BILLINGSLEY'S PACKING DIMENSION

MANAV DAS

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ABSTRACT. For a stochastic process on a finite state space, we define the notion of a packing measure based on the naturally defined cylinder sets. For any two measures ν, γ , corresponding to the same stochastic process, if

$$F \subseteq \left\{ \omega \in \Omega : \lim_n \frac{\log \gamma(c_n(\omega))}{\log \nu(c_n(\omega))} = \theta \right\},$$

then we prove that

$$\text{Dim}_\nu(F) = \theta \text{Dim}_\gamma(F).$$

INTRODUCTION

Let $x = \{x_1, x_2, \dots\}$ be a stochastic process defined on a probability measure space $(\Omega, \mathcal{B}, \mu)$. Suppose that the state space of the process is a finite set σ , the states of which are given by $\sigma = \{1, 2, \dots, N\}$. For $u \in \Omega$ an n -cylinder is a set of the form

$$c_n(u) = \{\omega : x_k(\omega) = u_k, k = 1, 2, \dots, n\}.$$

Let C denote the collection of all such n -cylinders for $n = 1, 2, \dots$. We may use this collection to define a cylinder packing measure for a set $F \subseteq \Omega$. Let $\text{Dim}_\mu(F)$ denote the corresponding cylinder packing dimension. We prove the following result:

Theorem 0.1. *Let ν and γ be any two measures corresponding to the same stochastic process. If*

$$M \subseteq \left\{ \omega \in \Omega : \lim_n \frac{\log \gamma(c_n(\omega))}{\log \nu(c_n(\omega))} = \theta \right\},$$

then

$$\text{Dim}_\nu M = \theta \text{Dim}_\gamma M.$$

This result is used in [7] to analyze the multifractal decomposition of graph-directed self-similar measures: let μ, ν be two graph-directed self-similar measures in $\mathbf{R}^d$ satisfying the open set condition. Let K be the associated invariant set. For $\alpha > 0$ we prove that if

$$F \subseteq K_\nu^{(\alpha)} = \left\{ x \in K : \lim_{\epsilon \rightarrow 0} \frac{\log \mu(B_\epsilon(x))}{\log \nu(B_\epsilon(x))} = \alpha \right\},$$

then

$$\text{Dim}_\nu(F : B) = \theta \text{Dim}_\mu(F : B).$$

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Here $\text{Dim}_\nu(F : B)$ denotes the packing dimension computed with respect to balls, rather than the cylinder sets.

Since we will be dealing with ratios of logarithms, we will need some conventions. For $0 < \varrho, \varsigma < 1$,

$$\begin{aligned}\frac{\log \varrho}{\log 0} &= \frac{\log 1}{\log \varsigma} = \frac{\log 1}{\log 0} = 0. \\ \frac{\log 0}{\log \varsigma} &= \frac{\log \varrho}{\log 1} = \frac{\log 0}{\log 1} = \infty. \\ \frac{\log 0}{\log 0} &= \frac{\log 1}{\log 1} = 1.\end{aligned}$$

1. DIMENSION IN PROBABILITY SPACES

We now introduce the notion of a cylinder packing measure in probability spaces and show its relation to the cylinder Hausdorff measure defined by Billingsley [2], and prove our main result.

1.1. Cylinder packing measure. Let F be a subset of Ω and let $\rho > 0$. If $\mathcal{V}$ is an enumerable collection of disjoint cylinders, then we say that $\mathcal{V}$ is a $\mu - \rho$ packing of F provided $\mu(v) < \rho$ and $F \cap v \neq \emptyset$, $\forall v \in \mathcal{V}$. If $\alpha \geq 0$, let

$$\widetilde{\mathcal{M}}_{\mu, \rho}^\alpha(E) = \sup\{\sum \mu(v)^\alpha : v \in \mathcal{V}\},$$

where the supremum is over all $\mu - \rho$ packings $\mathcal{V}$ of E . If there exist no $\mu - \rho$ packings of E , then we set $\widetilde{\mathcal{M}}_{\mu, \rho}^\alpha(E) = 0$. As $\rho \rightarrow 0$, $\widetilde{\mathcal{M}}_{\mu, \rho}^\alpha(E)$ decreases to a limit, $\widetilde{\mathcal{M}}_\mu^\alpha(E)$. Let

$$\overline{\mathcal{M}}_\mu^\alpha(F) = \inf_{\mathcal{C}} \sum_{E \in \mathcal{C}} \widetilde{\mathcal{M}}_\mu^\alpha(E)$$

where $\mathcal{C}$ is any countable cover of the set F . Since we use Method I (see [9]) we obtain an outer measure. Moreover, we assume that the stochastic process is such that every ∞ -cylinder has μ measure 0. Let $\mathcal{M}_\mu^\alpha$ denote the restriction of this outer measure to the measurable sets. We call $\mathcal{M}_\mu^\alpha$ the cylinder packing measure.

