ON LOCAL SOLVABILITY OF CERTAIN DIFFERENTIAL COMPLEXES

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(Communicated by David S. Tartakoff)

ABSTRACT. In any locally integrable structure a differential complex induced by the de Rham differential is naturally defined. We give necessary conditions, in terms of the signature of the Levi form, for its local solvability *with a prescribed rate of shrinking*.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The present paper deals with some invariants associated with locally integrable structures, recently introduced by C.D. Hill and M. Nacinovich [10] in the context of CR manifolds.

We recall (see Treves [22] for details) that a locally integrable structure on a manifold M is defined by a bundle $\mathcal{V} \subset \mathbb{C}TM$ satisfying the Frobenius condition and such that the subbundle $T' \subset \mathbb{C}T^*M$ orthogonal to $\mathcal{V}$ is locally spanned by exact differentials. As usual we will denote by $T^0 = T' \cap T^*M$ the so-called *characteristic set*. For any open subset $\Omega \subset M$ the space of (p, q) -forms $C^\infty(\Omega, \Lambda^{p,q})$ is then defined and the de Rham differential induces a map

$$\begin{aligned} d' : C^\infty(\Omega, \Lambda^{p,q}) &\rightarrow C^\infty(\Omega, \Lambda^{p,q+1}), \\ d' : \mathcal{D}'(\Omega, \Lambda^{p,q}) &\rightarrow \mathcal{D}'(\Omega, \Lambda^{p,q+1}). \end{aligned}$$

When $\mathcal{V} \cap \overline{\mathcal{V}} = 0$ the structure is called CR and d' is the tangential Cauchy-Riemann operator.

Now let $\mathbf{g}$ be a Riemannian metric near a given point $x_0 \in M$, and let $B(x_0, r)$ be the corresponding open ball of radius $r > 0$. We are concerned with the following local solvability property that involves two real numbers $r \geq r' > 0$ ($1 \leq q \leq \text{rank } \mathcal{V}$):

For any given cocycle $f \in C^\infty(B(x_0, r), \Lambda^{0,q})$ there exists a distribution
 (1.1) *section $u \in \mathcal{D}'(B(x_0, r'), \Lambda^{0,q-1})$ solving $d'u = f$ in $B(x_0, r')$.*

Following [10] we define¹ the functions

$$\kappa_{q,x_0}^{\mathbf{g}}(r) = \sup\{r' > 0 : (1.1) \text{ is valid}\},$$

Received by the editors November 15, 2005 and, in revised form, June 7, 2006.
 2000 *Mathematics Subject Classification*. Primary 58J10; Secondary 35N05.

¹With the convention that $\sup \emptyset = 0$ and $\log 0 = -\infty$.

and

$$\nu_q^-(x_0) = \liminf_{r \searrow 0} \frac{\log \kappa_{q,x_0}^{\mathbf{g}}(r)}{\log r} \quad \text{and} \quad \nu_q^+(x_0) = \limsup_{r \searrow 0} \frac{\log \kappa_{q,x_0}^{\mathbf{g}}(r)}{\log r}.$$

The functions $\nu_q^\pm$ take values in $[1, +\infty]$ and, as is shown in [10], are in fact independent of the choice of the Riemannian metric $\mathbf{g}$. We refer to Section 7 of [10] for geometric-cohomological interpretations of these invariants, which apply to our context too. In any case, here we do not use results proved in [10].

We now present our result. To this end, we recall that at any point $(x_0, \omega_0) \in T^0$ a sesquilinear form $\mathcal{B}_{(x_0, \omega_0)} : \mathcal{V}_{x_0} \times \mathcal{V}_{x_0} \rightarrow \mathbb{C}$ ($\mathcal{V}_{x_0}$ is the fibre above x_0) is well defined by

$$\mathcal{B}_{(x_0, \omega_0)}(\mathbf{v}_1, \mathbf{v}_2) = \langle \omega_0, (2\iota)^{-1}[V_1, \overline{V}_2]|_{x_0} \rangle,$$

with $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}_{x_0}$, where V_1 and V_2 are smooth sections of $\mathcal{V}$ such that $V_1|_{x_0} = \mathbf{v}_1$, $V_2|_{x_0} = \mathbf{v}_2$. The associated quadratic form $\mathcal{V}_{x_0} \ni \mathbf{v} \mapsto \mathcal{B}_{(x_0, \omega_0)}(\mathbf{v}, \mathbf{v})$, or $\mathcal{B}_{(x_0, \omega_0)}$ itself, is known as the *Levi form*.

