MEASURES ON INDEPENDENT SETS, A QUANTITATIVE VERSION OF RUDIN'S THEOREM

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ABSTRACT. We construct measures with independent support whose Fourier coefficients decrease as fast as possible.

1. INTRODUCTION

We work on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Throughout this paper *positive* will mean *strictly positive*. We need the following definition.

Definition 1.1. A closed set E is called independent if, given distinct points $x_1, x_2, \dots, x_q \in E$, the only solution to the equation

$$\sum_{j=1}^q m_j x_j = 0$$

with $m_j \in \mathbb{Z}$ is the trivial solution $m_1 = m_2 = \dots = m_q = 0$.

Lemma 1.2. Suppose that μ is a non-zero measure on $\mathbb{T}$ and q is a positive integer such that we can find an $\alpha > 1/q$ and an $A > 0$ with

$$|\hat{\mu}(r)| \leq A|r|^{-\alpha}$$

for all $r \neq 0$. Then we can find distinct points $x_1, x_2, \dots, x_q \in \text{supp } \mu$ and $m_j \in \mathbb{Z}$, not all zero, such that

$$\sum_{j=1}^q m_j x_j = 0.$$

Proof. Let $\mu_q = \mu * \mu * \dots * \mu$, the convolution of μ with itself q times. Then

$$|\hat{\mu}_q(r)| = |\hat{\mu}(r)|^q \leq A^q |r|^{-q\alpha}$$

for all $r \neq 0$. It follows that $\hat{\mu}_q \in l^1$ and so $d\mu_q(t) = f(t)dt$ for some continuous function f . Thus (since μ_q is non-zero) $\text{supp } \mu_q$ contains a non-trivial interval and so a non-zero rational number y and so we can find $y_1, y_2, \dots, y_q \in \text{supp } \mu$ such that

$$\sum_{j=1}^q y_j = y.$$

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Since we do not know that the y_j are distinct, we can only conclude that there exists a q' with $1 \leq q' \leq q$ distinct points $x_1, x_2, \dots, x_{q'} \in \text{supp } \mu$, and non-zero $n_j \in \mathbb{Z}$ such that

$$\sum_{j=1}^{q'} n_j x_j = y.$$

If we take $n_j = 0$ for $j > q'$, it now follows that there are distinct points $x_1, x_2, \dots, x_q \in \text{supp } \mu$ and $n_j \in \mathbb{Z}$, not all zero, such that

$$\sum_{j=1}^q n_j x_j = y.$$

The stated result follows if we choose a non-zero $M \in \mathbb{Z}$ such that $My = 0$ and we set $m_j = Mn_j$. $\square$

In the other direction, Rudin proved the following result (see [9]).

Theorem 1.3. *There exists a probability measure μ such that $\hat{\mu}(r) \rightarrow 0$ as $|r| \rightarrow \infty$, but $\text{supp } \mu$ is independent.*

Since Rudin's original paper was published, several different constructions have been given by Kahane [2], Kaufman [6], and others.

In this paper we prove the following version of Rudin's result.

Theorem 1.4. *Suppose that $\phi : \mathbb{N} \rightarrow \mathbb{R}$ is a sequence of positive numbers with $r^\alpha \phi(r) \rightarrow \infty$ as $r \rightarrow \infty$ whenever $\alpha > 0$. Then there exists a probability measure μ such that $\phi(|r|) \geq |\hat{\mu}(r)|$ for all $r \neq 0$, but $\text{supp } \mu$ is independent.*

In view of Lemma 1.2, this result is best possible.

Our method of proof also proves the following theorem.

Theorem 1.5. *If q is an integer with $q \geq 1$ and $\psi : \mathbb{N} \rightarrow \mathbb{R}$ is a sequence of positive numbers such that $\psi(r) \rightarrow \infty$ as $r \rightarrow \infty$, then there exists a probability measure μ such that*

$$|\hat{\mu}(r)| \leq |r|^{-1/(2q)} (\log(1 + |r|))^{1/2} \psi(|r|)$$

for all $r \neq 0$, but, given distinct points $x_1, x_2, \dots, x_q \in \text{supp } \mu$, the only solution to the equation

$$\sum_{j=1}^q m_j x_j = 0$$

with $m_j \in \mathbb{Z}$ is the trivial solution $m_1 = m_2 = \dots = m_q = 0$.

