NAYATANI'S METRIC AND CONFORMAL TRANSFORMATIONS OF A KLEINIAN MANIFOLD

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ABSTRACT. According to Schoen and Yau (1988), an extensive class of conformally flat manifolds is realized as Kleinian manifolds. Nayatani (1997) constructed a metric on a Kleinian manifold M which is compatible with the canonical flat conformal structure. He showed that this metric g_N has a large symmetry if g_N is a complete metric. Under certain assumptions including the completeness of g_N , the isometry group of (M, g_N) coincides with the conformal transformation group of M . In this paper, we show that g_N may have a large symmetry even if g_N is not complete. In particular, every conformal transformation is an isometry when (M, g_N) corresponds to a geometrically finite Kleinian group.

1. INTRODUCTION

Let M be a differentiable manifold of dimension $n \geq 3$ with a conformal structure $\mathcal{C}$ which is an equivalence class of metrics on M conformal to each other. $(M, \mathcal{C})$ is said to be *conformally flat* if for some (hence for any) $g \in \mathcal{C}$, g has an expression of the form

$$g = \lambda(x) \sum_{k=1}^n (dx^k)^2$$

for some coordinates and function $\lambda > 0$ locally. Schoen and Yau [9] showed that there is a large class of conformally flat manifolds that can be realized as Kleinian manifolds. In particular, this class contains all compact conformally flat manifolds $(M, \mathcal{C})$ of dimension $n \geq 4$ that satisfy $R_g > 0$ for some $g \in \mathcal{C}$, where R_g is the scalar curvature of g . Here a Kleinian manifold is of the form Ω/Γ where Ω is a domain of S^n and Γ is a Kleinian group acting freely and properly discontinuously on Ω (see the details below).

We let B^{n+1} be the Poincaré disk model of hyperbolic $n + 1$ space with the Poincaré metric g_H . Each isometry of (B^{n+1}, g_H) naturally extends to a diffeomorphism on the ideal boundary $S^n (= \partial B^{n+1})$ as a conformal transformation with respect to the standard metric g_0 . In this way, the isometry group of B^{n+1} may

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be identified with the conformal transformation group of S^n : $\text{Isom}(B^{n+1}, g_H) \cong \text{Conf}(S^n, g_0)$. Throughout this paper, we regard B^{n+1} and S^n as subsets of $\mathbb{R}^{n+1}$, that is,

$$B^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| < 1\}, \quad S^n = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}.$$

Let Γ be a Kleinian group, that is, an infinite discrete subgroup of the conformal transformation group $\text{Conf}(S^n, g_0)$. The largest open subset of S^n on which Γ acts properly discontinuously is called the *domain of discontinuity* of Γ , denoted by $\Omega(\Gamma)$. The set $\Omega(\Gamma)$ is characterized as the complement $S^n \setminus \Lambda(\Gamma)$, where $\Lambda(\Gamma)$ is the set of accumulation points of Γx , $x \in B^{n+1}$, in $\overline{B^{n+1}}$ ($= B^{n+1} \cup S^n$). Notice that $\Lambda(\Gamma)$, called the *limit set* of Γ , is independent of the choice of $x \in B^{n+1}$. Nayatani [6] constructed a metric on $\Omega(\Gamma)$ which is compatible with the canonical flat conformal structure induced by g_0 :

$$(1.1) \quad g_N = \left(\int_{\Lambda(\Gamma)} \left(\frac{2}{|y - \xi|^2} \right)^{\delta(\Gamma)} d\mu_{\delta(\Gamma)}(\xi) \right)^{2/\delta(\Gamma)} g_0, \quad y \in \Omega(\Gamma),$$

where $\delta(\Gamma)$ denotes the critical exponent of Γ and $\mu_{\delta(\Gamma)}$ is a Patterson-Sullivan measure of exponent $\delta(\Gamma)$ (see the definitions in Sect. 2).

Sullivan [11] proved the following theorem on the uniqueness of $\mu_{\delta(\Gamma)}$ for a large class of Kleinian groups.

Theorem 1.1 (Sullivan [11]). *Suppose that Γ is a geometrically finite Kleinian group. Then the critical exponent $\delta(\Gamma)$ equals the Hausdorff dimension $d(\Gamma)$ of the limit set $\Lambda(\Gamma)$. Moreover if Γ is nonelementary, there exists, up to a constant multiple, one and only one Patterson-Sullivan measure of exponent $\delta(\Gamma)$.*

We say that Γ is *convex cocompact* if the quotient $C(\Lambda(\Gamma))/\Gamma$ is compact, where $C(\Lambda(\Gamma))$ is the hyperbolic convex hull of $\Lambda(\Gamma)$ in B^{n+1} . Such a Γ is characterized as a geometrically finite group without parabolic elements. It is known that if Γ is nonelementary and convex cocompact, the unique Patterson-Sullivan measure $\mu_{\delta(\Gamma)}$ coincides (up to a constant multiple) with the $\delta(\Gamma)$ -dimensional Hausdorff measure $\mathcal{H}^{\delta(\Gamma)}$ on the limit set $\Lambda(\Gamma)$ of Γ .

