BLASCHKE REPRESENTATION OF FUNCTIONS ON THE CIRCLE

ELIAS WEGERT AND LOTHAR VON WOLFERSDORF

(Communicated by Joseph A. Ball)

ABSTRACT. We prove that every unimodularly bounded measurable function on the complex unit circle admits a representation

$$f = \frac{f_+ + f_-}{1 + \overline{f_-} f_+},$$

where f_+ and f_- extend holomorphically into the interior and the exterior of the circle, respectively, f_- vanishes at infinity, and both functions are unimodularly bounded. The representation is unique if $\|f\|_\infty < 1$.

Representations of functions on the complex unit circle $\mathbb{T}$ by boundary values of holomorphic functions have been known for a long time and play an important rôle in theoretical investigations as well as in applications.

One of the two most popular representations of this type is the additive Riesz decomposition

$$(0.1) \quad f = f_+ + f_-,$$

where f_+ and f_- are functions which extend holomorphically into the interior $\mathbb{D}$ and the exterior $\mathbb{E} := \mathbb{C} \setminus \overline{\mathbb{D}}$ of the unit circle, respectively, and f_- vanishes at infinity. Its multiplicative counterpart is the Wiener–Hopf factorization,

$$(0.2) \quad f = f_+ \cdot f_\kappa \cdot f_-,$$

where f_+ and f_- are additionally supposed to be nonvanishing in their domains, $f_-(\infty) = 1$, and $f_\kappa(t) = t^\kappa$ with $\kappa \in \mathbb{Z}$.

This paper is devoted to another representation which fits into this framework, and appears, to our knowledge, for the first time in the paper [6] by J. Sylvester and D. Winebrenner. Investigating inverse scattering in stratified media, they introduced a nonlinear decomposition of the form

$$(0.3) \quad f = f_- \circ f_+ := \frac{f_+ + f_-}{1 + \overline{f_-} f_+}$$

for functions f on the real line, where f_+ and f_- are holomorphic in the upper and lower half-plane, respectively. Sylvester and Winebrenner proved existence and uniqueness of this representation in the case where f is representable as a Fourier transform of a function supported in the interval $(-a, +\infty)$ with $a > 0$ and $-\int \log(1 - |f|^2) < \pi/(2a)$ (see Theorem 2.5 in [6]).

Received by the editors June 26, 2006 and, in revised form, September 22, 2006.

2000 *Mathematics Subject Classification*. Primary 30E25; Secondary 30D50, 81U40.

©2007 American Mathematical Society
 Reverts to public domain 28 years from publication

The goal of this paper is to show that in fact *every* unimodularly bounded measurable function f on $\mathbb{T}$ admits a representation of the form (0.3) with unimodularly bounded holomorphic functions f_+ and f_- in $\mathbb{D}$ and $\mathbb{E}$, respectively. Since the right-hand side of (0.3) has the form of a Blaschke factor we propose the name *Blaschke representation* of f . One may think of it as a hyperbolic counterpart of the Riesz decomposition. This is supported by the fact that $|w| = 1$ is a natural bound for the values of all involved functions.

We remark that substituting $\overline{f_-}$ by f_- in the denominator of (0.3) would yield a special case of a so-called linear fractional conjugation boundary value problem investigated by Mityushev and Rogosin (Section 3.6 of [4]). However, the nice properties of the Blaschke representation hinge exactly on this complex conjugation.

In order to give a precise statement we denote by $L^p(\mathbb{T})$ (or L^p for short), with $1 \leq p \leq \infty$, the Lebesgue spaces of measurable functions on $\mathbb{T}$ with norm $\|\cdot\|_p$. The Hardy spaces of holomorphic functions in $\mathbb{D}$ are denoted by H_+^p , while H_-^p stands for the corresponding Hardy spaces of holomorphic functions in $\mathbb{E}$ which vanish at infinity. As usual, functions in $H_\pm^p$ are identified with their boundary functions and the Hardy spaces $H_\pm^p$ are considered as subspaces of L^p .

Theorem 0.1. *Let $f \in L^\infty$ be a bounded measurable function on $\mathbb{T}$ with $\|f\|_\infty \leq 1$.*

- (i) *There exist functions $f_+ \in H_+^\infty$ and $f_- \in H_-^\infty$ with $\|f_\pm\|_\infty \leq 1$ such that almost everywhere on $\mathbb{T}$,*

$$(0.4) \quad f = f_- \circ f_+ := \frac{f_+ + f_-}{1 + \overline{f_-} f_+}.$$

- (ii) *If $\|f\|_\infty < 1$, then the functions $f_\pm$ in the representation (0.4) are uniquely determined.*

- (iii) *If $f \in C^1(\mathbb{T})$ and $\|f\|_\infty < 1$, then also $\|f_\pm\|_\infty < 1$.*

If we subsequently speak of a Blaschke representation, we always assume that f_+ and f_- satisfy the conditions of assertion (i). The following lemma shows that then the right-hand side of (0.3) is well defined.