Although we believe that Theorems 1.4 and 1.5 are new (compare the results of Chapter 17 in [3]), the idea behind the proof comes from [7]. (The papers [10] and [9] use probabilistic ideas in a similar context and, to give another example, so does the proof of the *maille* condition on page 146 of [4].)

2. THE BASIC CONSTRUCTION

We shall need the following well-known theorem (see, for example Chapter 6 of [3]).

Theorem 2.1. *Suppose that $Z, Z_1, Z_2, \dots, Z_n$ are independent identically distributed complex valued random variables with $|Z| = 1$ and $\mathbb{E}Z = 0$. Then*

$$\Pr \left(\left| \sum_{u=1}^n Z_u \right| \geq \lambda \right) \leq 4 \exp(-\lambda^2/n)$$

for all $\lambda > 0$.

The next lemma (with origins in [7]) forms the heart of this paper. We write δ_t for the unit Dirac point mass at t .

Lemma 2.2. *Let q be a positive integer and let $m_1, m_2, \dots, m_q$ be non-zero integers. Then, provided only that n is large enough, we can find distinct points $x_1, x_2, \dots, x_n$ with the following two properties.*

(i) *If we write $\mu = n^{-1} \sum_{u=1}^n \delta_{x_u}$, we have*

$$|\hat{\mu}(r)| \leq 4q^{1/2}n^{-1/2}(\log n)^{1/2}$$

for all $1 \leq |r| \leq n^{4q}$.

(ii) *If $j(1), j(2), \dots, j(q)$ are distinct integers with $1 \leq j(k) \leq n$, then*

$$\left| \sum_{k=1}^q m_k x_{j(k)} \right| \geq 8^{-1}n^{-q}.$$

Proof. Consider the independent random variables Y_u where each Y_u is uniformly distributed over $\mathbb{T}$. We look at the random measure

$$\sigma = n^{-1} \sum_{u=1}^n \delta_{Y_u}.$$

We note that

$$\hat{\sigma}(r) = n^{-1} \sum_{u=1}^n \exp(2\pi i r Y_u).$$

If $r \neq 0$, we see that the $\exp(2\pi i r Y_u)$ are independent identically distributed complex valued random variables with $|\exp(2\pi i r Y_u)| = 1$ and $\mathbb{E} \exp(2\pi i r Y_u) = 0$. Thus, by Theorem 2.1, with $\lambda = 4q^{1/2}n^{1/2}(\log n)^{1/2}$,

$$\begin{aligned} \Pr \left(|\hat{\sigma}(r)| \geq 4q^{1/2}n^{-1/2}(\log n)^{1/2} \right) \\ &= \Pr \left(\left| \sum_{u=1}^n \exp(2\pi i r Y_u) \right| \geq 4q^{1/2}n^{1/2}(\log n)^{1/2} \right) \\ &\leq 4 \exp(-16q \log n) = 4n^{-16q}. \end{aligned}$$

Thus, provided only that n is large enough,

$$\begin{aligned} \Pr \left(|\hat{\sigma}(r)| \geq 4q^{1/2}n^{-1/2}(\log n)^{1/2} \text{ for some } 1 \leq |r| \leq n^{4q} \right) \\ \leq \sum_{1 \leq |r| \leq n^{4q}} \Pr \left(|\hat{\sigma}(r)| \geq 4q^{1/2}n^{-1/2}(\log n)^{1/2} \right) \leq 2n^{4q}4n^{-16q} \leq 1/4. \end{aligned}$$

