A TROPICAL NULLSTELLENSATZ

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ABSTRACT. We suggest a version of Nullstellensatz over the tropical semiring, the real numbers equipped with operations of maximum and summation.

1. INTRODUCTION

Tropical mathematics takes place in *the tropical semiring*, the real numbers equipped with the operations of maximum and summation, corresponding to the classical operations of addition and multiplication, respectively (see in [7, 17] the representation of the tropical semiring as a limit of the semiring of non-negative real numbers $(\mathbb{R}_+, +, \cdot)$). Sometimes the tropical semiring is extended by $-\infty$, the neutral element for the maximum operation, but we shall not need it. We write the tropical operations in quotes, i.e.,

$$“a + b” = \max\{a, b\}, \quad “ab” = a + b.$$

A rapid development of tropical mathematics over the last years, especially of tropical algebraic geometry, has led to spectacular applications in classical algebraic geometry (see, for example, [2, 3, 4, 5, 8, 9, 10, 11, 13, 14, 15, 16, 17]). Tropical objects unexpectedly reveal much similarity with classical objects, which is based on the theory of large complex limits, logarithmic limits, non-Archimedean valuations, toric geometry, etc. Here we suggest the tropical analogue of Nullstellensatz (in a bit different context this problem was stated in [1], Question 16).

Similar to the classical case, we understand Nullstellensatz as a criterion for a polynomial to belong to the radical of an ideal. Let us give the necessary definitions (cf. [3, 11, 13, 14, 15, 17]). A *tropical polynomial* in n variables is a function $F : \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$(1.1) \quad F(x) = “\sum_{\omega \in \Omega} c_{\omega} x^{\omega}” = \max_{\omega \in \Omega} (\langle x, \omega \rangle + c_{\omega}),$$

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where Ω is a non-empty finite set of points in $\mathbb{Z}^n$ with non-negative coordinates, $\langle *, * \rangle$ denotes the scalar product, and $c_\omega \in \mathbb{R}$, $\omega \in \Omega$. This is a convex piecewise linear function. With a tropical polynomial F we associate two useful geometric objects: the *Newton polytope* $\Delta(F)$, defined as the convex hull of the set Ω , and the *tropical hypersurface* $Z(F)$, defined as the corner locus of the graph of F , i.e., the set of points $x \in \mathbb{R}^n$ such that F is not differentiable at x , or, equivalently, the set of points $x \in \mathbb{R}^n$ at which the maximum in the right-hand side of (1.1) is attained at least twice. A tropical hypersurface is a finite polyhedral object in $\mathbb{R}^n$ of pure dimension $n - 1$, whose complement consists of open convex polyhedra (see [2, 3, 15]). It is the tropical analogue of the zero locus of a classical polynomial. We notice in addition that a tropical polynomial F is differentiable (in the usual sense) outside $Z(F)$, and its partial derivatives are constant on each of the components of the complement of $Z(F)$.

The space of tropical polynomials in n variables is denoted by $\mathbb{R}[x_1, \dots, x_n]$.¹ For $F_1, \dots, F_k \in \mathbb{R}[x_1, \dots, x_n]$, we define the *tropical polynomial ideal* $I(F_1, \dots, F_k)$, generated by $F_1, \dots, F_k$ in $\mathbb{R}[x_1, \dots, x_n]$, as the set of tropical polynomials $G \in \mathbb{R}[x_1, \dots, x_n]$ representable in the form

$$(1.2) \quad G(x) = \sum_{i \in J} h_i(x) F_i(x) = \max_{i \in J} (h_i(x) + F_i(x)) ,$$

where J is a finite non-empty subset of $\{1, 2, \dots, k\}$, and $h_i \in \mathbb{R}[x_1, \dots, x_n]$, $i \in J$. The radical $\sqrt{I}$ of an ideal I is the set of polynomials G such that $G^m \in I$ for some natural m . We ask the following question:

Given a tropical polynomial $F \in \mathbb{R}[x_1, \dots, x_n]$, under what conditions does F belong to $\sqrt{I(F_1, \dots, F_k)}$?

The answer, presented in the next section, is given in terms of the “zero loci” of the tropical polynomials $F, F_1, \dots, F_k$, like in the classical case.

