATIONAL UNIVERSITY, DAEGU 702-702 KOREA

E-mail address: `lmr67@yumail.ac.kr`

INSTITUTE OF MATHEMATICAL SCIENCE, EWHA WOMANS UNIVERSITY, SEOUL, 120-750, KOREA

E-mail address: `pss4855@ewha.ac.kr`