Nayatani's metric g_N is Γ -invariant. Therefore it projects to a conformally flat metric on the quotient manifold Ω/Γ if Γ acts freely and properly discontinuously on a domain $\Omega \subset \Omega(\Gamma)$. We denote this metric on Ω/Γ by the same symbol g_N . Nayatani's metric g_N , by construction, has remarkable properties in connection with Kleinian group theory. For instance, there is a close relation between the curvature of g_N and the critical exponent $\delta(\Gamma)$ of Γ . Namely, if the limit set of Γ contains at least two points, the scalar curvature of g_N is positive (resp. zero, negative) if and only if $\delta(\Gamma) < (\text{resp. } =, >) (n-2)/2$ (see [6, Theorem 3.3]). Besides this, g_N has the following property, which gives the motivation of this paper.

Theorem 1.2 (Nayatani [6]). *Let $M = \Omega/\Gamma$ be a Kleinian manifold with $\delta(\Gamma) > 0$ and g_N a Nayatani's metric. Suppose that $\mu_{\delta(\Gamma)}$ is, up to a constant multiple, the unique Patterson-Sullivan measure of exponent $\delta(\Gamma)$ and that g_N is complete. Then the isometry group $\text{Isom}(M, g_N)$ coincides with the conformal transformation group $\text{Conf}(M, g_N)$.*

This theorem says that g_N has a large symmetry if g_N is a complete metric. In particular, by Theorem 1.1, we have that $\text{Conf}(M, g_N) = \text{Isom}(M, g_N)$ for a

Kleinian manifold M which corresponds to a convex cocompact group Γ . The purpose of this paper is to show that g_N may have a large symmetry even if g_N is not complete. More precisely speaking,

Theorem 3.5. *Let $M = \Omega/\Gamma$ be a Kleinian manifold with $\delta(\Gamma) > 0$ and g_N a Nayatani's metric on M . Suppose that Γ is geometrically finite. Then the isometry group $\text{Isom}(M, g_N)$ coincides with the conformal transformation group $\text{Conf}(M, g_N)$.*

Theorem 3.5 is not a direct consequence of Theorem 1.2. Indeed Nayatani's metric g_N is not necessarily complete even if Γ is geometrically finite. In fact, Maubon [5] gave the necessary and sufficient condition for g_N to be complete/incomplete when Γ is geometrically finite.

This paper is organized as follows. In Sect. 2, we review some facts on Patterson-Sullivan measures and Nayatani's metric. In Sect. 3, we shall prove Theorem 3.5, which is a result for geometrically finite groups. In Sect. 4, we consider the case that Kleinian groups are not necessarily geometrically finite.

2. PATTERSON-SULLIVAN MEASURES AND NAYATANI'S METRIC

We start this section with the definition and fundamental properties of Patterson-Sullivan measures. More details may be found in [8] for example.

For a Kleinian group Γ , the *critical exponent* $\delta(\Gamma)$ is defined by

$$\delta(\Gamma) = \inf \left\{ s > 0 \mid \sum_{\gamma \in \Gamma} \exp \{ -s d_H(x, \gamma y) \} < \infty \right\},$$

where $x, y \in B^{n+1}$ and d_H is the hyperbolic distance function on B^{n+1} . It is easy to see that $\delta(\Gamma)$ is independent of the choice of $x, y \in B^{n+1}$. It is known that $0 \leq \delta(\Gamma) \leq n$ (see [8, Theorem 1.6.1]). A Kleinian group Γ is called *nonelementary* if $\Lambda(\Gamma)$ contains at least three points. If Γ is nonelementary, $\delta(\Gamma) > 0$ (see [8, Corollary 3.4.5]).

Definition 2.1. For a Kleinian group Γ and $\rho > 0$, we define a *Patterson-Sullivan measure* μ_ρ of exponent ρ as a positive finite Borel measure on S^n which satisfies the following conditions:

- (i) μ_ρ is supported on $\Lambda(\Gamma)$.
- (ii)

$$(2.1) \quad \gamma^* \mu_\rho = |\gamma'|^\rho \mu_\rho, \quad \gamma \in \Gamma,$$

where $|\gamma'|$ denotes the linear stretch factor of γ defined by

$$\gamma^* g_0 = |\gamma'|^2 g_0.$$

The condition (2.1) implies that a Patterson-Sullivan measure of exponent ρ locally behaves like the ρ -dimensional Hausdorff measure on $\Lambda(\Gamma)$. Notice that a Patterson-Sullivan measure μ_ρ is nothing but a measure σ_o , where o is the origin of B^{n+1} and $\{\sigma_x\}_{x \in B^{n+1}}$ is a Γ -invariant conformal density of dimension ρ (see [8, Chapter 4]). Here a Γ -invariant conformal density of dimension ρ is a family of positive finite Borel measures $\sigma_x, x \in B^{n+1}$, on S^n that are supported on $\Lambda(\Gamma)$ and satisfy the following transformation laws:

$$(2.2) \quad \gamma^* \sigma_x = \sigma_{\gamma^{-1}x}, \quad x \in B^{n+1}, \gamma \in \Gamma,$$

$$(2.3) \quad \sigma_x = P(x, \cdot)^\rho \sigma_o, \quad x \in B^{n+1}, \gamma \in \Gamma,$$

where P is defined by

$$(2.4) \quad P(x, y) = \frac{1 - |x|^2}{|x - y|^2}, \quad x \in B^{n+1}, y \in S^n.$$