Theorem 1.1. *Let $(x_0, \omega_0) \in T^0$, $\omega_0 \neq 0$. Suppose that $\mathcal{B}_{(x_0, \omega_0)}$ has exactly q positive eigenvalues, $1 \leq q \leq \text{rank } \mathcal{V}$, and that its restriction to $\mathcal{V}_{x_0} \cap \overline{\mathcal{V}}_{x_0}$ is non-degenerate. Then for every Riemannian metric $\mathbf{g}$ it turns out that*

$$(1.2) \quad \kappa_{q,x_0}^{\mathbf{g}}(r) = O(r^{3/2}), \quad \text{as } r \searrow 0.$$

In particular, $\nu_q^-(x_0) \geq 3/2$.

The meaning of Theorem 1.1 is that, under its assumptions, either the system in (1.1) is not locally solvable even after shrinking the neighborhood or, if it is, the solvability neighborhood has to be taken much smaller than the initial one. This generalizes Theorem 7.3 of [10] (there $\mathcal{V}_x \cap \overline{\mathcal{V}}_x = \{0\}$ for every $x \in M$).

Indeed, several papers are devoted to the interplay between the signature of the Levi form and the possibility of locally solving the system $d'u = f$, possibly in a smaller neighborhood; see Lewy [13], Hörmander [11], Andreotti and Hill [2], Andreotti, Fredricks, and Nacinovich [1], Nacinovich [15, 16, 17], Kashiwara and Schapira [12], Michel [14], Treves [20, 21, 22, 23], Chen and Shaw [9], Peloso and Ricci [19], Hill and Nacinovich [10], Nicola [18]. We also refer, for other invariants, to the contributions by Cordaro and Hounie [4, 5, 6], Cordaro and Treves [7, 8], Chanillo and Treves [3].

One of the new ideas in [10], which inspired this work, was the measurement of the rate of shrinking of the neighborhood in solving the system in (1.1). In this connection Theorem 1.1 above may be of interest when the Levi form degenerates (on $\mathcal{V}_{x_0}$). In fact, it is known from Theorem XVIII.3.1 of [22] that if one supposes, in addition, that the Levi form at (x_0, ω_0) is non-degenerate on $\mathcal{V}_{x_0}$, then the Poincaré lemma does not hold at x_0 in degree q (with the expression “Poincaré lemma” we mean that for every $r > 0$ there exists $0 < r' \leq r$ such that (1.1) is valid for any Riemannian metric $\mathbf{g}$, i.e. $\kappa_{q,x_0}^{\mathbf{g}}(r) > 0$). This was a generalization of the classical result of [1].

It is important to observe that, generally, under the assumptions of Theorem 1.1 the Poincaré lemma at x_0 in degree q can hold or not; we refer to Nacinovich [16] for positive results, and to [10] for negative results.

An interesting question which remains open is an estimate of $\kappa_{q,x_0}^{\mathbf{g}}(r)$ as $r \searrow 0$ when the Levi form is allowed to degenerate on $\mathcal{V}_{x_0} \cap \overline{\mathcal{V}}_{x_0}$ as well. In fact, it seems

quite difficult in that case to give a general condition with an *invariant* meaning, although special situations can be treated with the techniques of [7] and [10].

This paper is organized as follows. Theorem 1.1 is proved in Section 3. Section 2 is devoted to some preliminary results. Eventually in Section 4 we discuss an example and the possibility of improving Theorem 1.1 in special cases.