2. A TROPICAL POLYNOMIAL NULLSTELLENSATZ

Theorem 2.1. *Let $F, F_1, \dots, F_k$, $k \geq 1$, be tropical polynomials in n variables. Then $F \in \sqrt{I(F_1, \dots, F_k)}$ if and only if, for any connected component D of $\mathbb{R}^n \setminus Z(F)$, there is $1 \leq i \leq k$ such that $D \cap Z(F_i) = \emptyset$, and, for each $j = 1, \dots, n$,*

$$(2.1) \quad \frac{\partial F}{\partial x_j} \Big|_D > 0 \quad \text{as far as} \quad \frac{\partial F_i}{\partial x_j} \Big|_D > 0 .$$

Proof. 1. *Necessity.* The function $mF|_D$ is linear. Hence it must coincide with one of the terms $(h_i + F_i)|_D$ in the expression $\max_i (h_i + F_i)|_D$, since otherwise the graph of the latter function would have a break inside D . Next, if $mF|_D = (h_i + F_i)|_D$, then both $h_i|_D$ and $F_i|_D$ must be linear in view of $Z(h_i + F_i) \supset Z(h_i) \cup Z(F_i)$. Thus, $D \cap Z(F_i) = \emptyset$. Observing that

$$m \frac{\partial F}{\partial x_j} \Big|_D = \frac{\partial h_i}{\partial x_j} \Big|_D + \frac{\partial F_i}{\partial x_j} \Big|_D \geq \frac{\partial F_i}{\partial x_j} \Big|_D ,$$

we obtain (2.1).

¹The quotation marks mean that we supply this space with the tropical operations, maximum and summation. In turn, operations without quotations (including derivatives) have the usual meaning.

2. *Sufficiency.* Distribute the connected components of $\mathbb{R}^n \setminus Z(F)$ into disjoint subsets Π_i , $i \in J$, where $J \subset \{1, \dots, k\}$, such that, for any $i \in J$ and $D \in \Pi_i$, we have $Z(F_i) \cap D = \emptyset$ and relation (2.1).

Fix some $i \in J$. Condition (2.1) yields that there is m_1 such that, for any $m \geq m_1$, one has

$$m \frac{\partial F}{\partial x_j} \Big|_D \geq \frac{\partial F_i}{\partial x_j} \Big|_D, \quad D \in \Pi_i, \quad j = 1, \dots, n,$$

and hence the gradients of the linear functions

$$(2.2) \quad L_{D,m} : \mathbb{R}^n \rightarrow \mathbb{R}, \quad L_{D,m} \Big|_D = mF \Big|_D - F_i \Big|_D, \quad D \in \Pi_i, \quad m \geq m_1,$$

have non-negative integral coordinates.

We claim that there exists m_2 such that, for any $D \in \Pi_i$, in the complement of the closure of D , we have

$$(2.3) \quad mF > L_{D,m} + F_i \quad \text{as far as} \quad m \geq m_1.$$

Indeed, write $F = L + \Phi$, $F_i = L' + \Phi'$, where L, L' are linear functions, and Φ, Φ' are convex piecewise linear functions, vanishing along D . Then $L_{D,m} = mL - L'$, and thus, $mF - L_{D,m} - F_i = m\Phi - \Phi'$. Since $\Phi > 0$ and $|\partial\Phi/\partial x_j| \geq 1$, $j = 1, \dots, n$, outside $\overline{D}$, we obtain (2.3) when m_2 exceeds all the absolute values of the partial derivatives of Φ' .

Define $h_i = \max_{D \in \Pi_i} L_{D,m}$. This is a tropical polynomial as $m \geq m_1$, and due to (2.2), (2.3) it satisfies

$$(h_i + F_i) \Big|_D = mF \Big|_D, \quad (h_i + F_i) \Big|_{\mathbb{R}^n \setminus \overline{D}} < F \Big|_{\mathbb{R}^n \setminus \overline{D}}, \quad D \in \Pi_i, \quad m \geq m_2.$$

That is, $G = "F^m" = mF$ satisfies (1.2) for all sufficiently large m . $\square$

Remark 2.2. From the above proof, one can extract an explicit upper bound to the minimal value of m such that $"F^m" \in I(F_1, \dots, F_k)$.