Set $\sigma = \sigma_x$. Then by (2.2) and (2.3) we have

$$(2.5) \quad \gamma^* \sigma = \left(\frac{P(\gamma^{-1}x, \cdot)}{P(x, \cdot)} \right)^\rho \sigma, \quad \gamma \in \Gamma.$$

We say that a measure σ on S^n is a *Patterson-Sullivan measure of exponent ρ with basepoint x* if σ is a positive finite Borel measure supported on $\Lambda(\Gamma)$ that satisfies (2.5). Let $x \in B^{n+1}$ and $\alpha \in \text{Conf}(S^n, g_0)$. We regard α as a diffeomorphism of $B^{n+1} \cup S^n$ through the identification $\text{Isom}(B^{n+1}, g_H) \cong \text{Conf}(S^n, g_0)$. Then the linear stretch factor of α with respect to g_0 is given by

$$(2.6) \quad |\alpha'(y)| = \frac{P(x, y)}{P(\alpha x, \alpha y)}$$

$$(2.7) \quad = P(\alpha^{-1}o, y),$$

where $x \in B^{n+1}$, $y \in S^n$ and o is the origin of B^{n+1} . Patterson-Sullivan measures were introduced by Patterson [7] and Sullivan [10]. They constructed such a measure in the case of exponent $\delta(\Gamma)$. It is known that if there exists a Patterson-Sullivan measure μ_ρ of exponent ρ , then $\rho \geq \delta(\Gamma)$ (see [8, Corollary 4.5.3]). However, complete answers are not known to the existence and uniqueness problems of Patterson-Sullivan measures μ_ρ of exponent $\rho > \delta(\Gamma)$.

Let Γ be a Kleinian group with $\delta(\Gamma) > 0$. Nayatani's metric (1.1) on $\Omega(\Gamma)$ is naturally extended to a metric on B^{n+1} as the following:

$$(2.8) \quad \begin{aligned} g_N &= \left(\int_{\Lambda(\Gamma)} \left(\frac{2}{|x - \xi|^2} \right)^{\delta(\Gamma)} d\mu_{\delta(\Gamma)}(\xi) \right)^{2/\delta(\Gamma)} g_E \\ &= \left(\int_{\Lambda(\Gamma)} \left(\frac{1 - |x|^2}{|x - \xi|^2} \right)^{\delta(\Gamma)} d\mu_{\delta(\Gamma)}(\xi) \right)^{2/\delta(\Gamma)} g_H, \end{aligned}$$

where $x \in B^{n+1}$ and g_E denotes the Euclidean metric (see [3]). This metric g_N is also Γ -invariant on B^{n+1} . Therefore if Γ acts freely and properly discontinuously on B^{n+1} , g_N projects to a metric, denoted by the same symbol g_N , on the quotient manifold B^{n+1}/Γ . Since Γ acts on B^{n+1} as isometries of (B^{n+1}, g_H) , B^{n+1}/Γ is a hyperbolic manifold.

Throughout this paper, we assume that a Kleinian manifold M is the quotient of a *connected* open subset $\Omega \subset \Omega(\Gamma)$ by Γ , $M = \Omega/\Gamma$, with the canonical flat conformal structure $\mathcal{C}_0$. Here Ω is Γ -invariant and Γ acts freely on Ω . We also assume $\dim M \geq 3$.

Our major concern in this paper is with the isometry group of a Kleinian manifold (M, g_N) . For the conformal transformation group of $(M, \mathcal{C}_0)$, we know the following.

Proposition 2.2 (Nayatani [6]). *Let $M = \Omega/\Gamma$ be a Kleinian manifold. Then the conformal transformation group $\text{Conf}(M)$ of M is isomorphic to $(N(\Gamma) \cap \text{Conf}(\Omega, g_0))/\Gamma$, where $N(\Gamma)$ denotes the normalizer of Γ in $\text{Conf}(S^n, g_0)$.*

We shall now sketch a proof of Proposition 2.2. For $\alpha \in N(\Gamma) \cap \text{Conf}(\Omega, g_0)$, we define $\underline{\alpha} \in \text{Conf}(M)$ as follows:

$$(2.9) \quad \underline{\alpha}(\Gamma y) = \Gamma \alpha y, \quad y \in \Omega,$$

where for $x \in \Omega$, Γx is regarded as a point in M . Note that $\underline{\alpha}$ is well defined. Letting $f(\alpha) = \underline{\alpha}$, we obtain a homomorphism $f : N(\Gamma) \cap \text{Conf}(\Omega, g_0) \rightarrow \text{Conf}(M)$. We can verify that f is surjective and the kernel of f coincides with Γ (see [6, Lemma 4.1]). Therefore $\text{Conf}(M)$ is isomorphic to $(N(\Gamma) \cap \text{Conf}(\Omega, g_0)) / \Gamma$.

