NONPOSITIVE SECTIONAL CURVATURE FOR (p, q, r) -COMPLEXES

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ABSTRACT. We give a criterion for the nonpositive sectional curvature of (p, q, r) -complexes. As a consequence, we show that certain 2-complexes have locally indicable, coherent and even locally quasiconvex fundamental groups.

1. INTRODUCTION

Definition 1.1. A (p, q, r) -complex is a combinatorial 2-complex X such that

- (1) the attaching map of each 2-cell of X has length $\geq p$,
- (2) for each $x \in X^0$, the graph $\text{link}(x)$ has girth $\geq q$,
- (3) each 1-cell e of X appears $\leq r$ times among the attaching maps of 2-cells; that is, $\text{link}(e)$ consists of $\leq r$ points.

To say that each attaching map has length p means that each 2-cell is a polygon with at least p sides. The *link* of $x \in X^0$ is the graph isomorphic to the boundary of an ϵ -ball around x in X . The vertices of $\text{link}(x)$ correspond to ends of 1-cells incident with x , and the edges of $\text{link}(x)$ correspond to corners of 2-cells incident with x . Recall that the *girth* of a graph is the infimum of the lengths of its simple cycles (the girth of a tree is ∞). Finally, the condition on the 1-cells means that a neighborhood of each open 1-cell looks like the spine of a book with at most r pages. This condition is equivalent to the requirement that for each $x \in X^0$, each vertex in $\text{link}(x)$ has valence $\leq r$.

The purpose of this note is to identify the values of (p, q, r) such that all finite (p, q, r) -complexes are either contractible or have nonpositive Euler characteristic. In fact, we will focus on a stronger but closely related local property: the property of being an angled 2-complex with nonpositive sectional curvature. We now briefly describe this property and refer the reader to [6] for more details.

A combinatorial 2-complex Y is an *angled 2-complex* if an angle is assigned to each corner of each 2-cell. Since the edges in links of 0-cells of Y are in one-to-one correspondence with the corners of 2-cells of Y , this is equivalent to an angle assignment to each edge of $\text{link}(y)$ for each $y \in Y^0$. Angles are allowed to be arbitrary real numbers in general, but all angles in this paper are positive. The

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curvature $\kappa(C)$ of an n -sided 2-cell C of Y is defined by

$$\kappa(C) = \sum_{c \in \text{Corners}(C)} \angle(c) - (n-2)\pi.$$

The curvature $\kappa(y)$ of a 0-cell $y \in Y^0$ is defined by

$$\kappa(y) = 2\pi - \pi\chi(\text{link}(y)) - \sum_{c \in \text{Corners}(y)} \angle(c).$$

The Gauss–Bonnet Theorem for angled 2-complexes states that

$$(1) \quad 2\pi\chi(Y) = \sum_{y \in Y^0} \kappa(y) + \sum_{C \in 2\text{-cells}(Y)} \kappa(C).$$

The idea of nonpositive sectional curvature is to require that for any immersion $Y \rightarrow X$ (where Y has no free faces and X is an angled 2-complex), when we consider the angles induced on corners of Y , each 0-cell and 2-cell of Y has nonpositive curvature. The condition on the 2-cells is immediately inherited from X , and it turns out that when all the angles are nonnegative it is sufficient to consider the case where Y is locally connected. We thus define the angled 2-complex X to have *nonpositive sectional curvature* if it satisfies the following two conditions: For each 2-cell C we have

$$(2) \quad \kappa(C) \leq 0,$$

and for each connected finite nonempty subgraph $\Gamma \subset \text{link}(x)$ where $x \in X^0$, if each vertex of Γ has valence ≥ 2 , then

$$(3) \quad 2\pi - \pi(\chi(\Gamma)) - \sum_{c \in \text{Edges}(\Gamma)} \angle(c) \leq 0.$$

The related notion of *negative sectional curvature* is defined similarly except that we require that the inequalities are strict in either Equation 2 or 3.

We are interested in nonpositive sectional curvature because of the following results proven in [6].

Theorem 1.2. *If X has nonpositive sectional curvature, then $\pi_1 X$ is locally indicable. Specifically, if X is compact, then $\chi(X) \leq 0$ unless X is contractible.*

If X is compact and has negative sectional curvature, then for any covering space $\hat{X}$ with $\pi_1 \hat{X}$ finitely generated, any compact subspace of $\hat{X}$ is contained in a compact core. In particular, $\pi_1 X$ is coherent.