2. PRELIMINARIES

In this section we briefly discuss the notation used in this paper and some basic results in the theory of locally integrable structure; we refer to Treves [22] for proofs. In particular, we now recall the existence of special coordinates that will play a crucial role in the next section (see section I.9 of [22]).

We regard x_0 as the origin of the coordinates, hence it will be denoted by 0. Let $n = \dim_{\mathbb{C}} \mathcal{V}_0$, $d = \dim_{\mathbb{R}} T_0^0$, $\nu = n - \dim_{\mathbb{C}} (\mathcal{V}_0 \cap \overline{\mathcal{V}}_0)$.

Proposition 2.1. *Let $(0, \omega_0) \in T^0$, $\omega_0 \neq 0$, and suppose that the restriction of the Levi form $\mathcal{B}_{(0, \omega_0)}$ to $\mathcal{V}_0 \cap \overline{\mathcal{V}}_0$ is non-degenerate. There exist coordinates x_j, y_j, s_k and t_l , $j = 1, \dots, \nu$, $k = 1, \dots, d$, $l = 1, \dots, n - \nu$, and smooth real functions $\phi_k(x, y, s, t)$, $k = 1, \dots, d$, in a neighborhood $\mathcal{O}$ of 0, satisfying*

$$(2.1) \quad \phi_k|_0 = 0 \quad \text{and} \quad d\phi_k|_0 = 0,$$

such that

$$\begin{cases} z_j := x_j + \iota y_j, & j = 1, \dots, \nu, \\ w_k := s_k + \iota \phi_k(x, y, s, t), & k = 1, \dots, d, \end{cases}$$

define a system of first integrals for $\mathcal{V}$, i.e. their differential span $T'|_{\mathcal{O}}$. Moreover, with respect to the basis

$$\left\{ \frac{\partial}{\partial \bar{z}_j} \Big|_0, \frac{\partial}{\partial t_l} \Big|_0 ; j = 1, \dots, \nu, \quad l = 1, \dots, n - \nu \right\}$$

of $\mathcal{V}_0$, the Levi form $\mathcal{B}_{(0, \omega_0)}$ reads

$$(2.2) \quad \sum_{j=1}^{p''} |\zeta_j|^2 - \sum_{j=p''+1}^{p''+q''} |\zeta_j|^2 + \sum_{l=1}^{p'} |\tau_l|^2 - \sum_{l=p'+1}^{n-\nu} |\tau_l|^2.$$

In particular

$$(2.3) \quad d'z_j = 0, \quad d'w_k = 0, \quad j = 1, \dots, \nu, \quad k = 1, \dots, d.$$

In these coordinates we have $T_0^0 = \text{span}_{\mathbb{R}} \{ds_k|_0; k = 1, \dots, d\}$, so that $\omega_0 = \sum_{k=1}^d \sigma_k ds_k|_0$, with $\sigma_k \in \mathbb{R}$. By (I.9.2) of [22] we have

$$\mathcal{B}_{(0, \omega_0)}(\mathbf{v}_1, \mathbf{v}_2) = \sum_{k=1}^d \sigma_k (V_1 \bar{V}_2 \phi_k)|_0,$$

with V_1 and V_2 smooth sections of $\mathcal{V}$ extending $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively. Hence, upon setting $\Phi = \sum_{k=1}^d \sigma_k \phi_k$, we see from (2.2) that

$$\begin{aligned}
 (2.4) \quad \Phi|_{s=0} &= \sum_{i,j=1}^{\nu} \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}_j + \frac{1}{2} \sum_{l,k=1}^{n-\nu} \frac{\partial^2 \Phi}{\partial t_l \partial t_k}(0) t_l t_k + \operatorname{Re} \sum_{j=1}^{\nu} \sum_{l=1}^k \frac{\partial^2 \Phi}{\partial \bar{z}_j \partial t_l}(0) \bar{z}_j t_l \\
 &\quad + \operatorname{Re} \sum_{i,j=1}^{\nu} \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}_j + O(|z|^3 + |t|^3) \\
 &= \sum_{j=1}^{p''} |z_j|^2 - \sum_{j=p''+1}^{p''+q''} |z_j|^2 + \frac{1}{2} \sum_{l=1}^{p'} t_l^2 - \frac{1}{2} \sum_{l=p'+1}^{n-\nu} t_l^2 \\
 &\quad + \operatorname{Re} \sum_{i,j=1}^{\nu} \frac{\partial^2 \Phi}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}_j + O(|z|^3 + |t|^3).
 \end{aligned}$$