3. GEOMETRICALLY FINITE CASE

In this section, we give a proof of Theorem 3.5 which is a result for geometrically finite Kleinian groups. A Kleinian group Γ is called *geometrically finite* if it has some convex fundamental polyhedron in B^{n+1} with finitely many faces. Geometrical finiteness is one of the most important notions in Kleinian group theory, particularly from the viewpoint of hyperbolic geometry. In the context of hyperbolic 3-manifolds, several equivalent definitions of geometrical finiteness are given by Beardon and Maskit [1], Marden [4], and Thurston [12]. Moreover Bowditch [2] gave equivalent definitions of geometrical finiteness in general dimension. The following is one of them:

(GF) For some $\epsilon > 0$, the ϵ -thick part of the convex core $C_\Gamma \subset B^{n+1}/\Gamma$ is compact. Here the convex core is defined by $C_\Gamma = C(\Lambda(\Gamma))/\Gamma$, where $C(\Lambda(\Gamma))$ is the hyperbolic convex hull of $\Lambda(\Gamma)$ in B^{n+1} .

We shall now show the following lemma, which is obtained by combining [6, Lemma 4.2] with [6, Lemma 4.3].

Lemma 3.1. *Let Γ be a Kleinian group with $\delta(\Gamma) > 0$, g_N a Nayatani's metric, and α an element of $N(\Gamma)$. Suppose that $\mu_{\delta(\Gamma)}$ is, up to a constant multiple, the unique Patterson-Sullivan measure of exponent $\delta(\Gamma)$. Then α is a homothety with respect to g_N ; that is,*

$$(3.1) \quad \alpha^* g_N = c g_N$$

for some constant $c > 0$. We also have, for this constant c ,

$$(3.2) \quad \alpha^* \mu_{\delta(\Gamma)} = c^{\delta(\Gamma)/2} |\alpha'|^{\delta(\Gamma)} \mu_{\delta(\Gamma)}.$$

Proof. It is known that $\alpha^* \mu_{\delta(\Gamma)}$ is a Patterson-Sullivan measure of exponent $\delta(\Gamma)$ with basepoint $\alpha^{-1}o$ (see [6, Lemma 4.2]). It follows from (2.3), (2.7) and the uniqueness of $\mu_{\delta(\Gamma)}$ that

$$(3.3) \quad \alpha^* \mu_{\delta(\Gamma)} = \tilde{c} |\alpha'|^{\delta(\Gamma)} \mu_{\delta(\Gamma)}$$

for some constant $\tilde{c} > 0$. We now recall the following formula:

$$(3.4) \quad |\gamma y - \gamma \xi|^2 = |\gamma'(y)| |\gamma'(\xi)| |y - \xi|^2, \quad y, \xi \in S^n, \gamma \in \text{Conf}(S^n, g_0).$$

Using these, we compute

$$\begin{aligned}
 \alpha^* g_N &= \left(\int_{\Lambda(\Gamma)} \left(\frac{2}{|\alpha y - \xi|^2} \right)^{\delta(\Gamma)} d\mu_{\delta(\Gamma)}(\xi) \right)^{2/\delta(\Gamma)} \alpha^* g_0 \\
 &= \left(\int_{\Lambda(\Gamma)} \left(\frac{2}{|\alpha y - \alpha \xi|^2} \right)^{\delta(\Gamma)} d(\alpha^* \mu_{\delta(\Gamma)})(\xi) \right)^{2/\delta(\Gamma)} |\alpha'(y)|^2 g_0 \\
 &= \left(\int_{\Lambda(\Gamma)} \left(\frac{2}{|\alpha'(y)| |\alpha'(\xi)| |y - \xi|^2} \right)^{\delta(\Gamma)} \tilde{c} |\alpha'(\xi)|^{\delta(\Gamma)} d\mu_{\delta(\Gamma)}(\xi) \right)^{2/\delta(\Gamma)} |\alpha'(y)|^2 g_0 \\
 &= \tilde{c}^{2/\delta(\Gamma)} g_N
 \end{aligned}$$