Moreover, if the angle assignment comes from a genuine piecewise Euclidean metric of negative curvature, then $\pi_1 X$ is a locally quasiconvex word-hyperbolic group.

Recall that a group G is *locally indicable* if every finitely generated nontrivial subgroup of G has an infinite cyclic quotient, and G is *coherent* if all its finitely generated subgroups are finitely presented. A *compact core* is a compact subspace whose inclusion induces a homotopy equivalence. A word-hyperbolic group is *locally-quasiconvex* if all its finitely generated subgroups are word-hyperbolic groups that embed by a quasi-isometry.

In [4], Jon McCammond and I gave an approach towards proving some of the properties in Theorem 1.2. Our approach hinged upon the notion of “perimeter”, and it became clear that in the study of coherence of fundamental groups, critical

attention should be paid to the relationship between p and r in any (p, q, r) -complex, in the sense that one expects coherence when p is sufficiently larger than r .

The purpose of this note is to prove sharp conditions on the nonpositive sectional curvature of (p, q, r) -complexes. We do this in Section 2 where we prove our main result, which is the following:

Theorem 2.2. *If X is a 2-complex satisfying one of the following conditions, then X has nonpositive sectional curvature:*

- (1) X is a $(p, 3, p - 3)$ -complex where $p \geq 6$;
- (2) X is a $(p, 4, p - 2)$ -complex where $p \geq 4$;
- (3) X is a $(p, 5, p - 1)$ -complex where $p \geq 4$.

In Section 3 we show that Theorem 2.2 is sharp by describing examples of compact $(p, 3, p - 2)$ -complexes, $(p, 4, p - 1)$ -complexes, and $(p, 5, p)$ -complexes with positive Euler characteristic, for arbitrarily large p .

A simple counting argument given in [6] shows that the (p, q, r) -complex X has nonpositive sectional curvature provided $p \geq \frac{q}{q-2}r$, and X has negative sectional curvature if the inequality is sharp. The possibility that stronger results might hold was suggested by the following analogous theorem proved in [8] about the $L^{(2)}$ second homology of (p, q, r) -complexes.

Theorem 1.3. $H_2^{(2)}(\tilde{X}) = 0$ provided any of the following holds:

- (1) X is a $(p, 3, p - 4)$ -complex where $p \geq 6$;
- (2) X is a $(p, 4, p - 3)$ -complex where $p \geq 4$;
- (3) X is a $(p, 5, p - 2)$ -complex where $p \geq 4$.

2. NONPOSITIVE SECTIONAL CURVATURE FOR (p, q, r) -COMPLEXES

Let X be a (p, q, r) -complex. We assign an angle of $\frac{p-2}{p}\pi$ to each corner of each 2-cell C of X , so $\kappa(C) \leq 0$. To show that X has nonpositive sectional curvature we will verify that for any $x \in X^0$, and for any finite spurless nontrivial subgraph of $\text{link}(x)$, we have $(2 - v) + e\frac{2}{p} \leq 0$, where v and e are the numbers of vertices and edges in $\text{link}(x)$.

Before proceeding with the proof of our main theorem, we will recall some useful graph-theoretical results.

Background 2.1 (Maximal number of edges in girth q graphs). It is immediate that a girth ≥ 3 graph with v vertices has at most $\binom{v}{2} = \frac{v(v-1)}{2}$ edges. In 1907 Mantel proved that a girth ≥ 4 graph has at most $v^2/4$ edges. This was subsequently reproven by Turan, who generalized this to an analogous theorem about graphs not containing K_m , the complete graph with m vertices. This led to the subject of extremal graph theory [1], and the reader can find other bounds there on the number of edges in graphs with restricted girths. We will employ a somewhat sharper theorem than described in [1] for graphs of girth ≥ 5 . Namely a graph of girth ≥ 5 has at most $\frac{1}{2}v\sqrt{v-1}$ edges. This was first proven in [2], and later reproved in [3], where there is also a table of explicitly computed lower bounds on the maximal number of edges in graphs with ≤ 200 vertices, and these values are achieved when there are ≤ 30 vertices.