Example 2.3. In the case $k = 1$, the criterion of Theorem 2.1 for $F \in \sqrt{I(F_1)}$ can be written as

- $Z(F) \supset Z(F_1)$, and
- for each $j = 1, \dots, n$, $\partial F / \partial x_j(x) > 0$ as far as $\partial F_1 / \partial x_j(x) > 0$, $x \notin Z(F)$.

We shall comment on the first condition. One can assign integer positive weights to the $(n-1)$ -cells of a tropical hypersurface so that it will satisfy an equilibrium condition (see [12], section 2.1). Taking m greater than the maximal ratio of the weights in $Z(F)$ and in $Z(F_1)$, we then subtract the weight of an $(n-1)$ -cell C of $Z(F_1)$ from the multiplied by m weights of those $(n-1)$ -cells of $Z(F)$, whose interior intersects with C . The equilibrium condition persists after the subtraction due to its linearity, and we again obtain a balanced complex supported at $Z(F)$. By [12], Proposition 2.4, it defines a convex piecewise linear function h_1 , which provides the relation $mF = h_1 + F_1$.

3. MODIFICATIONS

3.1. A tropical Laurent polynomial Nullstellensatz. A tropical Laurent polynomial in n variables is a function given by (1.1), where Ω is any non-empty finite subset of $\mathbb{Z}^n$. Denote the space of tropical Laurent polynomials by $"L[x_1, \dots, x_n]"$. Correspondingly we define the tropical Laurent polynomial ideal $I^{\text{lau}}(F_1, \dots, F_k) \subset$

" $L[x_1, \dots, x_n]$ ", generated by Laurent polynomials $F_1, \dots, F_k$, as the set of tropical Laurent polynomials representable in the form (1.2) for some non-empty finite $J \subset \{1, \dots, k\}$, and $h_i \in "L[x_1, \dots, x_n]"$, $i \in J$.

Theorem 3.1. *Let $F, F_1, \dots, F_k$, $k \geq 1$, be tropical Laurent polynomials in n variables. Then $F \in \sqrt{I^{\text{la}}(F_1, \dots, F_k)}$ if and only if, for any connected component D of $\mathbb{R}^n \setminus Z(F)$, there is $1 \leq i \leq k$ such that $D \cap Z(F_i) = \emptyset$.*

Proof. The necessity is established in the same way as in the proof of Theorem 2.1. The sufficiency follows from Theorem 2.1, if one "multiplies" $F, F_1, \dots, F_k$ by suitable tropical monomials, turning $F, F_1, \dots, F_k$ into tropical polynomials, and making all the partial derivatives of F positive. $\square$

3.2. Restricted ideals and restricted Nullstellensatz. In the space " $\mathbb{R}[x_1, \dots, x_n]$ " introduce the restricted ideal, generated by the tropical polynomials $F_1, \dots, F_k$, as

$$I^r(F_1, \dots, F_k) = \{G \in " \mathbb{R}[x_1, \dots, x_n] " : \\ G = " \sum_{i=1}^k h_i F_i " = \max_{1 \leq i \leq k} (h_i + F_i), \quad h_1, \dots, h_k \in " \mathbb{R}[x_1, \dots, x_n] " \}.$$

Notice that $I^r(F_1, \dots, F_k) \subset I(F_1, \dots, F_k)$. In general these ideals may differ, for example, $x \notin I^r(x, "x+1")$.

Theorem 3.2. *Let $F, F_1, \dots, F_k$, $k \geq 1$, be tropical polynomials in n variables. Then $F \in \sqrt{I^r(F_1, \dots, F_k)}$ if and only if the following conditions hold:*

- (i) *for any connected component D of $\mathbb{R}^n \setminus Z(F)$, there is $1 \leq i \leq k$ such that $D \cap Z(F_i) = \emptyset$ and (2.1) is fulfilled;*
- (ii) *there is m_0 such that $m\Delta(F)$ contains a translate of each Newton polytope $\Delta(F_1), \dots, \Delta(F_k)$ as far as $m \geq m_0$.*

Remark 3.3. Notice that condition (ii) always holds when $\dim \Delta(F) = n$.