on $\Omega(\Gamma)$. When we regard g_N as a metric on B^{n+1} , we also have

$$\begin{aligned}
 \alpha^* g_N &= \left(\int_{\Lambda(\Gamma)} P(\alpha x, \xi)^{\delta(\Gamma)} d\mu_{\delta(\Gamma)}(\xi) \right)^{2/\delta(\Gamma)} \alpha^* g_H \\
 &= \left(\int_{\Lambda(\Gamma)} P(\alpha x, \alpha \xi)^{\delta(\Gamma)} d(\alpha^* \mu_{\delta(\Gamma)})(\xi) \right)^{2/\delta(\Gamma)} g_H \\
 &= \left(\int_{\Lambda(\Gamma)} (|\alpha'(\xi)|^{-1} P(x, \xi))^{\delta(\Gamma)} \tilde{c} |\alpha'(\xi)|^{\delta(\Gamma)} d\mu_{\delta(\Gamma)}(\xi) \right)^{2/\delta(\Gamma)} g_H \\
 &= \tilde{c}^{2/\delta(\Gamma)} g_N.
 \end{aligned}$$

by (2.6) and (2.8). Putting $c = \tilde{c}^{2/\delta(\Gamma)}$ completes the proof. $\square$

We now prove

Proposition 3.2. *Let $M = \Omega/\Gamma$ be a Kleinian manifold with $\delta(\Gamma) > 0$, g_N a Nayatani's metric, and $\alpha \in N(\Gamma) \cap \text{Conf}(\Omega, g_0)$. Suppose that $\mu_{\delta(\Gamma)}$ is, up to a constant multiple, the unique Patterson-Sullivan measure of exponent $\delta(\Gamma)$ and that α has a fixed point in Ω or $\underline{\alpha}$ has a fixed point in M (see the definition of $\underline{\alpha}$ in Sect. 2). Then α (hence $\underline{\alpha}$) is an isometry with respect to g_N .*

Proof. It follows from Lemma 3.1 that α is a homothety. Therefore $\alpha^* g_N = c g_N$ (and $\underline{\alpha}^* g_N = c g_N$) for some constant $c > 0$. Suppose $c \neq 1$. Considering α^{-1} if necessary, we may assume $c < 1$. If α has a fixed point in Ω , then $\underline{\alpha}$ also has a fixed point in M . Hence we may assume that $\underline{\alpha}$ has a fixed point p in M . Then any loop in M is mapped into a simply connected neighborhood of p by $\underline{\alpha}^k$ for a sufficiently large integer k . So M is simply connected (actually contractible). Recalling that Γ is the image of $\pi_1(M)$ by the holonomy representation, one sees that Γ is trivial. This contradicts $\delta(\Gamma) > 0$. This completes the proof. $\square$

Remark 3.3. The proof of Proposition 3.2 slightly simplifies that of Theorem 1.2, because we can find a fixed point of a contracting homothety if (M, g_N) is complete. In fact $\{\underline{\alpha}^k(p)\}_{k=1}^\infty$, for $p \in M$ and a contracting homothety $\underline{\alpha}$, is a Cauchy sequence and hence converges to a point $q \in M$. This point q is fixed by α .

Let Γ be a Kleinian group. For $\alpha \in N(\Gamma)$, we extend the domain of $\underline{\alpha}$. Namely we define a homeomorphism $\underline{\alpha} : [B^{n+1} \cup \Omega(\Gamma)]/\Gamma \rightarrow [B^{n+1} \cup \Omega(\Gamma)]/\Gamma$ as the following:

$$\underline{\alpha}(\Gamma x) = \Gamma \alpha x, \quad x \in B^{n+1} \cup \Omega(\Gamma).$$

Proposition 3.4. *Let Γ be a finitely generated Kleinian group with the unique Patterson-Sullivan measure of exponent $\delta(\Gamma) > 0$, and let g_N be the Nayatani's metric. Suppose that α has a fixed point in B^{n+1} or $\underline{\alpha}$ has a fixed point in B^{n+1}/Γ . Then α is an isometry with respect to g_N .*

Proof. By the Selberg lemma, there exists a normal torsion-free subgroup $\Gamma' \subset \Gamma$ of finite index. Such a Γ' satisfies $\delta(\Gamma') = \delta(\Gamma)$, $\Lambda(\Gamma') = \Lambda(\Gamma)$, $N(\Gamma') = N(\Gamma)$ and a Patterson-Sullivan measure $\mu_{\delta(\Gamma')}$ of Γ' coincides with $\mu_{\delta(\Gamma)}$. The argument in the proof of Proposition 3.2 implies that Γ' is trivial and hence that Γ must be a finite group, a contradiction. This completes the proof. $\square$

By Proposition 3.4 and (GF), we have the following.