Now suppose that $j(1), j(2), \dots, j(q)$ are distinct integers with $1 \leq j(k) \leq n$. By symmetry or direct calculation, the random variable

$$\sum_{k=1}^q m_k Y_{j(k)}$$

is uniformly distributed and so

$$\Pr \left(\sum_{k=1}^q m_k Y_{j(k)} \in [-8^{-1}n^{-q}, 8^{-1}n^{-q}] \right) = 4^{-1}n^{-q}.$$

There are no more than n^q different q -tuples $j(1), j(2), \dots, j(q)$ of the type discussed. So, by the same kind of argument as we used in the previous paragraph, the probability that

$$\sum_{k=1}^q m_k Y_{j(k)} \in [-8^{-1}n^{-q}, 8^{-1}n^{-q}]$$

for any such q -tuple is no more than $1/4$.

Combining the results of our last two paragraphs, we see that, provided n is large enough, the probability that $x_u = Y_u$ will fail to satisfy the conditions of our lemma is at most $1/2$. Since there must be an instance of any event with positive probability, the required result follows. $\square$

The next lemma merely serves to establish notation.

Lemma 2.3. *Let $K : \mathbb{R} \rightarrow \mathbb{R}$ be an infinitely differentiable function with the following properties.*

(i') $K(x) \geq 0$ for all $x \in \mathbb{R}$.

(ii') $\int_{\mathbb{R}} K(x) dx = 1$.

(iii') $K(x) = 0$ for $|x| \geq 1/4$.

If N is a positive integer and we define $K_N : \mathbb{T} \rightarrow \mathbb{R}$ by

$$K_N(t) = \begin{cases} NK(Nt) & \text{if } |t| \leq 1/(4N), \\ 0 & \text{otherwise,} \end{cases}$$

then K_N is an infinitely differentiable function having the following properties.

(i) $K_N(t) \geq 0$ for all $t \in \mathbb{T}$.

(ii) $\int_{\mathbb{T}} K_N(t) dt = 1$.

(iii) $K_N(t) = 0$ for $|t| \geq 1/(4N)$.

(iv) $|\hat{K}_N(r)| \leq 1$ for all r .

(v) There exists a constant A , independent of N , such that $|\hat{K}_N(r)| \leq A(N/r)^2$ for all $r \neq 0$.

Proof. This is entirely straightforward. To obtain (v) we integrate by parts to get

$$\begin{aligned} |\hat{K}_N(r)| &= \frac{1}{|r|^2} \left| \int_{\mathbb{T}} K_N^{(2)}(t) \exp(-2\pi i r t) dt \right| \\ &\leq \frac{1}{|r|^2} \int_{\mathbb{T}} |K_N^{(2)}(t)| dt = \frac{N^2}{|r|^2} \int_{\mathbb{T}} |K^{(2)}(t)| dt. \end{aligned} \quad \square$$

We now “spread out” the measure of Lemma 2.2 to obtain the measure used in our construction.

Theorem 2.4. Suppose that $\psi : \mathbb{N} \rightarrow \mathbb{R}$ is a sequence of positive numbers such that $\psi(r) \rightarrow \infty$ as $r \rightarrow \infty$. Suppose that q is a positive integer, $\epsilon, \delta > 0$ and $\mathbf{m} = (m_1, m_2, \dots, m_q) \in \mathbb{Z}^q$ with $M = \sum_{k=1}^q |m_k| \neq 0$. Then we can find an infinitely differentiable function $f : \mathbb{T} \rightarrow \mathbb{R}$ with the following properties.

(i) $f(t) \geq 0$ for all t .

(ii) $\int_{\mathbb{T}} f(t) dt = 1$.

(iii) $|\hat{f}(r)| \leq \epsilon |r|^{-1/(2q)} (\log(1 + |r|))^{1/2} \psi(|r|)$ for all $r \neq 0$.

(iv) If $t_k \in \text{supp } f$ for $1 \leq k \leq q$ and $|t_k - t_l| \geq \delta$ for $1 \leq k < l \leq q$, then

$$\sum_{k=1}^q m_k t_k \neq 0.$$

Proof. Provided only that n is large enough, we can find x_u and μ satisfying the conditions of Lemma 2.2.