Lemma 0.2. *If $f_- \in H_-^\infty$, $f_+ \in H_+^\infty$ and $\|f_\pm\|_\infty \leq 1$, then $1 + \overline{f_-} f_+ \neq 0$ almost everywhere on $\mathbb{T}$.*

Proof. The function $1 + \overline{f_-} f_+$ is bounded on $\mathbb{T}$ and extends holomorphically into $\mathbb{D}$. If $\overline{f_-} f_+ = -1$ on a subset of positive measure, then, by Privalov's uniqueness theorem, $1 + \overline{f_-} f_+ = 0$ (almost everywhere) on $\mathbb{T}$. Since $|f_\pm| \leq 1$, this would imply that $|f_-| = |f_+| = 1$ and furthermore $f_+ = -f_-$ on $\mathbb{T}$, which is only possible if $f_+ \equiv f_- \equiv 0$. $\square$

The rest of the paper is devoted to the proof of Theorem 0.1. We start with proving assertion (ii) about uniqueness of f_- and f_+ .

Assume that a function $f \in L^\infty$ with $\|f\|_\infty < 1$ admits two Blaschke representations, $f = f_- \circ f_+ = g_- \circ g_+$. Then

$$(1 + \overline{f_-} f_+) f = f_+ + f_-, \quad (1 + \overline{g_-} g_+) f = g_+ + g_-,$$

and subtracting these equations yields

$$f \overline{f_-} (f_+ - g_+) + f g_+ (\overline{f_-} - \overline{g_-}) = f_+ - g_+ + f_- - g_-.$$

Setting $\Phi_+ := f_+ - g_+$, $\Phi_- := f_- - g_-$, and

$$a := -\frac{1}{1 - f\bar{f}_-}, \quad b := \frac{fg_+}{1 - f\bar{f}_-}$$

results in a (formally linear) transmission problem with conjugation,

$$(0.5) \quad \Phi_+ = a\Phi_- + b\bar{\Phi}_-.$$

Since $q := \|f\|_\infty < 1$ the functions a and b are bounded, $\operatorname{Re} a \leq -c < 0$, and $|b| \leq q|a|$ almost everywhere on $\mathbb{T}$. Consequently, by Corollary 4 of [5] (see also [3], Theorem 9.8 and Corollary 9.7), the homogeneous problem (0.5) has only the trivial solution, i.e., $\Phi_+ = f_+ - g_+ = 0$ and $\Phi_- = f_- - g_- = 0$.

Next we prove the existence of a Blaschke representation for continuously differentiable functions with $\|f\|_\infty < 1$. In preparing this proof we derive some auxiliary results.

For $1 < p < \infty$ let $H_\pm^\infty \cap W_p^1$ be the space of holomorphic functions in $H_\pm^\infty$ with boundary function in the Sobolev space $W_p^1(\mathbb{T})$, equipped with the norm of the boundary functions in $W_p^1(\mathbb{T})$. Since $W_p^1(\mathbb{T})$ is (compactly) embedded in $C(\mathbb{T})$ if $p > 1$, all boundary values of functions $f \in H_\pm^\infty \cap W_p^1$ exist as usual limits.

As a first result we show that holomorphic functions in $H_+^\infty \cap W_p^1$ that map the disk into a certain sector

$$S_\alpha := \{z \in \mathbb{D} : |\arg(1 - z)| < \alpha\}, \quad 0 < \alpha < \pi/2,$$

cannot access the corner of that sector (at $z = 1$) if p is sufficiently large.

Lemma 0.3. *Let $f \in H_+^\infty \cap W_p^1$ and $0 < \alpha < \pi/2$. If $f(z) \in \bar{S}_\alpha$ for all $z \in \mathbb{D}$ and $f(t) = 1$ for some $t \in \mathbb{T}$, then $p \leq \pi/(\pi - 2\alpha)$ or $f \equiv 1$.*

Proof. Assuming the contrary, we prove that $p > \pi/(\pi - 2\alpha)$ and $f \not\equiv 1$ lead to a contradiction. Without loss of generality we henceforth suppose that f equals one at $t = 1$.