We also observe that by (2.1) we have

$$(2.5) \quad \Phi = \Phi_{s=0} + O(|s|(|z| + |s| + |t|)) = \Phi|_{s=0} + \operatorname{Re} \sum_{j=1}^{\nu} \sum_{k=1}^d a_{jk} z_j s_k + O(|s|(|s| + |t|)).$$

If, say, $\sigma_1 \neq 0$, we may replace the functions w_k by the functions

$$(2.6) \quad \tilde{w}_1 := w_1 - \iota \sum_{i,j=1}^{\nu} \frac{\partial^2 \phi_1}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}_j - \iota \sigma_1^{-1} \sum_{j=1}^{\nu} \sum_{k=1}^d a_{jk} z_j w_k,$$

$$(2.7) \quad \tilde{w}_k := w_k - \iota \sum_{i,j=1}^{\nu} \frac{\partial^2 \phi_k}{\partial z_i \partial \bar{z}_j}(0) z_i \bar{z}_j, \quad k = 2, \dots, d,$$

so that, setting $\tilde{s}_k := \operatorname{Re} \tilde{w}_k = s_k + O(|z|(|z| + |s| + |t|))$ and $\tilde{\Phi} := \sum_{k=1}^d \sigma_k \operatorname{Im} \tilde{w}_k$, we obtain (after deleting the tildes)

$$(2.8) \quad \Phi = \sum_{j=1}^{p''} |z_j|^2 - \sum_{j=p''+1}^{p''+q''} |z_j|^2 + \frac{1}{2} \sum_{l=1}^{p'} t_l^2 - \frac{1}{2} \sum_{l=p'+1}^{n-\nu} t_l^2 + O(|s|(|s| + |t|) + |z|^3 + |t|^3).$$

We now present an a priori estimate which expresses a necessary condition for local solvability (Lemma VIII.1.1 of Treves [22]). It is a generalization of the classical estimate of Hörmander [11] and Andreotti, Fredricks, and Nacinovich [1].

Let us denote by $\|\cdot\|_{K,l}$, with $K \subset M$ compact and $l \in \mathbb{Z}_+$, the seminorms which define the topology of $C^\infty(M, \Lambda^{p,q})$. Let $m = \dim M - n$.

Lemma 2.2. *Let $1 \leq q \leq n$ and $\Omega' \subset \Omega \subset M$ be two open neighborhoods such that for every cocycle $f \in C^\infty(\Omega, \Lambda^{0,q})$ there exists a distribution section $u \in \mathcal{D}'(\Omega', \Lambda^{0,q-1})$ solving $d'u = f$ in Ω' .*

Then for every compact subset $K' \subset \Omega'$ there exist a compact $K \subset \Omega$ and constants $C > 0$, $l \in \mathbb{Z}_+$ such that, for every cocycle $f \in C^\infty(\Omega, \Lambda^{0,q})$ and every $v \in C_c^\infty(\Omega', \Lambda^{m,n-q})$, with $\operatorname{supp} v \subset K'$ we have

$$(2.9) \quad \left| \int_{\Omega} f \wedge v \right| \leq C \|f\|_{K,l} \|d'v\|_{K',l}.$$

3. PROOF OF THEOREM 1.1

First of all we observe that if the conclusion holds for a given Riemannian metric $\mathbf{g}$, then it holds for every other metric $\mathbf{g}'$, since there exist constants $C_1 > 0$, $C_2 > 0$, $r_0 > 0$ such that

$$\mathbf{d}(x_0, x) \leq C_1 \mathbf{d}'(x_0, x) \leq C_2 \mathbf{d}(x_0, x), \quad \forall x \in M \text{ with } \mathbf{d}(x_0, x) < r_0,$$

where $\mathbf{d}$ and $\mathbf{d}'$ are the corresponding distances.