Now take $N(n) = 4Mn^q$, let $K_{N(n)}$ be defined as in Lemma 2.3, and set $f = \mu * K_{N(n)}$. Conclusions (i) and (ii) are immediate.

Now suppose n sufficiently large that $n^{4q} > N(n)$ and $N(n) \geq \delta^{-1}$. If $t_k \in \text{supp } f$ for $1 \leq k \leq q$ and $|t_k - t_l| \geq \delta$ for $1 \leq k < l \leq q$, then, since

$$\text{supp } f \subseteq \bigcup_{u=1}^n [x_u - N(n)^{-1}/4, x_u + N(n)^{-1}/4],$$

it follows that we can find distinct integers $j(1), j(2), \dots, j(q)$ with $1 \leq j(k) \leq n$ such that

$$|x_{j(k)} - t_k| \leq N(n)^{-1}/4$$

for all $1 \leq k \leq q$. Condition (ii) of Lemma 2.2 now tells us that

$$\begin{aligned} \left| \sum_{k=1}^q m_k t_k \right| &\geq \left| \sum_{k=1}^q m_k x_{j(k)} \right| - \sum_{k=1}^q |m_k| |x_{j(k)} - t_k| \\ &\geq 8^{-1} n^{-q} - 4^{-1} M N(n)^{-1} = 16^{-1} n^{-q} > 0, \end{aligned}$$

and condition (iv) follows.

We bound $|\hat{f}(r)|$ using condition (i) of Lemma 2.2, Lemma 2.3, and the trivial bounds $|\hat{f}(r)|, |\hat{\mu}(r)| \leq 1$. If $1 \leq |r| \leq N(n)$, then

$$|\hat{f}(r)| \leq |\hat{\mu}(r)| \leq 4q^{1/2} n^{-1/2} (\log n)^{1/2} \leq C_1 N(n)^{-1/(2q)} (\log N(n))^{1/2}$$

for some constant C_1 independent of n . If $N(n) \leq |r| \leq n^{4q}$, then

$$\begin{aligned} |\hat{f}(r)| &\leq |\hat{\mu}(r)| |\hat{K}_{N(n)}(r)| \leq (4q^{1/2} n^{-1/2} (\log n)^{1/2}) (A(N(n)/r)^2) \\ &\leq C_2 N(n)^{-1/(2q)} (\log N(n))^{1/2} (N(n)/r)^2 \\ &= C_2 (N(n)/r)^{2-1/(2q)} r^{-(1/2q)} (\log N(n))^{1/2} \\ &\leq C_3 |r|^{-1/(2q)} (\log |r|)^{1/2} \end{aligned}$$

for some constants C_2 and C_3 independent of n . If $|r| \geq n^{4q}$, then

$$|\hat{f}(r)| \leq |\hat{K}_{N(n)}(r)| \leq A(N(n)/r)^2 = A|r|^{-1} (N(n)^2/|r|) \leq C_4 |r|^{-1/(2q)} (\log |r|)^{1/2}$$

for some constant C_4 independent of n .

Since $\psi(r) \rightarrow \infty$ as $r \rightarrow \infty$, it follows that, provided only that n is large enough,

$$|\hat{f}(r)| \leq \epsilon |r|^{-1/(2q)} (\log(1 + |r|))^{1/2} \psi(|r|)$$

for all $r \neq 0$ and condition (iv) holds. $\square$

The method of proof used in the next section requires us to make a further simple observation.