Since $f(1) = 1$ and $f \not\equiv 1$, the open mapping principle for nonconstant holomorphic functions yields that $f(z) \in S_\alpha \subset \mathbb{D}$ for all $z \in \mathbb{D}$. Consequently, the function g defined by

$$g(z) := \frac{1 - f(z)}{1 - z}, \quad z \in \mathbb{D},$$

is holomorphic and has no zeros in $\mathbb{D}$. Moreover, g belongs to the Hardy spaces H_+^r for all $r \in (0, 1)$ (for the definition of these spaces see, for instance, [1]) and has a continuous boundary function on $\mathbb{T} \setminus \{1\}$. By virtue of $1 - f(1) = 0$ and $f \in W_p^1(\mathbb{T})$ it follows from Hardy's inequality (see, for instance, [7], Chap. 1, Sect. 5.1) that the boundary function of g belongs to $L^p(\mathbb{T})$, which, together with $g \in H_+^r$, implies that $g \in H_+^p$ (see [1], Chap. II, Cor. 4.3).

Since $f(z) \in S_\alpha$, i.e. $|\arg(1 - f(z))| \leq \alpha$, and $|\arg(1 - z)| \leq \pi/2$, it follows that $|\arg g(z)| \leq \alpha + \pi/2 < \pi$. By the assumption on p , made at the beginning of the proof, there is a positive ε such that

$$p > \frac{\pi}{\pi - 2\alpha - 4\varepsilon}$$

and we can choose ε so small that $d := \pi/2 - (\alpha + \varepsilon) > 0$. If $t \in \mathbb{T}$ is close to 1, then $\arg(1 - t)$ is close to $-\operatorname{sign}(\operatorname{Im} t) \cdot \pi/2$ and therefore we find a positive δ such

FIGURE 1. The arguments of g and h

that

$$\begin{aligned} d \leq \arg g(t) \leq \alpha + \pi/2 & \quad \text{if } t \in T_+ := \{t \in \mathbb{T} : |t-1| \leq \delta, \operatorname{Im} t > 0\}, \\ -\alpha - \pi/2 \leq \arg g(t) \leq -d & \quad \text{if } t \in T_- := \{t \in \mathbb{T} : |t-1| \leq \delta, \operatorname{Im} t < 0\}, \end{aligned}$$

which implies

$$\left| \arg g + \frac{\pi - \varepsilon}{2} \right| \leq \alpha + \frac{\varepsilon}{2} \quad \text{if } t \in T_-, \quad \left| \arg g - \frac{\pi - \varepsilon}{2} \right| \leq \alpha + \frac{\varepsilon}{2} \quad \text{if } t \in T_+.$$

In summary, the graph of $\tau \mapsto \arg g(e^{i\tau})$ lies in the shaded region shown in the left part of Figure 1.

In order to reduce the jump of $\arg g$ at $t = 1$ we remark that the argument of the function h defined by $h(t) := (1-t)^{-1+\varepsilon/\pi}$ depends linearly on the polar angle $\tau \in (0, 2\pi)$ with $\lim_{\tau \rightarrow 0} \arg h(e^{i\tau}) = (\varepsilon - \pi)/2$ as $\tau \rightarrow 2\pi$ and $\lim_{\tau \rightarrow 0} \arg h(e^{i\tau}) = (\pi - \varepsilon)/2$ as $\tau \rightarrow 0$. Decreasing δ , if necessary, we can ensure that

$$\left| \arg h + \frac{\pi - \varepsilon}{2} \right| \leq \frac{\varepsilon}{2} \quad \text{if } t \in T_-, \quad \left| \arg h - \frac{\pi - \varepsilon}{2} \right| \leq \frac{\varepsilon}{2} \quad \text{if } t \in T_+.$$

The graph of $\arg h$ is shown in the right part of Figure 1. The function hg^{-1} is holomorphic in $\mathbb{D}$. Its boundary function is continuous on $\mathbb{T} \setminus \{1\}$ and

$$|\arg(h(t)g^{-1}(t))| \leq \alpha + \varepsilon \quad \text{if } |t-1| \leq \delta.$$

Then Zygmund's Lemma (see Garnett [1], Sect. 3.3., Cor. 2.6, Koosis [2], Sect. V:D, Thm. 1, for instance) tells us that $hg^{-1} \in L^r$, provided $r < \pi/(2\alpha + 2\varepsilon)$. By Hölder's inequality we obtain from $g \in L^p$ and $hg^{-1} \in L^r$ that $h \in L^s$ if $1/s = 1/r + 1/p$, which is the case for all $s < \pi/(\pi - 2\varepsilon)$. This is in conflict with $h \notin L^s$ for $s = \pi/(\pi - \varepsilon)$. $\square$

Lemma 0.4. *Let $f = f_- \circ f_+ \in W_p^1(\mathbb{T})$ with $\|f\|_\infty \leq q < 1$ and $f_- \in H_-^\infty \cap W_p^1$. If*

$$(0.6) \quad \frac{\pi}{\pi - 2\alpha} < p < \infty, \quad \text{with } \alpha := \arcsin \frac{2q}{1 + q^2},$$

then $f_+ \in H_+^\infty \cap W_p^1$ and $\|f_\pm\|_\infty < 1$.

