A NEW CONSTRUCTION OF THE UNSTABLE MANIFOLD FOR THE MEASURE-PRESERVING HÉNON MAP

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ABSTRACT. Let H denote the measure-preserving Hénon map with the parameter $a > 0$. The map H has a hyperbolic fixed point $\mathbf{p}$. The main result of this paper is that the unstable manifold of $\mathbf{p}$ is the iterated limit of a very simple set. Informally,

$$W^u(\mathbf{p}) = \lim_{n \rightarrow \infty} H^n(\mathcal{L})$$

where $\mathcal{L}$ is the line $y = -x$ and $W^u(\mathbf{p})$ denotes the unstable manifold of $\mathbf{p}$.

1. BACKGROUND MATERIAL

The Hénon map, in its original form, is written as

$$H_{\text{original}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + y - ax^2 \\ bx \end{pmatrix}$$

[H1] and has Jacobian

$$DH_{\text{original}} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix}$$

so that $\det DH_{\text{original}} = -b$.

Let H denote the measure-preserving Hénon map. Then

$$H \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 + y - ax^2 \\ -x \end{pmatrix}.$$

The equations for the fixed points of H are

$$\begin{aligned} x &= 1 + y - ax^2, \\ y &= -x. \end{aligned}$$

Solving these equations gives two fixed points:

$$\begin{aligned} \mathbf{p} &= \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{a}(-1 - \sqrt{1+a}) \\ \frac{1}{a}(1 + \sqrt{1+a}) \end{pmatrix}, \\ \mathbf{q} &= \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{a}(-1 + \sqrt{1+a}) \\ \frac{1}{a}(1 - \sqrt{1+a}) \end{pmatrix}. \end{aligned}$$

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To see that $\mathbf{p}$ is hyperbolic, note that

$$DH(\mathbf{p}) = \begin{pmatrix} -2ap_1 & 1 \\ -1 & 0 \end{pmatrix}$$

and has characteristic polynomial

$$\lambda^2 + 2ap_1 + 1$$

so that the eigenvalues of $DH(\mathbf{p})$ are

$$\begin{aligned} \lambda &= -ap_1 \pm \sqrt{a^2p_1^2 - 1} \\ &= 1 + \sqrt{1+a} \pm \sqrt{1+a+2\sqrt{1+a}} \end{aligned}$$

so that $\mathbf{p}$ is hyperbolic for $a > -1$.

The map H is the composition of two involutions. This fact is very important, since it will give us a geometric picture of H . Let $\mathcal{L}$ denote the line $y = -x$, and let $\mathcal{T}_{\mathcal{L}}$ denote a perpendicular reflection in $\mathcal{L}$. Let $\mathcal{P}$ denote the parabola $x = \frac{1}{2}(1 - ay^2)$, and let $\mathcal{T}_{\mathcal{P}}$ denote a horizontal reflection in $\mathcal{P}$. Then

$$\begin{aligned} \mathcal{T}_{\mathcal{L}} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -y \\ -x \end{pmatrix}, \\ \mathcal{T}_{\mathcal{P}} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 1 - x - ay^2 \\ y \end{pmatrix}, \end{aligned}$$

and

$$H = \mathcal{T}_{\mathcal{L}} \circ \mathcal{T}_{\mathcal{P}}.$$

Figure 1 is a diagram of the sets $\mathcal{L}$ and $\mathcal{P}$ for $a > 0$.

FIGURE 1. The map H is a perpendicular reflection in $\mathcal{L}$ followed by a horizontal reflection in $\mathcal{P}$.

Now, we calculate the slope of the unstable manifold of $\mathbf{p}$ at $\mathbf{p}$. Let s_1 denote the expanding eigenvalue of $DH(\mathbf{p})$, so that

$$s_1 = 1 + \sqrt{1+a} + \sqrt{1+a+\sqrt{1+a}}.$$

Then

$$\begin{aligned} DH(\mathbf{p}) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_1 x \\ s_1 y \end{pmatrix} &\Rightarrow \begin{pmatrix} -2ap_1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_1 x \\ s_1 y \end{pmatrix} \\ &\Rightarrow y = \frac{-1}{s_1} x \end{aligned}$$

so the slope of the unstable manifold of $\mathbf{p}$ at $\mathbf{p}$ is $\frac{-1}{s_1}$.