We work in a small neighborhood $\mathcal{O}$ of the point x_0 , that will be taken as the origin of the coordinates, i.e. $x_0 = 0$. There we can make use of the special coordinates introduced in the previous section. Hence $\omega_0 = \sum_{k=1}^d \sigma_k ds_k|_0$. We next consider the function $\Phi = \sum_{k=1}^d \sigma_k \phi_k$, which has the form in (2.8). We may assume, without loss of generality, that $\sigma = (1, 0, \dots, 0)$. Consequently, from (2.8) (after the change of variables $t \mapsto t/\sqrt{2}$) we have

$$(3.1) \quad \phi_1(x, y, s, t) = |z'|^2 - |z''|^2 + |t'|^2 - |t''|^2 + O(|s|(|s| + |t|) + |z|^3 + |t|^3),$$

where we set

$$\begin{cases} z' = (z_1, \dots, z_{p''}), \\ z'' = (z_{p''+1}, \dots, z_{p''+q''}), \\ z''' = (z_{p''+q''+1}, \dots, z_\nu), \\ t' = (t_1, \dots, t_{p'}), \\ t'' = (t_{p'+1}, \dots, t_{n-\nu}). \end{cases}$$

Then, we introduce the Riemannian metric induced by the Euclidean one by means of the coordinates in Proposition 2.1. Hence

$$B(0, r) = \{|z|^2 + |s|^2 + |t|^2 < r^2\}.$$

Moreover, we choose a function $\chi \in C^\infty(\mathbb{R}^{2\nu+d+(n-\nu)})$, with $\chi = 1$ in $B(0, 1/2)$ and $\chi = 0$ away from $B(0, 2/3)$. We set, for $\delta > 0$,

$$\chi_{\delta,r}(x, y, s, t) = \chi\left(x/(\delta r^{3/2}), y/(\delta r^{3/2}), s/(\delta r^{3/2}), t/(\delta r^{3/2})\right),$$

and, for $\rho > 0$,

$$\begin{aligned} f_{\rho,\lambda} &= e^{\rho h_{1,\lambda}} d\bar{z}' \wedge dt' \\ v_{\rho,\lambda} &= \rho^{(m+n)/2} \chi_{\delta,r} e^{\rho h_{2,\lambda}} d\bar{z}'' \wedge d\bar{z}''' \wedge dt'' \wedge dz \wedge dw, \end{aligned}$$

where, with $\lambda > 1$,

$$(3.2) \quad h_{1,\lambda} := -\iota s_1 + \phi_1 - 2|z'|^2 - 2|t'|^2 - \lambda \sum_{k=1}^d (s_k + i\phi_k)^2,$$

and

$$(3.3) \quad h_{2,\lambda} := \iota s_1 - \phi_1 - 2|z''|^2 - 2|z'''|^2 - 2|t''|^2 - \lambda \sum_{k=1}^d (s_k + \iota\phi_k)^2.$$

We are now going to apply Lemma 2.2 with $f_{\rho,\lambda}$ and $v_{\rho,\lambda}$ in place of f and v , respectively, with $\Omega = B(0, r)$, $\Omega' = B(0, \delta r^{3/2})$. Precisely we show that there exist r_0 and δ_0 such that (2.9) fails for every choice of C and l when $\rho \rightarrow +\infty$, if $r < r_0$, $\delta > \delta_0$, and λ is large enough.