Proposition 1.1. *Let $F \subseteq \Omega$. Let $0 < s < t$. Then*

- (i) $\widetilde{\mathcal{M}}_\mu^s(F) < \infty \Rightarrow \widetilde{\mathcal{M}}_\mu^t(F) = 0$.
- (ii) $\widetilde{\mathcal{M}}_\mu^t(F) > 0 \Rightarrow \widetilde{\mathcal{M}}_\mu^s(F) = \infty$.

Proof. (i) Suppose $\widetilde{\mathcal{M}}_\mu^s(F) < \infty$. Then there exist $\rho_0 > 0$ and $K < \infty$ such that $\widetilde{\mathcal{M}}_{\mu, \rho_0}^s(F) \leq K$. Let $0 < \rho < \rho_0$. If $\{v_i\}$ is a $\mu - \rho$ packing of F , then

$$\begin{aligned}\sum \mu(v_i)^t &\leq \sum \mu(v_i)^s \mu(v_i)^{t-s} \\ &\leq \rho^{t-s} \sum \mu(v_i)^s \\ &\leq \rho^{t-s} K \rightarrow 0, \quad \text{as } \rho \rightarrow 0.\end{aligned}$$

Thus $\widetilde{\mathcal{M}}_\mu^t(F) = 0$.

- (ii) This is just the contrapositive of the above statement. □

Thus there exists a critical point, $\tilde{\alpha}_0$, called the cylinder packing index, such that

$$\begin{aligned}\widetilde{\mathcal{M}}_\mu^\alpha(F) &= \infty, \quad \forall \alpha < \tilde{\alpha}_0. \\ \widetilde{\mathcal{M}}_\mu^\alpha(F) &= 0, \quad \forall \alpha > \tilde{\alpha}_0.\end{aligned}$$

Theorem 1.2. *Let $F \subseteq \Omega$. Let $0 < s < t$. Then*

- (i) $\mathcal{M}_\mu^s(F) < \infty \Rightarrow \mathcal{M}_\mu^t(F) = 0$.
- (ii) $\mathcal{M}_\mu^t(F) > 0 \Rightarrow \mathcal{M}_\mu^s(F) = \infty$.

Proof. (i) Suppose $\mathcal{M}_\mu^s(F) < \infty$. Then there is a covering $\mathcal{C} = \{C_i\}$ of F such that $\widetilde{\mathcal{M}}_\mu^s(C_i) < \infty$ for each i . If $t > s$, then by Proposition 1.1, $\widetilde{\mathcal{M}}_\mu^t(C_i) = 0 \forall i$ and so,

$$\mathcal{M}_\mu^t(F) = \inf_{\mathcal{C}} \sum_{C \in \mathcal{C}} \widetilde{\mathcal{M}}_\mu^t(C) \leq 0.$$

- (ii) This is the contrapositive of (i). □

Thus there exists a critical point α_0 called the cylinder packing dimension such that

$$\begin{aligned}\mathcal{M}_\mu^\alpha(F) &= \infty, \quad \forall \alpha < \alpha_0. \\ \mathcal{M}_\mu^\alpha(F) &= 0, \quad \forall \alpha > \alpha_0.\end{aligned}$$

Let

$$\text{Dim}_\mu F = \inf\{\alpha > 0 : \mathcal{M}_\mu^\alpha(F) = 0\}.$$

Note that it's possible to have $\mathcal{M}_\mu^\alpha(F) = 0$ for all $\alpha > 0$. In this case $\text{Dim}_\mu F = 0$. Similarly $\mathcal{M}_\mu^\alpha(F) = \infty$ for all $\alpha > 0$ would mean $\text{Dim}_\mu F = 1$.

1.2. Cylinder Hausdorff measure. This notion was introduced by Billingsley in [2]. Let

$$\begin{aligned}\overline{\mathcal{H}}_{\mu,\rho}^\alpha(F) &= \inf\left\{\sum \mu(v)^\alpha : F \subseteq \bigcup v, \mu(v) < \rho\right\}, \\ \mathcal{H}_\mu^\alpha(F) &= \lim_{\delta \rightarrow 0^+} \overline{\mathcal{H}}_{\mu,\rho}^\alpha(F).\end{aligned}$$

Let $\dim_\mu F$ denote the corresponding cylinder Hausdorff dimension.