Let C^u denote the branch of the unstable manifold of $\mathbf{p}$ that starts at $\mathbf{p}$ and initially moves to the right. Let $\mathbf{z}(t) = (z_1(t), z_2(t))$ denote a parameterization of C^u .

2. MOTIVATION

In 1995, Brown published a paper entitled ‘Horseshoes in the measure-preserving Hénon map’ [B]. In this paper, he proved that the measure-preserving Hénon map contains an embedded horseshoe for $a > 0$. To do this, Brown used geometric methods to prove that H has a homoclinic point for $a > 0$, and then he used a contradiction argument to show that this homoclinic point must be transverse. By a theorem of Smale [S1], a transverse homoclinic point implies the existence of a horseshoe.

The main result of the current paper was motivated in large part by the geometric methods Brown used in his proof. Specifically, Brown used the fact that C^u cannot cross $\mathcal{L}_1$ before crossing $\mathcal{L}_0$, where $\mathcal{L}_n$ denotes $H^n(\mathcal{L})$. We will prove this fact in Lemma 1.

Figure 2 shows the shape of C^u .

The point χ_c in Figure 2 is the intersection of $\mathcal{L}_0$ and C^u . By a result of Devaney [D1], χ_c must be a homoclinic point. Devaney, in his 1984 paper ‘Homoclinic bifurcations and the area-conserving Hénon mapping’ [D1], introduced the concept

FIGURE 2. $\mathcal{L}$, $\mathcal{P}$, $\mathcal{L}_1$, C^u , and χ_c

of T -reversibility. For our purposes, the Hénon map is $\mathcal{T}_{\mathcal{L}}$ -reversible and $\mathbf{p} \in \mathcal{L}$, so Devaney's result implies that $W^s(\mathbf{p}) = \mathcal{T}_{\mathcal{L}}(W^u(\mathbf{p}))$, where $W^s(\mathbf{p})$ denotes the stable manifold of $\mathbf{p}$ and $W^u(\mathbf{p})$ denotes the unstable manifold of $\mathbf{p}$. Since $\mathcal{T}_{\mathcal{L}}$ is a reflection in the line $\mathcal{L}$, then any point in $\mathcal{L} \cap W^u(\mathbf{p})$ must be in $W^s(\mathbf{p}) \cap W^u(\mathbf{p})$.

Brown used geometric methods to prove that C^u crosses $\mathcal{L}$. For the purposes of the current paper, the most important of these is the following lemma:

Lemma 1. *C^u cannot cross $H(\mathcal{L})$ before crossing $\mathcal{L}$.*

Proof. Suppose that C^u crosses $H(\mathcal{L})$ at $\mathbf{r}$ before crossing $\mathcal{L}$. Then $H^{-1}(\mathbf{r}) \in \mathcal{L}$ and $H^{-1}(\mathbf{r}) \in C^u$, and $H^{-1}(\mathbf{r})$ precedes $\mathbf{r}$ on C^u . This contradiction proves the lemma. $\square$

From Figure 2, one can see that the previous lemma gives a rough picture of the shape of C^u . This is the observation that led to the result of the current paper. Let $\mathcal{L}_n$ denote $H^n(\mathcal{L})$; Figure 3 is a graph of $\mathcal{L}_0$, $\mathcal{L}_1$, and $\mathcal{L}_2$.

In Brown's proof, the set $\mathcal{L}_1$ played an important role in the proof that H has a homoclinic point. The fact that C^u could not cross $\mathcal{L}_1$ before crossing $\mathcal{L}_0$ gave us a rough picture of the shape of C^u . Extending this idea, we find that C^u cannot cross $\mathcal{L}_n$ before crossing $\mathcal{L}_{n-1}$. It seems reasonable to investigate the possibility that each successive $\mathcal{L}_n$ is a better approximation of C^u . In fact, this is true. Further, one may obtain, in a precise way, a set $\mathcal{L}_{\infty}$ as a limit of the $\mathcal{L}_n$, with $\mathcal{L}_{\infty} = C^u$ between $\mathbf{p}$ and χ_c .