Theorem 2.2. *If X is a 2-complex satisfying one of the following conditions, then X has nonpositive sectional curvature:*

- (1) X is a $(p, 3, p-3)$ -complex where $p \geq 6$;
- (2) X is a $(p, 4, p-2)$ -complex where $p \geq 4$;
- (3) X is a $(p, 5, p-1)$ -complex where $p \geq 4$.

Proof. We assign angles of $\frac{2p-2}{p}\pi$ to each corner of a (p, q, r) -complex. To prove nonpositive sectional curvature, we show that for each finite (p, q, r) -complex X with no free faces, either X is a single 0-cell, or for each $x \in X^0$ we have $0 \geq \kappa(x) = 2\pi - \pi \chi \text{link}(x) - \sum_{e \in \text{link}(x)} \angle(e)$, or equivalently $2 - v + \frac{2e}{p} \leq 0$, where v and e are the numbers of vertices and edges in the graph $\text{link}(x)$. Accordingly, our goal will be to show that the following function is nonpositive subject to three constraints that we shall discuss below:

$$(4) \quad K(v, e) = 2 - v + \frac{2e}{p}.$$

The first two constraints are the following:

$$(5) \quad V_q = v - q \geq 0,$$

$$(6) \quad H = 2e - rv \leq 0.$$

We can assume constraint (5) because if $\text{link}(x)$ is finite and has girth $\geq q$, then it is either empty, consists of a single vertex, consists of the disjoint union of two or more vertices, or contains a cycle of length $\geq q$. The first few cases correspond to the degenerate cases where X is an isolated 0-cell, or X has a spur at v , or $e = 0$, so $\kappa(x) = 2 - v \leq 0$. The remaining case of a cycle obviously implies that $v \geq q$.

Constraint (6) arises from the “handshake lemma” since each vertex in $\text{link}(x)$ has valence $\leq r$.

Following the bounds described in Background 2.1, we have the following functions G_q , where the constraint $G_q \leq 0$ holds for any graph of girth $\geq q$ with v vertices and e edges:

$$(7) \quad G_3 = e - \frac{v(v-1)}{2},$$

$$(8) \quad G_4 = e - \frac{v^2}{4},$$

$$(9) \quad G_5 = e - \frac{1}{2}v\sqrt{v-1}.$$

Let $\bar{G}_q$ denote the curve in the (v, e) -plane corresponding to the constraint $G_q = 0$. Similarly, let $\bar{H}$, $\bar{K}$ and $\bar{V}_q$ denote the curves corresponding to the constraints $H = 0$, $K = 0$, and $V_q = v - q \geq 0$. Let $A = \bar{K} \cap \bar{V}_q$ and let $B = \bar{K} \cap \bar{H}$. We will prove the theorem by showing that $G_q(A) \geq 0$ and $G_q(B) \geq 0$, provided that (p, q, r) satisfy our hypothesis.

Consideration of the 1st and 2nd derivatives shows that $\bar{G}_3$ and $\bar{G}_4$ are concave-up for $v \geq 1$ and $\bar{G}_5$ is concave up for $v \geq \frac{4}{3}$. It seems this approach requires the fewest messy calculations, but there are various other ways of working things out. The point is that $\bar{K}$ has larger slope than $\bar{H}$, so if A and B lie above $\bar{G}_q$, then $\bar{K}$ lies above the infinite region R satisfying our constraints. Note that R is bounded by a finite curve in $\bar{G}_q$, bounded by $\bar{E}_q \cap \bar{G}_q$ and $\bar{H}_q \cap \bar{G}_q$ together with the bottom

FIGURE 1. $(p, q, r) = (7, 4, 5)$: The curves $\bar{G}_4$, $\bar{H}$, $\bar{V}_4$, and $\bar{K}$ are graphed above. The line $\bar{K}$ that passes through the points A and B is disjoint from the infinite region on the lower right determined by the constraints.

half of $\bar{E}_q$ and the right half of $\bar{H}_q$ (see Figure 1). Thus $K \leq 0$ on the region R and we are done. The points A and B are computed to be:

$$A = \left(q, \frac{1}{2}qp - p \right) \quad \text{and} \quad B = \left(\frac{2p}{p-r}, \frac{rp}{p-r} \right),$$

where $r = p - 3$, $r = p - 2$, and $r = p - 1$ according to whether $q = 3$, $q = 4$, or $q = 5$.