Proof. We assume that f_- and f_+ are not constant, because otherwise the result is trivial. Since

$$(0.7) \quad f_+ = \frac{f - f_-}{1 - \overline{f_-}f}$$

with $\|f\|_\infty < 1$, $\|f_-\|_\infty \leq 1$, and $W_1^p(\mathbb{T})$ is an algebra, we have $f_+ \in W_1^p(\mathbb{T})$. A direct computation of the absolute values of the Blaschke factors shows that $|f_\pm(z)| \leq 1$ and $|f_-(z) \circ f_+(z)| \leq q < 1$ imply that

$$g(z) := -\overline{f_-(z)} f_+(z) \in \overline{S}_\alpha, \quad z \in \mathbb{D}.$$

Assume now that $|f_-(t_0)| = 1$ for some $t_0 \in \mathbb{T}$. It can easily be seen that then $f = f_- \circ f_+$ with $\|f\|_\infty < 1$ necessarily implies $f_+(t_0) = -f_-(t_0)$ which yields $g(t_0) = |f_-(t_0)|^2 = 1$. Since g belongs to $H_+^\infty \cap W_p^1$ with $p > \pi/(\pi - 2\alpha)$ we infer from Lemma 0.3 that $g \equiv 1$; but this is impossible by Lemma 0.2. Finally, we conclude from the representation (0.7) and $\|f\|_\infty < 1$, $\|f_-\|_\infty < 1$ that also $\|f_+\|_\infty < 1$. $\square$

In order to prove the existence of a Blaschke representation we first assume that f belongs to $C^1(\mathbb{T})$ and $\|f\|_\infty \leq q < 1$. We fix p with $1 < p < \infty$ such that (0.6) is satisfied and work in the Sobolev space $W_p^1(\mathbb{T})$.

Step 1. With G defined by

$$G(t, z) := \frac{f(t) - z}{1 - f(t)\overline{z}}, \quad t \in \mathbb{T}, \quad z \in \overline{\mathbb{D}},$$

the Blaschke representation of f can be rewritten as

$$(0.8) \quad f_+(t) = G(t, f_-(t)), \quad t \in \mathbb{T}.$$

We prove that this *nonlinear Riemann-Hilbert problem* (or transmission problem, see [8]) has a (unique) solution $(f_+, f_-) \in H_+^\infty \times H_-^\infty$ with $\|f_\pm\|_\infty < 1$. In preparation we remark that the derivatives

$$\partial_z G(t, z) = \frac{1}{f(t)\overline{z} - 1}, \quad \partial_{\overline{z}} G(t, z) = \frac{f(t) - z}{(f(t)\overline{z} - 1)^2} f(t)$$

are continuous on $\mathbb{T} \times \overline{\mathbb{D}}$. The derivative $\partial_\tau G \equiv i t \partial_t G$ of G with respect to the polar angle τ of $t \equiv e^{i\tau}$ is equal to

$$\partial_\tau G(t, z) = \frac{1 - |z|^2}{(f(t)\overline{z} - 1)^2} \partial_\tau f(t).$$

This function is continuous in z for almost all $t \in \mathbb{T}$ and belongs to $L^p(\mathbb{T})$ with respect to t for all $z \in \overline{\mathbb{D}}$. Moreover, for all $(t, z) \in \mathbb{T} \times \overline{\mathbb{D}}$,

$$-\operatorname{Re} \partial_z G(t, z) \geq \frac{1}{2} \quad \text{and} \quad \frac{|\partial_{\overline{z}} G(t, z)|}{|\partial_z G(t, z)|} = \left| \frac{f(t) - z}{1 - f(t)\overline{z}} \right| \cdot |f(t)| \leq q < 1.$$

Step 2. We construct a compact operator that reduces (0.8) to a fixed point equation. For given $f \in W_p^1(\mathbb{T})$ with $\|f\|_\infty < 1$, the operator K_f is defined on the closed unit ball

$$B := \{u \in C(\mathbb{T}) : \|u\|_\infty \leq 1\}$$

of $C(\mathbb{T})$. For $u \in B$ we define

$$(0.9) \quad a(t) := \partial_z G(t, u(t)), \quad b(t) := \partial_{\overline{z}} G(t, u(t)), \quad c(t) := i t \partial_t G(t, u(t)).$$