So, we state the following theorem:

Theorem 1.

$$\mathcal{L}_{\infty} = C^u$$

between $\mathbf{p}$ and χ_c .

The next section is devoted to a proof.

FIGURE 3. $\mathcal{L}_0$, $\mathcal{L}_1$, and $\mathcal{L}_2$

3. PROOF

Theorem 1 can be proved via the following five steps:

- (1) Show that $\mathcal{L}_n \cap \mathcal{L}_{n+1} = \{\mathbf{p}, \mathbf{q}\}$ for all n .
- (2) Show that the slope of $\mathcal{L}_{n+1}$ at $\mathbf{p}$ is greater than the slope of $\mathcal{L}_n$ at $\mathbf{p}$ for all n .
- (3) Show that there exists a set $\mathcal{J}$ such that $\widetilde{\mathcal{L}}_n$ is in the region bounded by $\mathcal{J}$ and $\mathcal{L}_0$ for all n , where $\widetilde{\mathcal{L}}_n$ is the segment of $\mathcal{L}_n$ from $\mathbf{p}$ to the first intersection of $\mathcal{L}_n$ with $\mathcal{L}_0$. This is illustrated in Figure 4.
- (4) Define $\mathcal{L}_\infty$ in a precise way as a limit of the sets $\widetilde{\mathcal{L}}_n$.
- (5) Show that $\mathcal{L}_\infty = C^u$ between $\mathbf{p}$ and χ_c .

Step 1. Suppose that $\mathbf{v} \in \mathcal{L}_n \cap \mathcal{L}_{n+1}$. Then $H^{-n}(\mathbf{v}) \in \mathcal{L}_0 \cap \mathcal{L}_1$, and so either $H^{-n}(\mathbf{v}) = \mathbf{p}$, in which case $\mathbf{v} = \mathbf{p}$, or $H^{-n}(\mathbf{v}) = \mathbf{q}$, in which case $\mathbf{v} = \mathbf{q}$. So, $\mathcal{L}_n$ and $\mathcal{L}_{n+1}$ intersect only at $\mathbf{p}$ and $\mathbf{q}$. Combining this fact with the result of Step 2 proves that $\widetilde{\mathcal{L}}_{n+1}$ is completely outside of $\widetilde{\mathcal{L}}_n$ for all n .

One should note that $\mathcal{L}_n$ and $\mathcal{L}_{n+2}$ *do* intersect at points other than $\mathbf{p}$ and $\mathbf{q}$.

Step 2. Define

$$T_0(t) = t,$$

$$T_1(t) = t,$$

$$T_n(t) = 1 - T_{n-2}(t) - a[T_{n-1}(t)]^2 \quad \text{for } n \geq 2.$$

FIGURE 4. What an outer bound $\mathcal{J}$ for the $\mathcal{L}_n$ might look like

Then

$$\begin{aligned}\mathcal{L}_0(t) &= \begin{pmatrix} T_1(t) \\ -T_0(t) \end{pmatrix}, \\ \mathcal{L}_1(t) &= \begin{pmatrix} T_2(t) \\ -T_1(t) \end{pmatrix}, \\ \mathcal{L}_n(t) &= \begin{pmatrix} T_{n+1}(t) \\ -T_n(t) \end{pmatrix}.\end{aligned}$$

Let $m_n(t)$ be the slope of $\mathcal{L}_n(t)$. Then

$$m_n(t) = \frac{-\dot{T}_n(t)}{\dot{T}_{n+1}(t)}.$$

The $m_n(t)$ can be expressed recursively as

$$\begin{aligned}m_n(t) &= \frac{-\dot{T}_n(t)}{\dot{T}_{n+1}(t)} \\ &= \frac{-\dot{T}_n(t)}{-\dot{T}_{n-1}(t) - 2a\dot{T}_n(t)T_n(t)} \\ &= \frac{1}{2aT_n(t) + \frac{\dot{T}_{n-1}(t)}{\dot{T}_n(t)}} \\ &= \frac{1}{2aT_n(t) - m_{n-1}(t)}.\end{aligned}$$

Now we may prove that the slope of $\mathcal{L}_{n+1}$ at $\mathbf{p}$ is greater than the slope of $\mathcal{L}_n$ at $\mathbf{p}$ for all n . It is important to note that $T_n(p_1) = p_1$ for all n .