In the case $q = 3$, we have $G_3 = 2e - v(v - 1)$ and so

$$G_3(B) = \frac{1}{9}p^2 - \frac{2}{3}p,$$

so $G_3(B) \geq 0$ for $p \geq 6$. Also,

$$G_3(A) = \frac{1}{2} - p + \frac{qp}{2} - \frac{q(q-1)}{2},$$

so $G_3(A) \geq 0$ provided that $q = 3$ and $p \geq 6$.

In the case $q = 4$, we have $G_4(v, e) = e - v^2/4$ and so

$$G_4(B) = \frac{1}{4}p^2 - p,$$

so $G_4(B) \geq 0$ for $p \geq 4$. Also,

$$G_4(A) = \frac{1}{2}qp - p - \frac{1}{4}q^2,$$

so $G_4(A) \geq 0$ if $q = 4$ and $p \geq 4$.

We finally consider the case where $q = 5$ and $G_5(v, e) = e - v\sqrt{v-1}/2$. Then

$$G_5(B) = p^2 - p - p\sqrt{2p-1} = p(p-1-\sqrt{2p-1}),$$

so $G_5(B) \geq 0$ for $p \geq 4$. Also,

$$G_5(A) = \frac{1}{2}qp - p - \frac{1}{2}q\sqrt{q-1},$$

so $G_5(A) \geq 0$ if $q = 5$ and $p \geq 4$. □

Remark 2.3. In each case of Theorem 2.2, if p is smaller than required by the bound, then $\pi_1 X$ is free, and thus (incidentally) coherent. For instance, in the first case where $q = 3$, suppose that $p < 6$. If $p = 5$, then X is formed by gluing spheres and discs to a graph along 0-cells. If $p = 4$, then X is formed by gluing squares to a graph along 0-cells, and if $p \leq 3$, then X is a graph.

Theorem 2.4 (Negative sectional curvature). *Let X be a (p, q, r) -complex satisfying the hypotheses of Theorem 2.2, but assume that (p, q, r) is neither $(6, 3, 3)$ nor $(4, 4, 2)$. Then X has negative sectional curvature.*

Proof. Except for the $(6, 3, 3)$ and $(4, 4, 2)$ cases, we have strict inequalities $G_q(A) > 0$ and $G_q(B) > 0$, and so the line $\bar{K}$ is disjoint from the region R . Consequently, the sectional curvature is negative as claimed. $\square$

3. EXAMPLES OF (p, q, r) -COMPLEXES WITH POSITIVE EULER CHARACTERISTIC

Theorem 2.2 leads to two natural problems:

Problem 3.1. Find compact noncontractible complexes with $\chi(X) > 0$ satisfying:

- (1) X is a $(p, 3, p - 2)$ -complex and $p \geq 6$, or
- (2) X is a $(p, 4, p - 1)$ -complex and $p \geq 4$, or
- (3) X is a $(p, 5, p)$ -complex and $p \geq 4$.

Problem 3.2. Is the fundamental group of any/every such example incoherent?

It is not difficult to solve Problem 3.1. However, Problem 3.2 is more challenging, and we have not yet been able to answer it. Several families of incoherent groups which nearly have these properties are described in [5].

We first describe a family of examples of $(p, 3, p - 2)$ -complexes whose links are complete graphs on $p - 1$ vertices.

Example 3.3 (Branched cover of simplex 2-skeleton). For $p = 3n$ we let Y be the 2-skeleton of a $(p - 1)$ -simplex. Let Y' be obtained from Y by deleting a point at the center of each 2-simplex, and observe that $\pi_1 Y'$ is free, and the boundary of each 2-simplex represents a conjugacy class corresponding to a free generator. We can thus let $X = \hat{Y}$ be the branched cover of Y along the centers of 2-simplices that corresponds to the quotient $\pi_1 Y' \rightarrow H_1(Y', \mathbb{Z}_n)$. Then X is a finite $(p, 3, p - 2)$ -complex such that the link of each vertex of X is the complete graph on $p - 1$ vertices.