The functions a and b are continuous, while c belongs to the Lebesgue space $L^p(\mathbb{T})$. The norms $\|a\|_\infty$, $\|b\|_\infty$ and $\|c\|_p$ are bounded independently of u and

$$\frac{|b(t)|}{|a(t)|} \leq q < 1, \quad -\operatorname{Re} a \geq \frac{1}{2}.$$

In particular, the winding number of a about zero vanishes. These properties guarantee that the linear transmission problem with conjugation

$$(0.10) \quad \Phi_+ = a \Phi_- + b \overline{\Phi_-} + c$$

has a unique solution $\Phi_+ \in H_+^p$, $\Phi_- \in H_-^p$ for each p with $1 < p < \infty$ ([3], Theorem 9.8 and Corollary 9.7). Setting

$$(0.11) \quad \widehat{\Phi_-}(e^{i\tau}) := \int_0^\tau \Phi_-(e^{i\sigma}) d\sigma, \quad P_0 \widehat{\Phi_-} := \frac{1}{2\pi} \int_0^{2\pi} \widehat{\Phi_-}(e^{i\sigma}) d\sigma,$$

the operator $K_f: B \rightarrow W_p^1(\mathbb{T})$ is given by $K_f(u) := \widehat{\Phi_-} - P_0 \widehat{\Phi_-}$. The definition of $\widehat{\Phi_-}$ makes sense since $P_0 \Phi_- = 0$.

We summarize relevant properties of the operator K . Let W be the (nonclosed) subspace

$$W := \{f \in W_p^1(\mathbb{T}) : \|f\|_\infty < 1\}$$

of $W_p^1(\mathbb{T})$.

Lemma 0.5. *The following assertions hold for all p with $1 < p < \infty$:*

- (i) *For all $f \in W_p^1(\mathbb{T})$ with $\|f\|_\infty < 1$ the operator $K_f: B \subset C(\mathbb{T}) \rightarrow W_p^1(\mathbb{T})$ is continuous and its image is bounded.*
- (ii) *The mapping $K: W \times B \rightarrow W_p^1(\mathbb{T})$, $(f, u) \mapsto K_f(u)$ is continuous.*

Proof. To begin with we remark that the following superposition operators are continuous:

$$(0.12) \quad C(\mathbb{T}) \rightarrow C(\mathbb{T}): u \mapsto a := \partial_z G(., u),$$

$$(0.13) \quad C(\mathbb{T}) \rightarrow C(\mathbb{T}): u \mapsto b := \partial_{\bar{z}} G(., u),$$

$$(0.14) \quad C(\mathbb{T}) \rightarrow L^p(\mathbb{T}): u \mapsto c := i t \cdot \partial_t G(., u).$$

For fixed continuous functions a and b , the transmission problem (0.10) defines a (real linear) operator

$$T_{a,b}: H_+^p \times H_-^p \rightarrow L^p, (\Phi_+, \Phi_-) \mapsto c := \Phi_+ - a \Phi_- - b \overline{\Phi_-}.$$

The properties of a and b guarantee that $T_{a,b}$ is continuously invertible ([3] loc.cit.), which shows the continuity of the mapping

$$S_{a,b}: L^p \rightarrow H_-^p, c \mapsto \Phi_-.$$

Since $(a, b) \mapsto T_{a,b}$ is a continuous mapping of $C(\mathbb{T}) \times C(\mathbb{T})$ into the space of bounded linear operators $\mathcal{L}(H_+^p \times H_-^p, L^p)$, the mapping $(a, b) \mapsto S_{a,b}$ is continuous from $C(\mathbb{T}) \times C(\mathbb{T})$ into $\mathcal{L}(L^p, H_-^p)$, which together with the above results implies that $u \mapsto \Phi_-$ is continuous from $C(\mathbb{T})$ into H_-^p . Integrating Φ_- along $\mathbb{T}$ gives $K_f(u)$, and because integration is a continuous operator from H_-^p into $W_p^1(\mathbb{T})$ we get the continuity claimed in assertion (i).

A similar reasoning proves the continuity statement of assertion (ii).

In order to see that the image of K_f is bounded, we remark that the functions a and b defined by (0.12), (0.13) satisfy the estimates

$$\|a\|_\infty \leq C, \quad \|b\|_\infty \leq C, \quad |b|/|a| \leq q < 1$$

with constants not depending on u . This implies a uniform bound on the norm of the operator $S_{a(u), b(u)}$ (see [8]) and thus on the norm of $\Phi_- := S_{a,b} c$ in H_-^p . $\square$

Corollary 0.6. *The operator $K_f: B \subset C(\mathbb{T}) \rightarrow C(\mathbb{T})$ is compact.*

The assertion follows from the last lemma and the compactness of the embedding $W_p^1(\mathbb{T}) \rightarrow C(\mathbb{T})$.

