A LOWER BOUND FOR THE EQUILATERAL NUMBER OF NORMED SPACES

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ABSTRACT. We show that if the Banach-Mazur distance between an n -dimensional normed space X and ℓ_∞^n is at most $3/2$, then there exist $n+1$ equidistant points in X . By a well-known result of Alon and Milman, this implies that an arbitrary n -dimensional normed space admits at least $e^{c\sqrt{\log n}}$ equidistant points, where $c > 0$ is an absolute constant. We also show that there exist n equidistant points in spaces sufficiently close to ℓ_p^n , $1 < p < \infty$.

1. NOTATION

Throughout the paper we use the same symbol c for different absolute positive constants. Let X denote a normed space of finite dimension $\dim X = n$. Let $e(X)$ denote the largest size of an equilateral set in X . As usual, the space ℓ_p^n , $1 \leq p < \infty$, is defined as $\mathbb{R}^n$ with the norm $\|(x_1, x_2, \dots, x_n)\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$, and ℓ_∞^n is $\mathbb{R}^n$ with the norm $\|(x_1, x_2, \dots, x_n)\|_\infty = \max_i |x_i|$. The *Banach-Mazur distance* between two n -dimensional normed spaces is defined as $d(X, Y) = \inf \|T\| \|T^{-1}\|$, where the infimum is taken over all linear, invertible operators $T : X \rightarrow Y$. We say that X is a $(1 + \varepsilon)$ -copy of Y if $d(X, Y) \leq 1 + \varepsilon$.

2. THE MAIN THEOREMS

It is conjectured [4, 9, 11, 12, 17] that $e(X) \geq n+1$ for all n -dimensional normed spaces X . This is known for $n \leq 3$ [12] but open for $n \geq 4$. It is true for spaces sufficiently close to Euclidean:

Theorem 1 (Brass [3] and Dekster [6]). *Let X be an n -dimensional normed space with Banach-Mazur distance $d(X, \ell_2^n) \leq 1 + \frac{1}{n}$. Then an equilateral set in X of at most n points can be extended to one of $n+1$ points. In particular, $e(X) \geq n+1$.*

Combining this theorem with Schechtman's estimate [13, Theorem 3] in the Dvoretzky theorem [7, §4], the following general lower bound follows:

$$e(X) \geq \frac{c\sqrt{\log n}}{\log \log n}.$$

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(The previous best estimate in the Dvoretzky theorem, due to Gordon [8], gave $e(X) \geq c(\log n)^{1/3}$.) We improve this to the following:

Theorem A. *For any n -dimensional normed space X we have $e(X) \geq e^{c\sqrt{\log n}}$, where $c > 0$ is an absolute constant.*

The theoretical limit beyond which Dvoretzky's theorem cannot be improved in general (by considering ℓ_∞^n ; see the discussion above Theorem 3 in [13]) would give only $e(X) \geq c \log n / \log \log n$, still worse than Theorem A. Thus something more than Dvoretzky's theorem is needed. Indeed, the proof of Theorem A works by looking for large subspaces close to either ℓ_2^k or ℓ_∞^k . To this end we use the following theorem:

Theorem 2 (Alon and Milman [1]). *Let X be an n -dimensional normed space. Then for each $\varepsilon > 0$ there exists $c \in (0, 1)$ such that X has a subspace of dimension $m \geq e^{c\sqrt{\log n}}$ that is $(1 + \varepsilon)$ -isomorphic to either ℓ_2^m or ℓ_∞^m .*

If the second case occurs in the above theorem, i.e., if we find a $(1 + \varepsilon)$ -isomorphic copy of ℓ_∞^m in X , then we need a result similar to Theorem 1 for spaces near ℓ_∞^n . This we provide as follows:

Theorem B. *Let X be an n -dimensional normed space with Banach–Mazur distance $d(X, \ell_\infty^n) \leq 3/2$. Then $e(X) \geq n + 1$.*

We can then choose $\varepsilon = 1/2$ in Theorem 2 to obtain $e(X) \geq e^{c\sqrt{\log n}}$. On the other hand, if we find a $(1 + \frac{1}{2})$ -isomorphic copy of ℓ_2^m where $m \geq e^{c\sqrt{\log n}}$, we cannot yet apply Theorem 1. We first have to find a k -dimensional subspace of the m -dimensional space that is $(1 + \frac{1}{k})$ -isomorphic to ℓ_2^k . This is guaranteed by the following:

Theorem 3 (Milman [10]). *Let X be an m -dimensional normed space and $0 < \varepsilon < 1$. Then X contains a subspace Y of dimension $k \geq c\varepsilon^2 m / d^2(X, \ell_2^m)$ such that $d(Y, \ell_2^k) \leq 1 + \varepsilon$.*

See also [7, Corollary 4.2.2]. Putting $\varepsilon = 1/k$ and X the m -dimensional subspace that is $3/2$ -isomorphic to ℓ_2^m into the above theorem, we obtain a subspace Y of dimension $k > cm^{1/3}$, and Theorem 1 then gives $e(X) \geq e(Y) > e^{c\sqrt{\log n}}$. To complete the proof of Theorem A, it only remains to prove Theorem B.