Lemma 2.5. *Given $\epsilon > 0$, we can find an $\eta > 0$ such that, if μ is a probability measure with $|\hat{\mu}(r)| \leq \eta$ for $r \neq 0$, we know that $\text{supp } \mu$ intersects every interval of length ϵ .*

Proof. By translation, it suffices to show that $\text{supp } \mu$ intersects $(-\epsilon/2, \epsilon/2)$. Choose an integer N with $N \geq \epsilon^{-1}$. If $\text{supp } \mu$ does not intersect $(-\epsilon/2, \epsilon/2)$, then

$$\begin{aligned} 0 &= \left| \int_{\mathbb{T}} K_N(t) d\mu(t) \right| = \left| \sum_{r=-\infty}^{\infty} \hat{K}_N(-r) \hat{\mu}(r) \right| \\ &\geq |\hat{K}_N(0)| |\hat{\mu}(0)| - \sum_{r \neq 0} |\hat{K}_N(-r) \hat{\mu}(r)| \geq 1 - 2\eta A_2 N^2 \sum_{r=1}^{\infty} r^{-2}, \end{aligned}$$

which is impossible if η is sufficiently small. $\square$

Theorem 2.6. *Theorem 2.4 holds with the additional condition:*

(v) *If I is an interval of length ϵ , then $\text{supp } f \cap I \neq \emptyset$.*

3. PROOF OF THE MAIN THEOREM

We prove Theorem 1.4 by using a Baire category argument. (The reader will not need to be reminded that the introduction of Baire category methods to harmonic analysis is also due to Kaufman [5].) Like most Baire category constructions this can be replaced by a direct construction using careful bookkeeping. However, the author discovered Theorem 1.4 in the course of seeking a Baire category proof of Rudin's original result, and it seems reasonable to present it in this way.

Any Baire category argument requires an appropriate complete metric space.

Lemma 3.1. (i) *Consider the space $\mathcal{F}$ of non-empty closed subsets of $\mathbb{T}$. If we set*

$$d_{\mathcal{F}}(E, F) = \sup_{e \in E} \inf_{f \in F} |e - f| + \sup_{f \in F} \inf_{e \in E} |e - f|,$$

then $(\mathcal{F}, d_{\mathcal{F}})$ is a complete metric space.

(ii) *Suppose $\phi : \mathbb{N} \rightarrow \mathbb{R}$ is a positive sequence. Then the space of sequences $\mathbf{a} : \mathbb{Z} \rightarrow \mathbb{C}$ with $\sup_{r \in \mathbb{Z}} \phi(|r|)^{-1} |a_r|$ finite is a complete normed space under the norm*

$$\|\mathbf{a}\|_{\phi} = \sup_{r \in \mathbb{Z}} \phi(|r|)^{-1} |a_r|.$$

(iii) *Consider the space $\mathcal{P}_{\phi}$ consisting of ordered pairs (E, μ) where E is a closed subset of $\mathbb{T}$ and μ is a probability measure with $\text{supp } \mu \subseteq E$ and $\sup_{r \in \mathbb{Z}} \phi(|r|)^{-1} |\hat{\mu}(r)|$ finite. Then*

$$d_{\phi}((E, \mu), (F, \sigma)) = d(E, F) + \|\hat{\mu} - \hat{\sigma}\|_{\phi}$$

is a complete metric on $\mathcal{P}_{\phi}$.

(iv) *If*

$$\mathcal{G}_{\phi} = \{(E, \mu) \in \mathcal{P}_{\phi} : \phi(|r|)^{-1} |\hat{\mu}(r)| \rightarrow 0 \text{ as } |r| \rightarrow \infty\},$$

then $\mathcal{G}_{\phi}$ is a non-empty closed subset of $\mathcal{P}_{\phi}$.

Proof. (i) The metric d is called the Hausdorff metric and is introduced in [1]. Its properties are derived (usually as an exercise) in many textbooks.

(ii) Standard.

(iii) Use properties of weak convergence of measures.

(iv) Standard. Observe that, if m is a Lebesgue measure, $(\mathbb{T}, m) \in \mathcal{G}_\phi$. $\square$

We use the fact $(\mathcal{G}_\phi, d_\phi)$ is a non-empty complete metric space to obtain a Baire category version of Theorem 1.4.