Proposition 1.3. *For any $F \subseteq \Omega$, $\mathcal{H}_\mu^\alpha(F) \leq \mathcal{M}_\mu^\alpha(F)$.*

Proof. Let $\rho > 0$. Let $\mathcal{U} = \bigcup v_i$ be a $\mu - \rho$ cover of F and let $\mathcal{V}$ consist of those cylinders of $\mathcal{U}$ which are not subsets of other elements of $\mathcal{U}$. Then $\mathcal{V}$ covers F and its elements are disjoint. Moreover, $v \in \mathcal{V} \Rightarrow \mu(v) < \rho$. Thus $\mathcal{V}$ is also a $\mu - \rho$ -packing of F and so

$$\overline{\mathcal{H}}_{\mu,\rho}^\alpha(F) \leq \sum \mu(v)^\alpha : v \in \mathcal{V} \leq \widetilde{\mathcal{M}}_{\mu,\rho}^\alpha(F).$$

The result follows from this. □

Corollary 1.4. $\dim_\mu F \leq \text{Dim}_\mu F$.

2. THE MAIN RESULT

Proof of Theorem 0.1. We first prove the theorem under the assumptions that $\theta > 0$, $0 < s = \text{Dim}_\gamma M < 1$ and $\nu(c_n(\omega)) > 0$, $\forall n \in \mathbf{N}$. We will then show how these restrictions may be removed. Choose $\delta_0 > 0$ such that $\theta - \frac{\delta_0}{s} > 0$. For any $\delta \leq \delta_0$ let

$$\begin{aligned} S^{(k)} &= \left\{ \omega \in M : \theta - \frac{\delta}{s} \leq \frac{\log \gamma(c_k(\omega))}{\log \nu(c_k(\omega))} \leq \theta + \frac{\delta}{s} \right\}, \\ T^{(N)} &= \bigcap_{k=N}^{\infty} S^{(k)}, \\ M &= \bigcup_{N=1}^{\infty} T^{(N)}. \end{aligned}$$

It therefore suffices to show that $\text{Dim}_\nu T^{(N)} \leq s\theta + \delta$ for each $N \in \mathbf{N}$. Fix $N \in \mathbf{N}$. Choose $\rho > 0$ such that $\rho < \nu(c_N(\omega))$, $\forall \omega \in \Omega$. Let $\mathcal{V}$ be a $\nu - \rho$ packing of $T^{(N)}$. By our choice of $\rho > 0$, for any $c_{n_i}(\omega_i) \in \mathcal{V}$, we must have

$$\nu(c_{n_i}(\omega_i)) < \rho < \nu(c_N(\omega_i)),$$

for each i . Thus $n_i > N$. By the definition of $T^{(N)}$,

$$\begin{aligned} \gamma(c_{n_i}(\omega_i)) &\geq \nu(c_{n_i}(\omega_i))^{(\theta + \frac{\delta}{s})}, \quad \text{and} \\ \gamma(c_{n_i}(\omega_i)) &\leq \nu(c_{n_i}(\omega_i))^{(\theta - \frac{\delta}{s})} < \rho^{(\theta - \frac{\delta}{s})}, \end{aligned}$$

and so $\mathcal{V}$ is also a $\gamma - \rho^{(\theta - \frac{\delta}{s})}$ packing of $T^{(N)}$. Also $\theta - \frac{\delta}{s} > 0$, by our choice of δ , and hence

$$\rho^{(\theta - \frac{\delta}{s})} \rightarrow 0, \text{ as } \rho \rightarrow 0.$$

Thus

$$\widehat{\mathcal{M}}_{\gamma, \rho^{(\theta - \frac{\delta}{s})}}^s(T^{(N)}) \geq \widehat{\mathcal{M}}_{\nu, \rho}^{s\theta + \delta}(T^{(N)}),$$

and letting $\rho \rightarrow 0$, since $0 < s < 1$, we obtain

$$\widehat{\mathcal{M}}_{\nu}^{s\theta + \delta}(T^{(N)}) \leq \widehat{\mathcal{M}}_{\gamma}^s(T^{(N)}) < \infty.$$

And so,

$$\text{Dim}_\nu(T^{(N)}) \leq s\theta + \delta.$$

But $\delta > 0$ was arbitrary. So

$$\text{Dim}_\nu(T^{(N)}) \leq \theta \text{Dim}_\gamma M.$$

Since this is true for each N , we get

$$\text{Dim}_\nu M \leq \theta \text{Dim}_\gamma M.$$

Now we remove the restrictions imposed at the beginning of the proof. We begin by considering the cases $s = 0, 1$. Choose $\delta > 0$ such that $\theta - \delta > 0$, and redefine $S^{(k)}$ as follows:

$$S^{(k)} = \left\{ \omega \in M : \theta - \delta \leq \frac{\log \gamma(c_k(\omega))}{\log \nu(c_k(\omega))} \leq \theta + \delta \right\}.$$