We have that

$$\begin{aligned}m_0(p_1) &= -1, \\ m_1(p_1) &= \frac{1}{2aT_1(p_1) + 1} = \frac{1}{-1 - 2\sqrt{1+a}} > -1.\end{aligned}$$

We proceed by induction on n :

$$\begin{aligned}& m_n(p_1) > m_{n-1}(p_1) \\ \iff & -m_n(p_1) < -m_{n-1}(p_1) \\ \iff & 2ap_1 - m_n(p_1) < 2ap_1 - m_{n-1}(p_1) \\ \iff & \frac{1}{2ap_1 - m_n(p_1)} > \frac{1}{2ap_1 - m_{n-1}(p_1)} \\ \iff & \frac{1}{2aT_{n+1}(p_1) - m_n(p_1)} > \frac{1}{2aT_n(p_1) - m_{n-1}(p_1)} \\ \iff & m_{n+1}(p_1) > m_n(p_1).\end{aligned}$$

In order to justify the operation between lines 3 and 4 in the above inequalities, we must show that $2ap_1 - m_n(p_1)$ and $2ap_1 - m_{n-1}(p_1)$ have the same sign. As a consequence of the proof of Step 3, we have that

$$0 > m_n(p_1) > m_{n-1}(p_1) > -1$$

which implies that

$$2ap_1 < 2ap_1 - m_n(p_1) < 2ap_1 - m_{n-1}(p_1) < 2ap_1 + 1.$$

Since $2ap_1$ and $2ap_1 + 1$ are both negative, then $2ap_1 - m_n(p_1)$ and $2ap_1 - m_{n-1}(p_1)$ are both negative.

So, $m_{n+1}(p_1) > m_n(p_1)$ for all n .

Step 3. Figure 4 is an illustration of what a set $\mathcal{J}$ might look like.

We will show that C^u is an appropriate $\mathcal{J}$. First, note that the slope of C^u at $\mathbf{p}$ is greater than the slope of $\mathcal{L}_1$ at $\mathbf{p}$, so that C^u starts above $\mathcal{L}_1$. If C^u is not above all of the $\mathcal{L}_n$, then it must start between two consecutive $\mathcal{L}_n$. So, suppose that C^u starts between $\mathcal{L}_n$ and $\mathcal{L}_{n+1}$. This leads to a contradiction as follows. Let $\mathbf{v} \in C^u$. Since $\mathbf{v}$ is between $\mathcal{L}_n$ and $\mathcal{L}_{n+1}$, then $H^{-1}(\mathbf{v})$ is between $\mathcal{L}_{n-1}$ and $\mathcal{L}_n$. But, since $\mathbf{v} \in C^u$, then $H^{-1}(\mathbf{v}) \in C^u$, and so $H^{-1}(\mathbf{v})$ is between $\mathcal{L}_n$ and $\mathcal{L}_{n+1}$. This is a contradiction, and so C^u must be above $\mathcal{L}_n$ for all n .

Since C^u crosses $\mathcal{L}_0$ (for a proof of this see [B] or [D1]), the previous argument also shows that all the $\mathcal{L}_n$ are bounded.

Step 4. Now we define the set $\mathcal{L}_\infty$. First, let $\mathcal{V}_\theta$ be a ray from the origin with angle θ , where $\theta \in [-\frac{\pi}{4}, \frac{3\pi}{4}]$. Then, let $\theta_n = \widetilde{\mathcal{L}_n} \cap \mathcal{V}_\theta$. We know that $\lim_{n \rightarrow \infty} \theta_n$ exists since the sequence $\{\theta_n\}$ is strictly increasing and bounded. So, let

$$\theta_\infty = \lim_{n \rightarrow \infty} \theta_n.$$

Then $\mathcal{L}_\infty$ is defined by

$$\mathcal{L}_\infty = \left\{ \theta_\infty : \theta \in \left[-\frac{\pi}{4}, \frac{3\pi}{4} \right] \right\}.$$

This construction is illustrated in Figure 5.

Step 5. Let $\mathbf{v} \in \mathcal{L}_\infty$. We must show that $\lim_{n \rightarrow \infty} H^{-n}(\mathbf{v}) = \mathbf{p}$.