Theorem 3.2. *Suppose that $\phi : \mathbb{N} \rightarrow \mathbb{R}$ is such that $\phi(0) = 1$, $\phi(r) > 0$ for all r , and $r^\alpha \phi(r) \rightarrow \infty$ as $r \rightarrow \infty$ whenever $\alpha > 0$. Then the set of $(E, \mu) \in \mathcal{G}_\phi$ such that E has a complement of first category in $(\mathcal{G}_\phi, d_\phi)$.*

We obtain Theorem 3.2 by studying the set $\mathcal{H}(q, p, \mathbf{m})$ defined as follows.

Definition 3.3. Suppose that ϕ is as in Theorem 3.2, q and p are positive integers, and $\mathbf{m} = (m_1, m_2, \dots, m_q) \in \mathbb{Z}^q$ with

$$M = \sum_{j=1}^q |m_j| \neq 0.$$

Then the set $\mathcal{H}(q, p, \mathbf{m})$ consists of those $(E, \mu) \in \mathcal{G}_\phi$ such that $\sum_{j=1}^q m_j x_j \neq 0$ whenever $x_j \in E$ and $|x_i - x_j| \geq 1/p$ for $i \neq j$.

Since the set of finite sequences of integers is countable, Theorem 3.2 follows from the following lemma.

Lemma 3.4. *The set $\mathcal{H}(q, p, \mathbf{m})$ is open and dense in $(\mathcal{G}_\phi, d_\phi)$.*

We split the proof of Lemma 3.4 into two parts.

Lemma 3.5. *The set $\mathcal{H}(q, p, \mathbf{m})$ is open.*

Proof. We show that the complement of $\mathcal{H}(q, p, \mathbf{m})$ is closed. Suppose that $(E_r, \mu_r) \notin \mathcal{H}(q, p, \mathbf{m})$ and $(E_r, \mu_r) \rightarrow (E, \mu)$. We can find $x_j(r) \in E_r$ such that $|x_i(r) - x_j(r)| \geq 1/p$ for $i \neq j$ and

$$\sum_{j=1}^q m_j x_j(r) = 0.$$

By an appropriate form of the Bolzano–Weierstrass theorem, we can find $x_j \in \mathbb{T}$ and $r(k) \rightarrow \infty$ such that $x_j(r(k)) \rightarrow x_j$ for each $1 \leq j \leq q$. Automatically, $|x_i - x_j| \geq 1/p$ for $i \neq j$ and

$$\sum_{j=1}^q m_j x_j = 0.$$

Since $d_{\mathcal{F}}(E_{r(k)}, E) \rightarrow 0$ it follows that $x_j \in E$ for $1 \leq j \leq q$ and so $(E, \mu) \notin \mathcal{H}(q, p, \mathbf{m})$ as required. $\square$

Our proof of the remaining part of Lemma 3.4 uses the following two simple observations.

Lemma 3.6. *Given $(E, \mu) \in \mathcal{G}_\phi$ and $\epsilon > 0$ we can find an $(F, \sigma) \in \mathcal{G}_\phi$ with $d_\phi((E, \mu), (F, \sigma)) < \epsilon$ having the following properties.*

(i) $d\sigma(x) = g(x)dm(x)$, where g is infinitely differentiable and m is a Lebesgue measure.

(ii) *There exists an $\alpha > 0$ such that, whenever $x \in F$, we can find an interval $I = [y - \alpha, y + \alpha]$ such that $x \in I \subseteq F$.*

Proof. Choose $u_n : \mathbb{T} \rightarrow \mathbb{R}$, a non-negative, infinitely differentiable function, such that $\text{supp } u_n \subseteq [-1/n, 1/n]$ and $\int_{\mathbb{T}} u_n(t) dt = 1$. Provided that n is large enough, standard theorems show that $g = u_n * \sigma$, $d\sigma(x) = g(x)dm(x)$, and $F = E + [-1/n, 1/n]$ satisfy the conclusions of the lemma. $\square$

Lemma 3.7. *There exists a constant A with the following property. Suppose that $\omega : \mathbb{N} \rightarrow \mathbb{R}$ is a sequence of positive numbers with $\omega(0) = 1$,*

$$K^{-1}n^{-1} \leq \omega(n)$$

for all $n \neq 0$, and

$$K^{-1}\omega(n) \leq \omega(r) \leq K\omega(n)$$

for all $1 \leq n \leq r \leq 2n$ and some constant $K > 1$. Suppose also that f and g are continuous functions with $\hat{f}(0) = 1$ and

$$|\hat{g}(r)| \leq Br^{-2}, \quad |\hat{f}(r)| \leq C\omega(|r|)$$

for all $r \neq 0$. Then

$$|\widehat{f \times g}(r) - \hat{g}(r)| \leq ABCK\omega(r)$$

for all r .