First of all we observe that $f_{\rho,\lambda} \in C^\infty(\mathcal{O}, \Lambda^{0,q})$ since $p'' + p' = q$ by hypothesis, and $v_{\rho,\lambda} \in C_0^\infty(\mathcal{O}, \Lambda^{m,n-q})$, where $\text{supp } v_{\rho,\lambda} \subset B(0, \delta r^{3/2})$ is independent of ρ, λ (recall $m = \dim M - n$). Moreover, by (2.3),

$$d'f_{\rho,\lambda} = 0,$$

and

$$(3.4) \quad d'v_{\rho,\lambda} = \rho^{(m+n)/2} e^{\rho h_{2,\lambda}} d'\chi_{\delta,r} \wedge d\bar{z}'' \wedge d\bar{z}''' \wedge dt'' \wedge dz \wedge dw.$$

In order to estimate the right-hand side of (2.9) we observe that, by (3.1) and (3.2),

$$\text{Re } h_{1,\lambda} = -|z'|^2 - |z''|^2 - |t|^2 - \lambda|s|^2 + \mathcal{R}(s, t) + O(|z|^3 + |t|^3) + \lambda O(|z|^4 + |t|^4),$$

where

$$(3.5) \quad |\mathcal{R}(s, t)| = O(|s|(|s| + |t|)) \leq \tilde{C} \left(\frac{\epsilon}{2} (|s| + |t|)^2 + \frac{1}{2\epsilon} |s|^2 \right),$$

for every $\epsilon > 0$. Hence, if ϵ and then $1/\lambda$ are small enough, we see that there exist $r_0 > 0$ and $a > 0$ such that

$$(3.6) \quad \sup_{B(0,r)} \text{Re } h_{1,\lambda} \leq ar^3, \quad \forall 0 < r < r_0.$$

Similarly,

$$\begin{aligned} \text{Re } h_{2,\lambda} = & -|z'|^2 - |z''|^2 - 2|z'''|^2 - |t|^2 - \lambda|s|^2 \\ & + \mathcal{R}'(s, t) + O(|z|^3 + |t|^3) + \lambda O(|z|^4 + |t|^4), \end{aligned}$$

with $\mathcal{R}'$ satisfying the same estimate (3.5). Therefore if λ is sufficiently large, in $\mathcal{O}$ we have

$$\text{Re } h_{2,\lambda} \leq -\frac{1}{2}(|z|^2 + |t|^2 + \lambda|s|^2) + C_1(|z|^3 + |t|^3) + C_2\lambda(|z|^4 + |t|^4).$$

Hence, possibly for a smaller r_0 , there exist constants $b > 0$, $c > 0$ such that

$$(3.7) \quad \sup_{B(0,r) \setminus B(0,\delta r^{3/2})} \text{Re } h_{2,\lambda} \leq (-b\delta^2 + c)r^3, \quad \forall 0 < r < r_0.$$

As a consequence of (3.6), (3.7), and (3.4), if $r < r_0$, for every compact subset $K \subset B(0, r)$ and any integer $l \geq 0$ it turns out that

$$(3.8) \quad \|f_{\rho,\lambda}\|_{K,l} \leq C' \rho^l e^{ar^3 \rho},$$

and

$$(3.9) \quad \|d'v_{\rho,\lambda}\|_{K,l} \leq C'' \rho^{(m+n)/2+l} e^{(-b\delta^2+c)r^3 \rho},$$

where the constants C', C'' depend on λ and r but are independent of ρ . It follows from (3.8) and (3.9) that if $\delta > \sqrt{(a+c)/b}$ for every $0 < r < r_0$, it turns out that

$$(3.10) \quad \|f_{\rho,\lambda}\|_{K,l} \|d'v_{\rho,\lambda}\|_{K,l} \leq C' C'' \rho^{(m+n)/2+2l} e^{-(b\delta^2-a-c)r^3 \rho} \longrightarrow 0, \text{ as } \rho \rightarrow +\infty.$$

We look now at the left-hand side of (2.9). We have

$$(3.11) \quad \int f_{\rho,\lambda} \wedge v_{\rho,\lambda} = c \rho^{(m+n)/2} \int e^{\rho(h_{1,\lambda}+h_{2,\lambda})} \chi_{\delta,r} \det \left(\text{Id}_{d \times d} + \iota \frac{\partial \phi}{\partial s} \right) dx dy ds dt,$$

where $0 \neq c \in \mathbb{R}$. Observe that

$$(h_{1,\lambda} + h_{2,\lambda})(x, y, s, t) = -2(|z|^2 + |t|^2) - 2\lambda|s|^2 + O(|z|^3 + |s|^3 + |t|^3).$$