Theorem 3.5. *Let $M = \Omega/\Gamma$ be a Kleinian manifold with $\delta(\Gamma) > 0$, and let g_N be a Nayatani's metric on M . Suppose that Γ is geometrically finite. Then the isometry group $\text{Isom}(M, g_N)$ coincides with the conformal transformation group $\text{Conf}(M, g_N)$.*

Proof. By Proposition 2.2, it is sufficient to show that $\alpha^*g_N = g_N$ for any $\alpha \in N(\Gamma) \cap \text{Conf}(\Omega, g_0)$. It follows from Lemma 3.1 that α is a homothety; that is, $\alpha^*g_N = cg_N$ for some constant $c > 0$. Suppose $c \neq 1$. We may assume $c < 1$. The ϵ -thick part of the convex core C_Γ , by (GF), is compact and is preserved by the action of α because α acts isometrically on C_Γ in the hyperbolic sense. Here we used the fact that α leaves $\Lambda(\Gamma)$ invariant. Therefore for a point x in the ϵ -thick part of C_Γ , $\{\alpha^k(x)\}_{k=1}^\infty$, taking a subsequence if necessary, converges to some point $\tilde{x} \in C_\Gamma \subset B^{n+1}/\Gamma$. Then $\tilde{x}$ is a fixed point of $\underline{\alpha}$. In fact, since $\underline{\alpha}$ is a homothety, the sequence $\{\alpha^k(x)\}_{k=1}^\infty$ itself converges, and the limit must be fixed by $\underline{\alpha}$. Since geometrically finite Kleinian groups are finitely generated, Proposition 3.4 completes the proof. $\square$

4. GENERAL CASE

From what we have seen so far, it is plausible that $\text{Isom}(M, g_N) = \text{Conf}(M, g_N)$ holds for every Kleinian manifold (M, g_N) corresponding to a Kleinian group Γ with the unique Patterson-Sullivan measure of exponent $\delta(\Gamma)$. In this section, we show a result which may be considered as a partial evidence to this statement. First, we show the following.

Theorem 4.1. *Let $M = \Omega/\Gamma$ be a Kleinian manifold with $\delta(\Gamma) > 0$, and let g_N be a Nayatani's metric. Suppose that there is no Patterson-Sullivan measure μ_ρ of exponent $\rho \neq \delta(\Gamma)$ and that $\mu_{\delta(\Gamma)}$ is unique up to a constant multiple. Then the isometry group $\text{Isom}(M, g_N)$ coincides with the conformal transformation group $\text{Conf}(M, g_N)$.*

Proof. By Proposition 2.2, it is sufficient to show that $\alpha^*g_N = g_N$ for any $\alpha \in N(\Gamma) \cap \text{Conf}(\Omega, g_0)$. It follows from Lemma 3.1 that α is a homothety; that is, $\alpha^*g_N = cg_N$ for some constant $c > 0$. Suppose $c \neq 1$. Considering α^{-1} if necessary, we may assume $c < 1$. By Proposition 3.2, we may also assume that neither α nor

$\underline{\alpha}$ have a fixed point in Ω or M . Then $\langle \alpha \rangle$ (resp. $\langle \underline{\alpha} \rangle$) acts properly discontinuously on Ω (resp. M). If not, since α (hence $\underline{\alpha}$) is a contracting homothety, α or $\underline{\alpha}$ has a fixed point in Ω or M . In fact, for a compact subset K of Ω such that

$$\sharp \{k \in \mathbb{Z} \mid \alpha^k K \cap K \neq \emptyset\} = \infty,$$

$\{\alpha^k(y)\}_{k=1}^\infty, y \in K$, has an accumulation point $y' \in K$ and this point y' is fixed by α . We now let $\tilde{\Gamma} = \langle \Gamma, \alpha \rangle$. Suppose $\tilde{\Gamma}$ is discrete. Then we have $\delta(\tilde{\Gamma}) = \delta(\Gamma)$ as follows. First, we recall that the action of $N(\Gamma)$ leaves $\Lambda(\Gamma)$ invariant and if $\delta(\Gamma) > 0$, then $\Lambda(\Gamma)$ is the smallest closed subset of S^n that is Γ -invariant. This means that $\Lambda(\tilde{\Gamma}) = \Lambda(\Gamma)$. Therefore each Patterson-Sullivan measure $\tilde{\mu}_{\delta(\tilde{\Gamma})}$ of $\tilde{\Gamma}$ is also a Patterson-Sullivan measure of Γ . By the hypothesis that the exponent of Patterson-Sullivan measures is unique, we have $\delta(\tilde{\Gamma}) = \delta(\Gamma)$. Moreover it follows from the uniqueness of $\mu_{\delta(\Gamma)}$ that $\tilde{\mu}_{\delta(\tilde{\Gamma})}$ coincides with $\mu_{\delta(\Gamma)}$ up to a constant multiple. The transformation law (2.1) of Patterson-Sullivan measures, with (3.2), implies that $c = 1$. This is a contradiction.