Proof. We have

$$\begin{aligned} |\widehat{f \times g}(r) - \hat{g}(r)| &= \left| \sum_{u \neq 0} \hat{f}(r-u)\hat{g}(u) \right| \leq \sum_{u \neq 0} |\hat{f}(r-u)\hat{g}(u)| \\ &\leq BC \sum_{u \neq 0} \frac{\omega(|r-u|)}{u^2} \\ &= BC \sum_{0 < |u| \leq |r|/2} \frac{\omega(|r-u|)}{u^2} + BC \sum_{|u| > |r|/2} \frac{\omega(|r-u|)}{u^2} \\ &\leq BC \sum_{|u| \leq |r|/2} \frac{K\omega(|r|)}{u^2} + BC \sum_{|u| > |r|/2} \frac{1}{u^2} \\ &\leq A_1 BCK\omega(|r|) + A_2 BC \frac{1}{|r|} \leq ABCK\omega(|r|) \end{aligned}$$

for appropriate constants A_1 , A_2 , and A and all $r \neq 0$. A similar calculation works for $r = 0$. $\square$

We can now complete the proof of Lemma 3.4.

Lemma 3.8. *The set $\mathcal{H}(q, p, \mathbf{m})$ is dense in $(\mathcal{G}_\phi, d_\phi)$.*

Proof. We wish to show that, given any $1/10 > \eta > 0$ and any $(E, \mu) \in \mathcal{G}_\phi$, we can find an $(F, \sigma) \in \mathcal{H}(q, p, \mathbf{m})$ with

$$d_\phi((E, \mu), (F, \sigma)) < \eta.$$

In view of Lemma 3.6, we may suppose that $d\mu(x) = g(x)dm(x)$, where g is infinitely differentiable and there exists an $\alpha > 0$ such that every point of $\text{supp } g$ lies in an

interval $I \subseteq \text{supp } g$ of length at least $\alpha > 0$. Thus there exists a constant B such that

$$|\hat{g}(r)| \leq B|r|^{-2}$$

for all $r \neq 0$.

Theorem 2.6 tells us that, if $\epsilon > 0$, we can find an infinitely differentiable function $f_\epsilon : \mathbb{T} \rightarrow \mathbb{R}$ with the following properties.

- (i) $f_\epsilon(t) \geq 0$ for all t .
- (ii) $\int_{\mathbb{T}} f_\epsilon(t) dt = 1$.
- (iii) $|\hat{f}_\epsilon(r)| \leq \epsilon|r|^{-1/(4q)}$ for all $r \neq 0$.
- (iv) If $x_j \in \text{supp } f_\epsilon$ for $1 \leq j \leq q$, and $|x_j - x_k| \geq 1/p$ for $1 \leq j < k \leq q$, then

$$\sum_{j=1}^q m_j x_j \neq 0.$$

- (v) If I is an interval of length ϵ , then $\text{supp } f_\epsilon \cap I \neq \emptyset$.