Proof of Theorem B. We use the Brouwer fixed point theorem [5, §14.3], as in Brass' proof of Theorem 1. Without loss of generality we may assume $X = (\mathbb{R}^n, \|\cdot\|)$ and

$$\|x\| \leq \|x\|_\infty \leq \frac{3}{2}\|x\| \quad \text{for all } x \in X.$$

Let $I = \{(i, j) : 1 \leq i < j \leq n + 1\}$, with $|I| = n(n + 1)/2 = N$. For $\varepsilon = (\varepsilon_{i,j})_{(i,j) \in I} \in [0, \frac{1}{2}]^N$, let

$$\begin{aligned} p_1(\varepsilon) &= (-1, 0, \dots, 0), \\ p_j(\varepsilon) &= (\varepsilon_{1,j}, \dots, \varepsilon_{j-1,j}, -1, 0, \dots, 0), \quad 2 \leq j \leq n - 1, \\ p_n(\varepsilon) &= (\varepsilon_{1,n}, \dots, \varepsilon_{n-1,n}, -1), \\ p_{n+1}(\varepsilon) &= (\varepsilon_{1,n+1}, \dots, \varepsilon_{n,n+1}). \end{aligned}$$

For $1 \leq i < j \leq n$ we have $\|p_i(\varepsilon) - p_j(\varepsilon)\|_\infty = 1 + \varepsilon_{i,j}$. Define $\varphi : [0, 1/2]^N \rightarrow [0, 1/2]^N$ by $\varphi_{i,j}(\varepsilon) = 1 + \varepsilon_{i,j} - \|p_i(\varepsilon) - p_j(\varepsilon)\|$, $1 \leq i < j \leq n$. Note that

$$\varphi_{i,j}(\varepsilon) \geq 1 + \varepsilon_{i,j} - \|p_i - p_j\|_\infty = 0$$

and

$$\varphi_{i,j}(\varepsilon) \leq 1 + \varepsilon_{i,j} - \frac{2}{3}\|p_i - p_j\|_\infty = \frac{1}{3}(1 + \varepsilon_{i,j}) \leq \frac{1}{2},$$

so φ is well defined. Brouwer now gives the existence of a point $\varepsilon' = (\varepsilon'_{i,j}) \in [0, 1/2]^N$ with $\varphi(\varepsilon') = \varepsilon'$, which implies that $\|p_i(\varepsilon') - p_j(\varepsilon')\| = 1$ for all $1 \leq i < j \leq n$. We have obtained $n + 1$ equilateral points. $\square$

3. A GENERALIZATION TO ℓ_p^n

The following theorem partially generalizes Theorem B to all ℓ_p^n spaces with $1 < p < \infty$.

Theorem C. *For each $n > 2$ and $p \in (1, \infty)$ there exists $R(p, n) > 1$ such that for any n -dimensional normed space X with Banach-Mazur distance $d(X, \ell_p^n) \leq R(p, n)$ we have $e(X) \geq n$. In fact,*

$$\begin{aligned} R(p, n) &= \max_{\theta > 0} \left(\frac{1 + (1 + \theta)^p}{2 + (n - 2)\theta^p} \right)^{1/p} \\ &\sim 1 + \frac{p-1}{2p} n^{-\frac{1}{p-1}} \text{ as } n \rightarrow \infty \text{ with } p \text{ fixed.} \end{aligned}$$

Proof. We follow the proof of Theorem B. Assume $X = (\mathbb{R}^n, \|\cdot\|)$ and

$$\|x\| \leq \|x\|_p \leq R\|x\| \quad \text{for all } x \in X.$$

Fix $\beta, \gamma > 0$. Let $I = \{(i, j) : 1 \leq i < j \leq n\}$, with $|I| = n(n-1)/2 = N$. For $\varepsilon = (\varepsilon_{i,j})_{(i,j) \in I} \in [0, \beta]^N$, let

$$\begin{aligned} p_1(\varepsilon) &= (-\gamma, 0, \dots, 0), \\ p_j(\varepsilon) &= (\varepsilon_{1,j}, \dots, \varepsilon_{j-1,j}, -\gamma, 0, \dots, 0), \quad 2 \leq j \leq n-1, \\ p_n(\varepsilon) &= (\varepsilon_{1,n}, \dots, \varepsilon_{n-1,n}, -\gamma). \end{aligned}$$