We next consider the case that $\tilde{\Gamma}$ is not discrete. Let $\{\tilde{\gamma}_i\}_{i=1}^\infty \subset \tilde{\Gamma}$ be a sequence that converges to the identity. Since α is an element of $N(\Gamma)$, each $\tilde{\gamma}_i$ is represented by the form $\gamma_i \alpha^{k_i}$, where $\gamma_i \in \Gamma$ and k_i is some integer. If $\{k_i\}_{i=1}^\infty$ is bounded, then there exist some $k \in \mathbb{Z}$ and subsequence $\{\gamma_{i(j)}\}_{j=1}^\infty$ such that

$$\lim_{j \rightarrow \infty} \gamma_{i(j)} = \alpha^k.$$

This means that α^k is an element of Γ , and hence $c = 1$. It remains to consider the case that $\{k_i\}_{i=1}^\infty$ is not bounded. Notice that each $\tilde{\gamma}_i = \gamma_i \alpha^{k_i}$ equals $\underline{\alpha}^{k_i}$ as an element of $\text{Conf}(M, g_N)$. However $\langle \underline{\alpha} \rangle$ acts properly discontinuously on Ω/Γ . This contradicts the assumption that $\tilde{\gamma}_i$ converges to the identity. This completes the proof. $\square$

Remark 4.2. Actually, we can show that Kleinian groups satisfying the conditions in Theorem 4.1 are convex cocompact. If we can show that $\delta(\Gamma) = \delta(\tilde{\Gamma})$ (see the proof of Theorem 4.1), we can drop the assumption in Theorem 4.1 that there is no Patterson-Sullivan measure of exponent $\rho > \delta(\Gamma)$.

Next we recall the classification of elements of $\text{Conf}(S^n, g_0)$ according to their dynamics on $B^{n+1} \cup S^n$. A transformation $\gamma \in \text{Conf}(S^n, g_0)$ is called *elliptic* if γ has fixed points in B^{n+1} , *loxodromic* if γ is not elliptic and has exactly two fixed points in S^n , and *parabolic* otherwise. If γ is parabolic, γ has exactly one fixed point in S^n . Let Γ be a Kleinian group with $\delta(\Gamma) > 0$, g_N a Nayatani's metric on $\Omega(\Gamma)$ and $\alpha \in N(\Gamma)$. We now suppose that $\mu_{\delta(\Gamma)}$ is, up to a constant multiple, the unique Patterson-Sullivan measure of exponent $\delta(\Gamma)$. It is not difficult to see that for α elliptic, α is an isometry of $(\Omega(\Gamma), g_N)$ (see Proposition 3.4 in the case Γ is finitely generated). Moreover we can show the following.

Proposition 4.3. *Let Γ be a Kleinian group with $\delta = \delta(\Gamma) > 0$, g_N a Nayatani's metric on $\Omega(\Gamma)$, and $\alpha \in N(\Gamma)$. Suppose that μ is, up to a constant multiple, the unique Patterson-Sullivan measure of exponent $\delta(\Gamma)$ and that α is a parabolic element. Then α is an isometry of $(\Omega(\Gamma), g_N)$ and hence if $M = \Omega/\Gamma$ is a Kleinian manifold and $\alpha \in N(\Gamma) \cap \text{Conf}(\Omega, g_0)$, $\underline{\alpha}$ also acts isometrically on (M, g_N) .*

Proof. By Lemma 3.1, α is a homothety; that is, $\alpha^*g_N = cg_N$ for some constant $c > 0$. Suppose $c \neq 1$. We may assume $c < 1$. Since $(\alpha^k)^*g_N = c^k g_N$ for $k \in \mathbb{Z}$, by the formula (2.8), we have the following equation:

$$(4.1) \quad \left(\int_{\Lambda(\Gamma)} \left(\frac{1}{|\alpha^k x - \xi|^2} \right)^\delta d\mu(\xi) \right)^{2/\delta} = \frac{c^k}{(1 - |\alpha^k x|^2)^2} \left(\int_{\Lambda(\Gamma)} \left(\frac{1 - |x|^2}{|x - \xi|^2} \right)^\delta d\mu(\xi) \right)^{2/\delta},$$

where $x \in B^{n+1}$. We now compute the order of $(1 - |\alpha^k x|^2)^2$ as $k \rightarrow \infty$. To do this, it is convenient to work with the upper half-space model $H^{n+1} = \{z \in \mathbb{R}^{n+1} \mid z_{n+1} > 0\}$ of the hyperbolic space. The identification $J : H^{n+1} \rightarrow B^{n+1}$ is given by the composition of the following two maps:

- (i) $H^{n+1} \rightarrow \{y_{n+1} < \frac{1}{2}\}; z \mapsto y = z + (\frac{1}{2} - 2z_{n+1})e_{n+1}$,
- (ii) $\{y_{n+1} < \frac{1}{2}\} \rightarrow B^{n+1}; y \mapsto x = e_{n+1} + (y - e_{n+1})|y - e_{n+1}|^{-2}$,

where e_{n+1} denotes the $(n+1)$ -th element of the standard basis of $\mathbb{R}^{n+1}$. By direct calculation, we have

$$(4.2) \quad 1 - |x|^2 = \frac{2z_{n+1}}{|[z]_n|^2 + (\frac{1}{2} + z_{n+1})^2},$$