If we set $g_\epsilon(t) = g(t)f_\epsilon(t)$, and $E_\epsilon = E \cap \text{supp } f_\epsilon$, then, automatically

- (i') $f_\epsilon(t) \geq 0$ for all t ,

and, since $\text{supp } g_\epsilon = \text{supp } f_\epsilon \cap \text{supp } g$, it follows that $\text{supp } g_\epsilon \subseteq E_\epsilon$, and

- (iv') if $x_j \in \text{supp } g_\epsilon$ for $1 \leq j \leq q$, and $|x_j - x_k| \geq 1/p$ for $1 \leq j < k \leq q$, then

$$\sum_{j=1}^q m_j x_j \neq 0.$$

On the other hand, condition (v) tells us that, provided only ϵ is small enough,

$$d(E_\epsilon, E) < \eta/2.$$

If we take $\omega(0) = 1$, $\omega(r) = r^{-1/4q}$ for $r \geq 1$ and $C = \epsilon$ in Lemma 3.7, the inequality proved there shows that, if $\gamma > 0$ is fixed, $|\hat{g}_\epsilon(0) - \hat{g}(0)| \leq \gamma$ and

$$|\hat{g}_\epsilon(r) - \hat{g}(r)| \leq \gamma r^{-1/4q}$$

for all $r \neq 0$ and all sufficiently small ϵ . Since $r^{1/4q}\phi(r) \rightarrow \infty$ as $r \rightarrow \infty$, it follows that, if $\beta > 0$ is fixed with $1/10 > \beta > 0$,

$$\|\hat{g}_\epsilon - \hat{g}\|_\phi < \beta$$

for all ϵ sufficiently small. In particular, we know that

$$|\hat{g}_\epsilon(0) - 1| = |\hat{g}_\epsilon(0) - \hat{g}(0)| < \beta$$

for ϵ sufficiently small. Writing $G_\epsilon = \hat{g}_\epsilon(0)^{-1}g_\epsilon$, we have

$$\begin{aligned} \|\hat{G}_\epsilon - \hat{g}\|_\phi &= \left\| \frac{\hat{g}_\epsilon}{\hat{g}(0)} - \hat{g} \right\|_\phi \\ &\leq \|\hat{g}_\epsilon - \hat{g}\|_\phi + \left(1 - \frac{1}{\hat{g}(0)} \right) (\|\hat{g}\|_\phi + \|\hat{g}_\epsilon - \hat{g}\|_\phi) \\ &\leq \beta + 2\beta(\|\hat{g}\|_\phi + \beta). \end{aligned}$$

It follows that $G_\epsilon \in \mathcal{G}_\phi$ and, provided only that β (and so ϵ) is small enough,

$$\|\hat{G}_\epsilon - \hat{g}\|_\phi < \eta/2.$$

Thus, provided only that ϵ is small enough, $F = E_\epsilon$ and $d\sigma(x) = G_\epsilon(x)dm(x)$ satisfy the conclusions required by the first sentence of this proof. $\square$

We have thus proved Theorem 3.2 and so Theorem 1.4. We can prove Lemma 1.5 in the same way, obtaining the following Baire category version.

Lemma 3.9. *Suppose q is an integer with $q \geq 1$ and $\psi : \mathbb{N} \rightarrow \mathbb{R}$ is a sequence of strictly positive numbers with $\psi(r) \rightarrow \infty$ as $r \rightarrow \infty$. Then, if we write $\phi(r) = r^{-1/(2q)}(\log(1 + |r|))^{1/2}\psi(|r|)$ for $r \geq 1$ and set $\phi(0) = 1$, the set of $(E, \mu) \in \mathcal{G}_\phi$ with the property that, given distinct points $x_1, x_2, \dots, x_q \in E$, the only solution to the equation*

$$\sum_{j=1}^q m_j x_j = 0$$

with the $m_j \in \mathbb{Z}$ is the trivial solution $m_1 = m_2 = \dots = m_q = 0$, has a complement of first category in $(\mathcal{G}_\phi, d_\phi)$.

There is large gap between the results of Lemma 1.2 and Theorem 1.5. When $q = 1$ we can improve Theorem 1.5 using the arguments of [8]. These show that, if $\omega : \mathbb{N} \rightarrow \mathbb{R}$ is a sequence of positive numbers with $\sum_{r=1}^\infty \omega(r)$ divergent and ω satisfies certain regularity conditions, then we can find a probability measure μ with $|\hat{\mu}(r)| \leq \omega(r)$ for all $r \neq 0$ whose support contains no rational number.

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