Step 3. The next result shows in which way the functions f_+ and f_- of the representation (0.3) are related to the fixed points of K_f .

Lemma 0.7. *Let $f \in C^1(\mathbb{T})$ with $\|f\|_\infty \leq q < 1$ and choose $p > 1$ according to (0.6).*

- (i) *If $f = f_- \circ f_+$ is a Blaschke representation of f and if $f_- \in W_p^1(\mathbb{T})$ satisfies $\|f_-\|_\infty < 1$, then f_- is a fixed point of K_f .*
- (ii) *If $u \in B$ is a fixed point of K_f , then $f_- := u$ belongs to $H_-^\infty \cap W_p^1$, $\|f_-\|_\infty < 1$, $f_+ := (f - u)/(1 - f\bar{u})$ lies in $H_+^\infty \cap W_p^1$, and $f = f_- \circ f_+$ is a Blaschke representation of f .*

Proof. (i) If $f = f_- \circ f_+$ with $f_- \in W_p^1(\mathbb{T})$, then also $f_+ \in W_p^1(\mathbb{T})$. Differentiating the boundary relation $f_+(t) = G(t, f_-(t))$ with respect to the polar angle τ we obtain that the pair $(\Phi_+, \Phi_-) := (\partial_\tau f_+, \partial_\tau f_-)$ is the (unique) solution of the transmission problem (0.10). Consequently,

$$K_f(f_-)(e^{i\tau}) = q + \int_0^\tau \Phi_-(e^{i\sigma}) d\sigma = q + \int_0^\tau \partial_\sigma f_-(e^{i\sigma}) d\sigma,$$

where the constant q is chosen so that the mean value of $K_f(f_-)$ over $\mathbb{T}$ vanishes. The right-hand side differs from f_- at most by a constant, and since

$$\int f_-(e^{i\tau}) d\tau = 2\pi f_-(\infty) = 0$$

this constant must be zero, i.e. $K_f(f_-) = f_-$.

(ii) Now let $f_- \equiv u \in B$ be a fixed point of K_f and $f_+ := G(., f_-)$. First of all, f_- and consequently f_+ belong to $W_p^1(\mathbb{T})$.

The solutions Φ_+ and Φ_- of the transmission problem (0.10) are holomorphic in $\mathbb{D}$ and $\mathbb{E}$, respectively, and $\hat{\Phi}_-$ defined by (0.11) is holomorphic in $\mathbb{E}$. Since

$$(0.15) \quad f_- = K_f(f_-) = \hat{\Phi}_- + q,$$

the function f_- extends holomorphically into $\mathbb{E}$, and since the mean value of $K_f(f_-)$ over $\mathbb{T}$ vanishes by definition of K_f , we have $f_- \in H_-^\infty \cap W_p^1$.

The relation (0.15) implies that $\Phi_- = \partial_\tau f_-$, and inserting this into (0.10) shows that

$$\Phi_+ = a\Phi_- + b\bar{\Phi}_- + c = \partial_\tau (G(., f_-)) = \partial_\tau f_+.$$

Consequently $\partial_\tau f_+$ and hence f_+ is holomorphic in $\mathbb{D}$. Finally, Lemma 0.4 tells us that $\|f_\pm\|_\infty < 1$. $\square$

Step 4. In particular, it follows from the preceding step that K has no fixed points on the boundary of the ball B and thus the Leray–Schauder fixed point index $\text{fix}_B K$ of K with respect to B is well defined. In order to determine this index we consider the homotopy $\lambda \mapsto f_\lambda := \lambda f$ with $\lambda \in [0, 1]$.

With any function f_λ we associate the corresponding operator $K_{\lambda f}$. The operators $K_{\lambda f}$ are compact and depend continuously on λ . Since, by Lemma 0.7 (ii), there are no fixed points of $K_{\lambda f}$ on ∂B , the fixed point index $\text{fix}_B K_{\lambda f}$ of $K_{\lambda f}$ with respect to B is well defined for all λ and, by homotopy invariance, it does not depend on λ .

For $\lambda = 0$ we have $f_0 \equiv 0$ and $K_0 = 0$, which gives $\text{fix}_B K_0 = 1$, so that also $\text{fix}_B K_f = 1$. Hence K_f has a fixed point in B , which gives rise to a Blaschke representation of f .