Hence, if r is small enough we have

$$(3.12) \quad \operatorname{Re}(h_{1,\lambda} + h_{2,\lambda})(x, y, s, t) \leq -(|z|^2 + |t|^2 + |s|^2) \quad \text{in } B(0, r).$$

We now perform the change of variables $(x, y, s, t) \rightarrow \rho^{-1/2}(x, y, s, t)$ in (3.11). Then $\rho \cdot (h_{1,\lambda} + h_{2,\lambda})(\rho^{-1/2}x, \rho^{-1/2}y, \rho^{-1/2}s, \rho^{-1/2}t)$ converges pointwise to

$$-2(|z|^2 + |t|^2) - 2\lambda|s|^2$$

as $\rho \rightarrow +\infty$. Hence, by virtue of (3.11), (2.1), (3.12) and the Lebesgue convergence theorem, we deduce

$$\int f_{\rho,\lambda} \wedge v_{\rho,\lambda} \longrightarrow c \int e^{-2(|z|^2 + |t|^2) - 2\lambda|s|^2} dx dy ds dt \neq 0,$$

which, together with (3.10), contradicts (2.9).

This completes the proof of Theorem 1.1.

4. SOME EXAMPLES

This example is, in a sense, the model case when $\dim T_{x_0}^0 = 1$.

Consider $\mathbb{R}^{2\nu+1+(n-\nu)}$, with coordinates $x_j, y_j, s, t_l, j = 1, \dots, \nu, l = 1, \dots, n-\nu$, endowed with the locally integrable structure, of the hypersurface type, defined by the first integrals $z_j = x_j + \iota y_j$ and $w = s + \iota \phi(x, y, s, t)$, where

$$\phi(x, y, s, t) = \sum_{j=1}^{p''} |z_j|^2 - \sum_{j=p''+1}^{p''+q''} |z_j|^2 + \sum_{l=1}^{p'} t_l^2 - \sum_{l=p'+1}^{n-\nu} t_l^2 + O(|z|^3 + |t|^3 + |s|(|s| + |t|)),$$

near the origin, with $p'' + p' > 0$.

If $p'' + q'' = \nu$, then Theorem XVIII.3.1 of [22] applies to any characteristic point of the type $(0, \alpha ds|_0)$, $\alpha > 0$, above the origin, and yields $\kappa_{q,0}^{\mathbf{g}}(r) = 0$ for every $r < r_0$, $q = p' + p''$ (and $q = n - p' - p''$ if $n - p' - p'' > 0$).

If $p'' + q'' < \nu$, then Theorem 1.1 applies to any characteristic point of the type $(0, \alpha ds|_0)$, $\alpha > 0$, as before, and $\kappa_{q,0}^{\mathbf{g}}(r) = O(r^{3/2})$ as $r \searrow 0$, for $q = p' + p''$ (and $q = q'' + n - \nu - p'$ if $q'' + n - \nu - p' > 0$).

We observe that, of course, Theorem 1.1 can be improved in special cases. For example, if ϕ in the example above has instead the form

$$\phi(x, y, s, t) = \sum_{j=1}^{p''} |z_j|^2 - \sum_{j=p''+1}^{p''+q''} |z_j|^2 + \sum_{l=1}^{p'} t_l^2 - \sum_{l=p'+1}^{n-\nu} t_l^2 + O(|z|^4 + |t|^4 + |s|(|s| + |t|)),$$

then an argument similar to the proof of Theorem 1.1 shows that

$$\kappa_{q,0}^{\mathbf{g}}(r) = O(r^2), \quad \text{as } r \searrow 0, \quad q = p' + p''.$$

ACKNOWLEDGMENTS

The author wishes to thank P. Cordaro for correspondence on the subject of this paper. The author is also very grateful to M. Nacinovich for providing his paper [10] and to L. Rodino for helpful discussions.

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