We start with some definitions.

FIGURE 5. The construction of $\mathcal{L}_\infty$

FIGURE 6. The set $\mathcal{A}_n$ is bounded by $\mathcal{L}_n$, $H(\Upsilon)$, and C^u

FIGURE 7. The set $\mathcal{B}_{n-1}$ is bounded by $\mathcal{L}_{n-1}$, Υ , and C^u

FIGURE 8. The set $\mathcal{C}_{n-2}$ is bounded by $\mathcal{L}_{n-2}$, the positive vertical axis, and C^u

Let Υ be the line from the first intersection of $\mathcal{L}_1$ with the x -axis to the first intersection of C^u with the x -axis. So, Υ is the line from $(1, 0)$ to $(x_+, 0)$. Consider

the image of Υ under H .

$$H(\Upsilon) = H\begin{pmatrix} t \\ 0 \end{pmatrix} = \begin{pmatrix} 1 - at^2 \\ -t \end{pmatrix} \quad \text{for } t \in [1, x_+].$$

We will show that $H(\Upsilon)$ is below the line $\mathcal{L}_0$.

$$-t < -1 + at^2 \iff 0 < at^2 + t - 1$$

but $at^2 + t - 1$ is an increasing function for $t \geq 1$, and $a \cdot 1^2 + 1 - 1 = a > 0$. So, $H(\Upsilon)$ is below the line $\mathcal{L}_0$.

Now we define the following sets:

$\mathcal{A}_n$: the area bounded by $\mathcal{L}_n$, $H(\Upsilon)$, and C^u ,

$\mathcal{B}_n$: the area bounded by $\mathcal{L}_n$, Υ , and C^u ,

$\mathcal{C}_n$: the area bounded by $\mathcal{L}_n$, the positive y -axis, and C^u .

These sets are illustrated in Figures 6, 7, and 8.

We note that

$$H^{-1}(\mathcal{A}_n) \subset \mathcal{B}_{n-1} \quad \text{for all } n,$$

$$H^{-2}(\mathcal{A}_n) \subset \mathcal{C}_{n-2} \quad \text{for all } n.$$

Now, define

$$\mathcal{A}_\infty = \bigcap_{n=1}^{\infty} \mathcal{A}_n,$$

$$\mathcal{B}_\infty = \bigcap_{n=1}^{\infty} \mathcal{B}_n,$$

$$\mathcal{C}_\infty = \bigcap_{n=1}^{\infty} \mathcal{C}_n.$$

It is clear that $\mathcal{L}_\infty \subset \mathcal{A}_\infty$. We will show that for any $v \in \mathcal{A}_\infty$, $\lim_{n \rightarrow \infty} H^{-n}(v) = p$.

FIGURE 9. If $w \in \mathcal{C}_\infty$, then $H^{-1}(w) \in \mathcal{C}_\infty$ and $H^{-1}(w)$ is to the left of w

First, note that since $\mathbf{v} \in \mathcal{A}_\infty$, then $\mathbf{v} \in \mathcal{A}_n$ for all n . We will show that $H^{-2-k}(\mathbf{v}) \in \mathcal{C}_\infty$ for all nonnegative integers k .

Since $\mathbf{v} \in \mathcal{A}_n$ for all n , then $H^{-2}(\mathbf{v}) \in \mathcal{C}_{n-2}$ for all n . So, $H^{-2}(\mathbf{v}) \in \mathcal{C}_\infty$.

Also, $H^{-1}(\mathcal{C}_\infty) \subset \mathcal{C}_\infty$, since

$$\begin{aligned} \mathbf{w} \in \mathcal{C}_\infty &\Rightarrow \mathbf{w} \in \mathcal{C}_2 \cap \mathcal{C}_3 \cap \mathcal{C}_4 \cdots \\ &\Rightarrow H^{-1}(\mathbf{w}) \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3 \cdots \\ &\Rightarrow H^{-1}(\mathbf{w}) \in \mathcal{C}_\infty. \end{aligned}$$

So, $H^{-2-k}(\mathbf{v}) \in \mathcal{C}_\infty$ for all nonnegative integers k .