In the final part of the proof of Theorem 0.1 we drop the additional assumptions on f and assume that $f \in L^\infty$ with $\|f\|_\infty \leq 1$. Then there exists a sequence (f_n) of infinitely differentiable functions on $\mathbb{T}$ such that $\|f_n\|_\infty < 1$ and $f_n \rightarrow f$ in L^2 . For instance, let $f_n(t) := rPf(rt)$, where Pf is the harmonic extension of f to $\mathbb{D}$ and $r := 1 - 1/n$.

The smooth functions f_n admit a (unique) Blaschke representation $f_n = f_n^- \circ f_n^+$ with $\|f_n^\pm\|_\infty \leq 1$. Since the sequences (f_n^+) and (f_n^-) are also bounded in L^2 they contain subsequences converging weakly in L^2 , and we can assume that $f_n^+ \rightharpoonup f_+$ and $f_n^- \rightharpoonup f_-$. In order to show that $f_n^- \circ f_n^+ \rightharpoonup f_- \circ f_+$ we need the following results, which can easily be verified.

Lemma 0.8. *Let $f_n^\pm \in L^\infty$ and $\|f_n^\pm\|_\infty \leq 1$ and assume that $f_n^\pm \rightharpoonup f_\pm$ in L^2 . Then*

- (i) *the limits $f_\pm$ are in L^∞ and $\|f_\pm\|_\infty \leq 1$;*
- (ii) *the product $f_n^+ f_n^-$ converges weakly to $f_+ f_-$ in L^2 ;*
- (iii) *if $f_n^\pm \in H_\pm^\infty$, then $f_\pm \in H_\pm^\infty$.*

Using these facts we take the weak limits in $f(1 + f_n^+ \overline{f_n^-}) = f_n^+ + f_n^-$ and obtain

$$(0.16) \quad f(1 + f_+ \overline{f_-}) = f_+ + f_-.$$

Since $f_- \in H_-^\infty$, the function $1 + f_+ \overline{f_-}$ belongs to H_+^∞ . So, by Privalov's uniqueness theorem, either $1 + f_+ \overline{f_-} \equiv 0$ or $1 + f_+ \overline{f_-} \neq 0$ almost everywhere on $\mathbb{T}$. The first case can be excluded, since then equation (0.16) implies that $f_+ \equiv f_- \equiv 0$; in the second case $f = f_- \circ f_+$ almost everywhere on $\mathbb{T}$. This completes the proof of Theorem 0.1.

The following examples illustrate the limitations of the results. The first one shows that the Blaschke representation need not be unique if $\|f\|_\infty = 1$.

Example 1. The function $f(t) = \overline{t}$ admits two representations $f = f_- \circ f_+ = g_- \circ g_+$ with $f_-(z) = g_-(z) = 1/z$, $f_+(z) = 1$, and $g_+(z) = z$. Indeed,

$$f(t) = \overline{t} = \frac{1 + 1/t}{1 + 1/\overline{t} \cdot 1} = \frac{t + 1/t}{1 + 1/\overline{t} \cdot t} \quad \text{almost everywhere on } \mathbb{T}.$$

The second example shows that there are in fact Blaschke representations of functions $f \in L^\infty$ with $\|f\|_\infty < 1$ and $\|f_\pm\|_\infty = 1$.

Example 2. If $p < 1/\lambda$ the functions $f_+(t) = 1 - (1-t)^\lambda$ and $f_-(t) = -1 + (1-1/t)^\lambda$ with $0 < \lambda < 1$ belong to $H_\pm^\infty \cap W_p^1$ and satisfy $\|f_\pm\|_\infty = 1$. A straightforward calculation shows that $f := f_- \circ f_+$ has norm $\|f\|_\infty$ less than one.

Note that the function f is not continuous, but has a jump of absolute value $\sin(\lambda\pi)$ at $t = 1$. So this example also demonstrates that f_+ and f_- can both have higher regularity than f . Figure 2 shows the images of the unit circle under the mappings f , f_+ and f_- for $\lambda = 1/2$.

We have no counterexample of this type where the function f itself belongs to $W_p^1(\mathbb{T})$, and we conjecture that such functions do not exist.

Conjecture. Let $f \in W_p^1(\mathbb{T})$ with $1 < p < \infty$. If f has the Blaschke representation $f = f_- \circ f_+$ and $\|f\|_\infty < 1$, then $f_\pm \in W_p^1(\mathbb{T})$ and $\|f_\pm\|_\infty < 1$.

FIGURE 2.

FIGURE 3.

On the other hand, the third example demonstrates that the conjecture cannot be extended to continuous functions f .