Proof. 1. *Auxiliary statement.* Let $G, H \in " \mathbb{R}[x_1, \dots, x_n] "$. We claim that

- if $G \geq H$, then $\Delta(G) \supset \Delta(H)$;
- if $\Delta(G) \supset \Delta(H)$, then there is $c \in \mathbb{R}$ such that $G \geq H + c$.

Represent $\Delta(G)$ as the intersection of finitely many halfspaces. Pick one of these halfspaces and apply an integral-affine automorphism Q of $\mathbb{R}^n$, taking the halfspace to $x_n \geq 0$. Since $G \circ Q^{-1} \geq H \circ Q^{-1}$, the Newton polytope $\Delta(H \circ Q^{-1}) = Q(\Delta(H))$ cannot contain points with a negative n -th coordinate. Indeed, otherwise we would have that, for $x_1, \dots, x_{n-1} = \text{const}$, $x_n \rightarrow -\infty$, the function $G(Q^{-1}(x))$ does not increase, whereas $H(Q^{-1}(x))$ tends to $+\infty$. Running over all halfspaces forming $\Delta(G)$, we conclude that $\Delta(G) \supset \Delta(H)$.

For the second statement, write

$$G(x) = \max_{\omega \in \Delta(G) \cap \mathbb{Z}^n} (\langle x, \omega \rangle + a_\omega), \quad H(x) = \max_{\omega \in \Delta(H) \cap \mathbb{Z}^n} (\langle x, \omega \rangle + b_\omega).$$

Then one can take

$$c = \min_{\omega \in \Delta(H) \cap \mathbb{Z}^n} (a_\omega - b_\omega).$$

2. *Necessity.* We have to prove only (ii). Observing that $\Delta(h_i + F_i)$ is the convex hull of few translates of $\Delta(F_i)$, we derive (ii) from the above auxiliary statement.

3. *Sufficiency.* A translate of $\Delta(F_i)$ is the Newton polytope of a tropical polynomial “ $h_i F_i$ ”, where h_i is a tropical monomial (i.e., a linear function). Then, according to the auxiliary statement, $mF_i \geq (h_i + c_i) + F_i$ for sufficiently large m and certain constant c_i . Then, using Theorem 2.1, we obtain

$$mF = \max_{i \in J} (h_i + F_i) = \max \left\{ \max_{i \in J} (h_i + F_i), \max_{i \notin J} (h_i + c_i + F_i) \right\}. \quad \square$$

3.3. Nullstellensatz for convex piecewise linear functions of finite type. A function given by (1.1), where Ω is a finite subset of $\mathbb{R}^n$, we call a convex piecewise linear function of finite type in n variables. Denote the space of such functions by “ $PL(x_1, \dots, x_n)$ ”. As in the polynomial case, we define tropical hypersurfaces $Z(F)$ in $\mathbb{R}^n$ and finitely generated ideals $I^{\text{fun}}(F_1, \dots, F_k)$ in “ $PL(x_1, \dots, x_n)$ ”, and obtain a corresponding Nullstellensatz:

Theorem 3.4. *Let $F, F_1, \dots, F_k \in \text{“}PL(x_1, \dots, x_n)\text{”}$. Then $F \in \sqrt{I^{\text{fun}}(F_1, \dots, F_k)}$ if and only if, for any connected component D of $\mathbb{R}^n \setminus Z(F)$, there is $1 \leq i \leq k$ such that $D \cap Z(F_i) = \emptyset$.*

The proof coincides with that of Theorem 3.1.

3.4. Nullstellensatz for polynomials over an extended tropical semiring.

In [6], the second author introduced an extension $\mathbb{T}$ of the tropical semiring $\mathbb{R}$. Namely, $\mathbb{T}$ is obtained by adding another copy of $\mathbb{R}$, which we denote by $\mathbb{R}^\nu$. That means $\mathbb{T}$ is the disjoint union of $\mathbb{R}$ and $\mathbb{R}^\nu$. Denoting elements of $\mathbb{R}^\nu$ by $a^\nu \in \mathbb{R}^\nu$, where $a \in \mathbb{R}$, we equip $\mathbb{T}$ with the following tropical operations:

$$\begin{aligned} “a + b” &= \begin{cases} \max\{a, b\}, & \text{if } a \neq b, \\ a^\nu, & \text{if } a = b, \end{cases} & “a^\nu + b^\nu” &= (\max\{a, b\})^\nu, \\ “a + b^\nu” &= \begin{cases} a, & \text{if } a > b, \\ b^\nu, & \text{if } a \leq b, \end{cases} \\ “ab” &= a + b, \quad “a^\nu b^\nu” &= (a + b)^\nu, \quad “ab^\nu” &= (a + b)^\nu \end{aligned}$$

for all $a, b \in \mathbb{R}$. There is a natural epimorphism of tropical semirings:

$$\pi : \mathbb{T} \rightarrow \mathbb{R}, \quad \pi(a) = a, \quad \pi(a^\nu) = a, \quad \text{for all } a \in \mathbb{R},$$

which induces epimorphisms π_* of the tropical polynomial and tropical Laurent polynomial rings. The above variants of Nullstellensatz can easily be translated to the case of base semiring $\mathbb{T}$. Here we present such a translation of Theorem 2.1.

For a polynomial $F = “\sum_{\omega \in \Omega} c_\omega x^\omega” \in “\mathbb{T}[x_1, \dots, x_n]”$, where Ω is a non-empty finite set of points with non-negative integral coordinates, we introduce a tropical hypersurface $Z(F) := Z(\pi_* F) \subset \mathbb{R}^n$. The restriction of $\pi_* F$ to a connected component D of $\mathbb{R}^n \setminus Z(F)$ is a linear function $\langle \omega, x \rangle + \pi(c_\omega)$ for some $\omega = \omega(D) \in \Omega$. So we divide the set of the connected components of $\mathbb{R}^n \setminus Z(F)$ into two disjoint subsets $\Pi(F)$ and $\Pi^\nu(F)$, letting $D \in \Pi(F)$ or $D \in \Pi^\nu(F)$ according as $c_{\omega(D)} \in \mathbb{R}$ or $c_{\omega(D)} \in \mathbb{R}^\nu$.

Theorem 3.5. *Let $F, F_1, \dots, F_k \in “\mathbb{T}[x_1, \dots, x_n]”$, $k \geq 1$. Then $F \in \sqrt{I(F_1, \dots, F_k)}$ if and only if*

- (i) *for any $D \in \Pi(F)$, there is $1 \leq i \leq k$ such that D is contained in some component $D_i \in \Pi(F_i)$, and, for each $j = 1, \dots, n$, relation (2.1) holds true;*

- (ii) for any $D \in \Pi^\nu(F)$, there is $1 \leq i \leq k$ such that $D \cap Z(F_i) = \emptyset$, and, for each $j = 1, \dots, n$, relation (2.1) holds true.

Proof. The necessity part is immediate in view of the proof of Theorem 2.1 and the fact that $\mathbb{R}\mathbb{R}^\nu = \mathbb{R}^\nu$. To prove the sufficiency, we construct coefficients h_i of the expansion (1.2) to be in $\mathbb{R}[x_1, \dots, x_n] \subset \mathbb{T}[x_1, \dots, x_n]$ by the following modification of the procedure from the proof of Theorem 2.1:

- if $D \in \Pi(F)$, and $1 \leq i \leq k$ satisfies condition (i), then we put $h_i|_D = mF|_D - F_i|_x D$;
- if $D \in \Pi^\nu(F)$, and $1 \leq i \leq k$ satisfies condition (ii), then we put $h_i|_D = m \cdot \pi_* F|_D - \pi_* F_i|_D$.

Choosing a sufficiently large m , we complete the proof as that of Theorem 2.1. $\square$

Corollary 3.6. *Let $F, F_1, \dots, F_k \in \mathbb{T}[x_1, \dots, x_n]$, $k \geq 1$, and $I(F_1, \dots, F_k) \supset \mathbb{R}^\nu[x_1, \dots, x_n]$. Then $F \in \sqrt{I(F_1, \dots, F_k)}$ if and only if condition (i) of Theorem 3.5 holds true.*

Indeed, the relation $I(F_1, \dots, F_k) \supset \mathbb{R}^\nu[x_1, \dots, x_n]$ means that the set of generators $\{F_1, \dots, F_k\}$ contains a constant polynomial $F_i = a^\nu$, $a \in \mathbb{R}$, and thus, condition (ii) always holds.

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