where $x = J(z)$ and $[z]_n = (z_1, \dots, z_n) \in \mathbb{R}^n$. We may assume $\alpha(e_{n+1}) = e_{n+1}$ in the Poincaré model B^{n+1} for considering $(1 - |\alpha^k x|^2)^2$. Because each parabolic element has a decomposition of the form $P^{-1}\alpha'P$, where P is an orthogonal transformation of $\mathbb{R}^{n+1}$ and α' is a parabolic element of $\text{Conf}(S^n, g_0)$ that fixes e_{n+1} . Correspondingly $\alpha(\infty) = \infty$ in the upper half-space model H^{n+1} . Then α has the following expression in H^{n+1} :

$$\alpha(z) = \begin{bmatrix} T & 0 \\ 0 & 1 \end{bmatrix} z + \begin{bmatrix} a \\ 0 \end{bmatrix},$$

where T is an $n \times n$ orthogonal matrix and $a \in \mathbb{R}^n \setminus \{o\}$ (see [8, p. 13], for example). Taking x as $\alpha^k o$ in the formula (4.2), since we have $J^{-1}(o) = \frac{1}{2}e_{n+1}$,

$$1 - |\alpha^k o|^2 \geq \frac{1}{|a|^2 k^2 + 1} = O(k^{-2}).$$

Therefore the right-hand side of (4.1) tends to 0 as $k \rightarrow \infty$. On the other hand, the left-hand side of (4.1) is larger than $\frac{\mu(\Lambda(\Gamma))}{16} > 0$ because $|\alpha^k x - \xi| < 2$ for any $\xi \in \Lambda(\Gamma)$, a contradiction. Thus c must be 1, and this completes the proof. $\square$

Remark 4.4. For $\alpha \in N(\Gamma)$ loxodromic, we cannot use the same argument as above, because, in this case, $(1 - |\alpha^k x|^2)^2$ tends to 0 exponentially for any $x \in B^{n+1}$. In fact the action of α on $\overline{B}^{n+1}$ has a fixed point at e_{n+1} , taking conjugates by some orthogonal transformation of $\mathbb{R}^{n+1}$ if necessary. Accordingly, in H^{n+1} , α is represented by

$$\alpha(z) = \lambda \begin{bmatrix} T & 0 \\ 0 & 1 \end{bmatrix} \left(z - \begin{bmatrix} a \\ 0 \end{bmatrix} \right),$$

where $\lambda \in \mathbb{R}_{>0} \setminus \{1\}$, T is an $n \times n$ orthogonal matrix and $a \in \mathbb{R}^n$. This, with (4.2), means that $(1 - |\alpha^k x|^2)^2 \rightarrow 0$ exponentially. If we can show, under the same conditions as Proposition 4.3, that every loxodromic element of $N(\Gamma)$ acts isometrically on $(\Omega(\Gamma), g_N)$, we can remove the assumption in Theorem 4.1 that there is no Patterson-Sullivan measure of exponent $\rho > \delta(\Gamma)$.

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REFERENCES

1. A. Beardon and B. Maskit, *Limit sets of Kleinian groups and finite sided fundamental polyhedra*, Acta Math. **132** (1974), 1–12. MR0333164 (48:11489)
2. B. H. Bowditch, *Geometrical finiteness for hyperbolic groups*, J. Funct. Anal. **113** (1993), 245–317. MR1218098 (94e:57016)
3. H. Izeki and S. Nayatani, *Canonical metric on the domain of discontinuity of a Kleinian group*, Séminaire de théorie spectrale et géométrie GRENOBLE **16** (1998), 9–32. MR1666506 (2000a:53018)
4. A. Marden, *The geometry of finitely generated Kleinian groups*, Ann. of Math. (2) **99** (1974), 383–462. MR0349992 (50:2485)
5. J. Maubon, *Geometrically finite Kleinian groups: The completeness of Nayatani’s metric*, C. R. Acad. Sci. Paris, **325** (1997), 1065–1070. MR1614003 (99a:30041)
6. S. Nayatani, *Patterson-Sullivan measure and conformally flat metrics*, Math. Z. **225** (1997), 115–131. MR1451336 (98g:53072)
7. S. J. Patterson, *The limit set of a Fuchsian group*, Acta Math. **136** (1976), 241–273. MR0450547 (56:8841)
8. P. J. Nicholls, *The ergodic theory of discrete groups*, LMS Lect. Notes Ser. **143**, Cambridge University Press, Cambridge, 1989. MR1041575 (91i:58104)
9. R. Schoen and S. T. Yau, *Conformally flat manifolds, Kleinian groups and scalar curvature*, Invent. Math. **92**, (1988) 47–71. MR931204 (89c:58139)
10. D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, I. H. E. S. Publ. Math. **50** (1979), 171–202. MR556586 (81b:58031)
11. D. Sullivan, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Math. **153** (1984), 259–277. MR766265 (86c:58093)
12. W. P. Thurston, *The Geometry and Topology of 3-manifolds*, notes, Princeton Univ. Math. Department, 1979.

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