Example 3. Let S be the domain bounded by three circular arcs with centers at $1+i$, $1-i$, 0 , and radii 1 , 1 , and $\sqrt{2}-1$, respectively (S is the white domain in Figure 3). Furthermore, let f_+ be the conformal mapping of $\mathbb{D}$ onto S , normalized so that $f_+(0) = 0$, $f_+(1) = 1$, and let $f_-(t) := -\overline{f_+}(t)$. By the Carathéodory–Osgood theorem the boundary function of f_+ , and consequently of f_- , are continuous on $\mathbb{T}$. It can be easily seen that $f := f_- \circ f_+$ is continuous on $\mathbb{T}$ and satisfies $\|f\|_\infty < 1$, while $\|f_\pm\|_\infty = 1$.

Finally, we show that the hyperbolic *Parseval equation*, obtained by Sylvester and Winebrenner [6] for the half-plane, also holds for the unit disc, and that it remains in force even if the energy defined by

$$E(f) := - \int_{\mathbb{T}} \ln(1 - |f|^2) |dt|$$

is infinite.

Theorem 0.9. *If $f = f_- \circ f_+$ is a Blaschke representation of $f \in L^\infty$ with $\|f\|_\infty \leq 1$, then*

$$(0.17) \quad E(f) = E(f_-) + E(f_+).$$

Proof. By definition, the energy is always nonnegative, so the sum $E(f_-) + E(f_+)$ makes sense also if one or both summands are infinite. If $|f| = 1$ on a set of positive measure, then at least one of $|f_+|$ or $|f_-|$ must be one on this set and both sides of (0.17) are infinite. If $|f| < 1$ almost everywhere on $\mathbb{T}$, we have

$$|1 + \overline{f_-} f_+|^2 (1 - |f|^2) = |1 + \overline{f_-} f_+|^2 - |f_+ + f_-|^2 = (1 - |f_-|^2)(1 - |f_+|^2)$$

and hence

$$(0.18) \quad \ln(1 - |f|^2) = \ln(1 - |f_-|^2) + \ln(1 - |f_+|^2) - 2 \ln |1 + \bar{f}_- f_+|.$$

By the maximum principle, $|f_{\pm}(z)| < 1$ in $\mathbb{D}$. The function $g := \log(1 + \bar{f}_- f_+)$ extends holomorphically into $\mathbb{D}$ with $g(0) = 0$ and its imaginary part is bounded by $\pm\pi/2$. Thus g belongs to H^p for all $p < \infty$ and

$$\int_{\mathbb{T}} \ln |1 + \bar{f}_- f_+| |dt| = \operatorname{Re} \int_{\mathbb{T}} g(t) |dt| = 2\pi \operatorname{Re} g(0) = 0.$$

The two remaining functions on the right-hand side of (0.18) are measurable and nonpositive. So their integrals are either both finite, in which case $E(f) = E(f_+) + E(f_-) < +\infty$, or at least one integral is $-\infty$, in which case $E(f_+) + E(f_-) = +\infty = E(f)$. $\square$

REFERENCES

1. Garnett, J.B.: Bounded Analytic Functions. Academic Press, 1981. MR628971 (83g:30037)
2. Koosis, P.: Introduction to H_p spaces (2nd edition). Cambridge University Press, 1998. MR565451 (81c:30062)
3. Litvinchuk, G.M., and I.M. Spitkovsky: Factorization of Measurable Matrix Functions. Birkhäuser, Basel, 1987. MR1015716 (90g:47030)
4. Mityushev, V.V., and S.V. Rogosin: Constructive Methods for Linear and Nonlinear Boundary Value Problems for Analytic Functions. Chapman and Hall/CRC Press, 2000. MR1739063 (2001d:30075)
5. Spitkovsky, I.M.: On the theory of the generalized Riemann boundary value problem in L_p spaces (Russian). Ukrain. Mat. J. **31**, 1 (1979) 63–73. MR523157 (81e:30053)
6. Sylvester, J., and D.P. Winebrenner: Linear and nonlinear inverse scattering. SIAM J. Appl. Math. **59**, No.2, 669–699 (1999). MR1654391 (2000a:34021)
7. Triebel, H.: The Structure of Functions. Birkhäuser, 2001. MR1851996 (2002k:46087)
8. Wegert, E., Khimshiashvili, G., Spitkovsky, I.: Nonlinear transmission problems. Mem. Differ. Equ. Math. Phys. **12**, 223–230 (1997). MR1636880 (99g:47058)

INSTITUTE OF APPLIED ANALYSIS, TU BERGAKADEMIE FREIBERG, 09596 FREIBERG, GERMANY
E-mail address: `wegert@math.tu-freiberg.de`

INSTITUTE OF APPLIED ANALYSIS, TU BERGAKADEMIE FREIBERG, 09596 FREIBERG, GERMANY
E-mail address: `wolfersd@math.tu-freiberg.de`