Consider a $\nu - \rho^{(\theta - \delta)}$ packing of $T^{(N)}$. Suppose $\text{Dim}_\gamma M = 0$. Then as before, we would obtain for any $\alpha > 0$,

$$\widehat{\mathcal{M}}_{\gamma, \rho^{(\theta - \delta)}}^\alpha(T^{(N)}) \geq \widehat{\mathcal{M}}_{\nu, \rho}^{\alpha(\theta + \delta)}(T^{(N)}).$$

Letting $\rho \rightarrow 0$, we obtain

$$\widehat{\mathcal{M}}_\nu^{\alpha(\theta+\delta)}(T^{(N)}) \leq \widehat{\mathcal{M}}_\gamma^\alpha(T^{(N)}).$$

But since $\text{Dim}_\gamma M = 0$ we obtain $\text{Dim}_\nu M \leq \alpha(\theta + \delta) \rightarrow 0$ as $\alpha \rightarrow 0$. Next suppose $\text{Dim}_\gamma M = 1$. By the definition of $T^{(N)}$,

$$\begin{aligned} \gamma(c_{n_i}(\omega_i)) &\geq \nu(c_{n_i}(\omega_i))^{(\theta+\delta)}, \quad \text{and} \\ \gamma(c_{n_i}(\omega_i)) &\leq \nu(c_{n_i}(\omega_i))^{(\theta-\delta)} < \rho^{(\theta-\delta)}. \end{aligned}$$

Thus $\mathcal{V}$ is also a $\gamma - \rho^{(\theta-\delta)}$ packing of $T^{(N)}$. As before for any $\alpha > 0$ we have

$$\widehat{\mathcal{M}}_{\gamma, \rho^{(\theta-\delta)}}^\alpha(T^{(N)}) \leq \widehat{\mathcal{M}}_{\nu, \rho}^{\alpha(\theta-\delta)}(T^{(N)}).$$

Letting $\rho \rightarrow 0$, we obtain

$$\widehat{\mathcal{M}}_\nu^{\alpha(\theta-\delta)}(T^{(N)}) \geq \widehat{\mathcal{M}}_\gamma^\alpha(T^{(N)}).$$

Since $\text{Dim}_\gamma M = 1$, we obtain $\widehat{\mathcal{M}}_\nu^{\alpha(\theta-\delta)}(T^{(N)}) = \infty$, $\forall \alpha > 0$, and hence $\text{Dim}_\nu M = 1$. Since $\theta > 0$, we may interchange the roles of ν and γ , and replace θ by $1/\theta$ in the definition of M , to obtain the reverse inequality.

Next we remove the restriction $\nu(v) > 0$ for all cylinders v . Let E_ν be the union of all cylinders of ν measure 0. If there is ω such that $\nu(v_n(\omega)) = 0$ for some n , then by our conventions in Section 1, since $\theta > 0$, there must be some m so that $\gamma(v_m(\omega)) = 0$. So $M \cap E_\nu \subseteq M \cap E_\gamma$. We have already proved that $\text{Dim}_\nu(M \setminus E_\nu) = \theta \text{Dim}_\gamma(M \setminus E_\nu)$. Thus,

$$\text{Dim}_\nu M = \text{Dim}_\nu(M \setminus E_\nu) = \theta \text{Dim}_\gamma(M \setminus E_\nu) \geq \theta \text{Dim}_\gamma(M \setminus E_\gamma) = \theta \text{Dim}_\gamma M.$$

Again, we may interchange the roles of ν and γ , and replace θ by $1/\theta$ in the definition of M , to obtain the reverse inequality.

Finally suppose $\theta = 0$. By our conventions in Section 1, and since $\gamma(c_{n_i}(\omega_i)) < 1$ for n large, we must have $\nu(c_{n_i}(\omega_i)) = 0$ for large n . It then follows that $\text{Dim}_\nu M = 0$. $\square$

Corollary 2.1. *If $\gamma M > 0$, then $\dim_\nu M = \text{Dim}_\nu M = \theta$.*

Proof. This follows from Theorem 0.1 and Billingsley's result [2]. $\square$

In particular we see that such subsets of a probability space are actually Taylor fractals [10] in the sense that their Hausdorff and packing dimensions coincide. Packing dimension in probability spaces has also been defined and studied in [5]. Their approach is more general and the result stated above may be deduced from their results. However we present a direct proof akin to Billingsley's original formulation.

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DEPARTMENT OF MATHEMATICS, 328 NATURAL SCIENCES BUILDING, UNIVERSITY OF LOUISVILLE, LOUISVILLE, KENTUCKY 40292

E-mail address: `manav@louisville.edu`