Now, let $\mathbf{w} = H^{-2}(\mathbf{v})$, so that $\mathbf{w} \in \mathcal{C}_\infty$. Let τ be the x co-ordinate of $\mathbf{w}$, and let N_τ denote the vertical line through $\mathbf{w}$. Consider Figure 9.

The map H^{-1} maps N_τ to the parabola $y = ax^2 - 1 + \tau$. Clearly, $H^{-1}(\mathbf{w})$ is to the left of the point $\mathbf{u}$, where $\mathbf{u}$ is the intersection of $\mathcal{L}_0$ and $H^{-1}(N_\tau)$.

By direct computation, the point $\mathbf{u}$ has x co-ordinate

$$-\frac{1}{2a} - \frac{1}{2a} \sqrt{1 + 4a(1 - \tau)}.$$

Let

$$\xi(\tau) = -\frac{1}{2a} - \frac{1}{2a} \sqrt{1 + 4a(1 - \tau)}.$$

FIGURE 10. For $p_1 \leq \tau \leq 0$, $\xi^k(\tau)$ approaches p_1

Then we have that

$$\begin{aligned} \mathbf{w} &\in \mathcal{C}_\infty \cap \{x \leq \tau\}, \\ H^{-1}(\mathbf{w}) &\in \mathcal{C}_\infty \cap \{x \leq \xi(\tau)\}, \\ H^{-2}(\mathbf{w}) &\in \mathcal{C}_\infty \cap \{x \leq \xi^2(\tau)\}, \\ &\vdots \\ H^{-k}(\mathbf{w}) &\in \mathcal{C}_\infty \cap \{x \leq \xi^k(\tau)\}. \end{aligned}$$

So, to complete the proof, we must show that $\lim_{k \rightarrow \infty} \xi^k(\tau) = p_1$. However, this is clear if we consider $\xi : \mathbb{R} \rightarrow \mathbb{R}$ as a dynamical system. The point p_1 is an attracting fixed point for this system, and all $\tau \in [p_1, 0]$ go to p_1 under iteration of ξ . This is shown in Figure 10.

So, we have shown that for $\mathbf{v} \in \mathcal{L}_\infty$, $\lim_{n \rightarrow \infty} H^{-n}(\mathbf{v}) = \mathbf{p}$. Thus, $\mathcal{L}_\infty = C^u$ from $\mathbf{p}$ to χ_c .

4. POSSIBLE APPLICATIONS

Theorem 1 gives a new way of looking at the unstable manifold. This result could have many applications in the field of chaotic dynamics, since a better understanding of the unstable manifold could lead to new techniques for finding homoclinic points.

In the following example, Theorem 1 is used to determine the slope of C^u at $\mathbf{p}$ for the quadratic Hénon map H .

Example: Determining the slope of C^u at $\mathbf{p}$. The slope of C^u at $\mathbf{p}$ is the limit of the slopes of the $\mathcal{L}_n$ at $\mathbf{p}$.

Using the construction from Step 2 of the proof, one can express this limit as the following continued fraction:

$$c = \frac{1}{2ap_1 - \frac{1}{2ap_1 - \frac{1}{2ap_1 - \dots}}}$$

Since $c = \frac{1}{2ap_1 - c}$, then

$$\begin{aligned} c &= \frac{2ap_1 - \sqrt{(2ap_1)^2 - 4}}{2} \\ &= ap_1 + \sqrt{a^2 p_1^2 - 1}. \end{aligned}$$

Also, since $p_1 = \frac{-1 - \sqrt{1+a}}{a}$, then

$$c = -1 - \sqrt{1+a} + \sqrt{1+a+2\sqrt{1+a}}.$$

So, the slope of C^u at $\mathbf{p}$ is $-1 - \sqrt{1+a} + \sqrt{1+a+2\sqrt{1+a}}$.

In Section 1, the slope s_1 of C^u at $\mathbf{p}$ was determined using eigenvectors of the Jacobian. The result was

$$s_1 = \frac{-1}{1 + \sqrt{1+a} + \sqrt{1+a+2\sqrt{1+a}}}.$$

To double check that $c = s_1$:

$$s_1 \cdot \frac{1 + \sqrt{1+a} - \sqrt{1+a+2\sqrt{1+a}}}{1 + \sqrt{1+a} - \sqrt{1+a+2\sqrt{1+a}}} = c.$$

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