For $1 \leq i < j \leq n$ we have

$$\|p_j - p_i\|_p^p = \sum_{k=1}^{i-1} |\varepsilon_{k,j} - \varepsilon_{k,i}|^p + (\varepsilon_{i,j} + \gamma)^p + \sum_{k=i+1}^j \varepsilon_{k,j}^p + \gamma^p.$$

Define $\varphi : [0, \beta]^N \rightarrow [0, \beta]^N$ by $\varphi_{i,j}(\varepsilon) = 1 + \varepsilon_{i,j} - \|p_i - p_j\|$ for $1 \leq i < j \leq n$. On the one hand,

$$\begin{aligned} \varphi_{i,j}(\varepsilon) &\leq 1 + \varepsilon_{i,j} - R^{-1}\|p_i - p_j\|_p \\ &\leq 1 + \varepsilon_{i,j} - R^{-1}[(\gamma + \varepsilon_{i,j})^p + \gamma^p]^{1/p}. \end{aligned}$$

Taking into account that the latter is increasing with respect to $\varepsilon_{i,j}$, the inequality $\varepsilon_{i,j} \leq \beta$ implies that

$$\varphi_{i,j}(\varepsilon) \leq 1 + \beta - R^{-1}[(\gamma + \beta)^p + \gamma^p]^{1/p}.$$

Therefore, if $(\gamma + \beta)^p + \gamma^p \geq R^p$, then $\varphi_{i,j}(\varepsilon) \leq \beta$. On the other hand,

$$\begin{aligned}\varphi_{i,j}(\varepsilon) &\geq 1 + \varepsilon_{i,j} - \|p_i - p_j\|_p \\ &\geq 1 + \varepsilon_{i,j} - [(n-2)\beta^p + (\gamma + \varepsilon_{i,j})^p + \gamma^p]^{1/p}.\end{aligned}$$

Again the latter is increasing with respect to $\varepsilon_{i,j}$, so using $\varepsilon_{i,j} \geq 0$ we have

$$\varphi_{i,j}(\varepsilon) \geq 1 - [(n-2)\beta^p + 2\gamma^p]^{1/p}.$$

Then $\varphi_{i,j}(\varepsilon_1, \dots, \varepsilon_m) \geq 0$ would follow if $(n-2)\beta^p + 2\gamma^p \leq 1$. Subsequently, if

$$(*) \quad (\gamma + \beta)^p + \gamma^p \geq R^p \quad \text{and} \quad (n-2)\beta^p + 2\gamma^p \leq 1,$$

then φ is well defined. Brouwer now gives a point $\varepsilon' = (\varepsilon'_{i,j}) \in [0, \beta]^N$ such that $\varphi(\varepsilon') = \varepsilon'$, implying that the points $p_1(\varepsilon'), \dots, p_n(\varepsilon')$ are equilateral.

Finally, to take the best choice for the parameters in $(*)$ we have to maximize the expression $(\gamma + \beta)^p + \gamma^p$ under the constraints $(n-2)\beta^p + 2\gamma^p \leq 1$ and $\beta, \gamma \geq 0$. Setting $\theta = \beta/\gamma$, we obtain

$$R^p = \max_{\theta > 0} \frac{1 + (1 + \theta)^p}{2 + (n-2)\theta^p}.$$

It is not difficult to see that for θ close to $n^{-1/(p-1)}$, the right-hand side is > 1 and $R - 1 \sim \frac{p-1}{2p} n^{-\frac{1}{p-1}}$. $\square$

4. CONCLUDING REMARKS

For $p = 2$ the estimate in the above theorem is $d(X, \ell_2^n) \lesssim 1 + \frac{1}{4n}$, slightly worse than Theorem 1. However, we don't know how to obtain $n+1$ equidistant points as in Theorem B. It would also be interesting to know whether arbitrary equilateral sets of at most n points in spaces near ℓ_p^n can be extended as in Theorem 1. A different idea will be needed to extend the above theorem to the case $p = 1$. See [16] for a survey on equilateral sets, as well as [2, 14, 15] for further results on equilateral sets in ℓ_p^n .

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