We will use the Gauss–Bonnet Theorem in equation (1) to verify that $\chi(X) > 0$. We assign an angle of $\frac{3n-2}{3n}\pi$ to each corner so that the curvature of each 2-cell is zero. We then observe that for each $x \in X^0$, $\kappa(x) = 2\pi - \pi(v - e) + e\frac{3n-2}{3n}\pi = \pi(2 - v + \frac{2}{3n}e)$. Thus $\kappa(x) > 0$ if $2 - v + \frac{2}{3n}e > 0$, which holds since $v = p - 1 = 3n - 1$ and $e = \binom{v}{2} = (3n - 1)(3n - 2)/2$.

We now describe a construction with a somewhat more general flavor that allows us to produce the desired examples with girth ≥ 3 .

Example 3.4. Given a graph Λ , let L be the suspension of Λ . So L is the 2-complex obtained from $\Lambda \times [0, 1]$ by identifying each of $\Lambda \times \{0\}$ and $\Lambda \times \{1\}$ to a point. Note that each 2-cell of L is a bigon, and the link of each 0-cell of L is isomorphic to Λ . For each n , we let X_n denote the branched cover of L (branched

along centers of 2-cells of L) that corresponds to the cover of L^1 associated to the kernel of $\pi_1 L^1 \rightarrow H_1(L^1, \mathbb{Z}_n)$. Of course, if Λ and hence L^1 is finite, then X_n is a finite 2-complex.

Each 2-cell of X_n has $2n$ sides, and for each $x \in X_n^0$, we have $\text{link}(x) \cong \Lambda$. Consequently, X_n is a $(2n, q, r)$ -complex where q is the girth of Λ and r is the maximum valence of Λ .

By choosing Λ appropriately, for each n we obtain examples of compact $(2n, 3, 2n - 2)$ -complexes, $(2n, 4, 2n - 1)$ -complexes, and $(2n, q, 2n)$ -complexes (for each $q \geq 5$) with positive Euler characteristic.

In each case, to see that $\chi(X_n) > 0$, we assign angles of $\frac{2n-2}{2n}\pi$ at each corner, so each 2-cell has zero curvature and for each $x \in X^0$ we have

$$\kappa(x) = 2\pi - \pi\chi(\Lambda) - \sum_{c \in \text{Edges}(\Lambda)} \angle(c) = \pi(2 - v + \frac{e}{n}),$$

where Λ has v vertices and e edges. Thus $\kappa(x) > 0$ provided $2 - v + \frac{e}{n} > 0$, which holds if $2e > 2n(v - 2)$.

The $q = 3$ case. For the case $q = 3$ we let Λ be the complete graph on $2n - 1$ vertices, then the girth of Λ equals 3, and so X_n is a $(2n, 3, 2n - 2)$ -complex.

Since Λ has $v = 2n - 1$ vertices and $e = \frac{(2n-1)(2n-2)}{2}$ edges we are done since

$$2e = (2n - 1)(2n - 2) = 4n^2 - 6n + 3 > 4n^2 - 6n = 2n(2n - 3) = 2n(v - 2).$$

The $q = 4$ case. In this case we let Λ be isomorphic to the complete bipartite graph $K(2n - 1, 2n - 1)$. Then the girth of Λ is ≥ 4 , and each vertex of Λ has valence $2n - 1$. Thus X_n is a $(2n, 4, 2n - 1)$ -complex. However $\chi(X_n) > 0$ since for each $x \in X^0$, we have $\kappa(x) = \pi(2 - v + \frac{e}{n})$, where Λ has $v = 2(2n - 1)$ vertices and $e = (2n - 1)^2$ edges. As before $\kappa(x) > 0$ if and only if $e > n(v - 2)$, which holds since

$$e = (2n - 1)^2 = 4n^2 - 4n + 1 > 4n^2 - 4n = n(2(2n - 1) - 2) = n(v - 2).$$

The $q \geq 5$ case. We first produce a finite graph Λ with girth $q \geq 5$ such that each vertex of Λ has valence $2n$. For instance, let Λ equal the finite covering space of a bouquet of n circles B_n corresponding to the intersection of all subgroups of index $\leq q$ in $\pi_1 B_n$. Since each vertex of Λ has valence $2n$, the handshake lemma tells us that $2e = (2n)v$ and so $v = \frac{e}{n}$. Letting X_n be the branched cover of X as before, we see that $\kappa(x) = \pi(2 - v + \frac{e}{n}) = 2\pi > 0$ for each $x \in X^0$.

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