A MÜNTZ SPACE HAVING NO COMPLEMENT IN L_1

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ABSTRACT. in $C([0, 1])$. In the present paper, we prove that there is a Müntz space not complemented in $L_1([0, 1])$.

1. INTRODUCTION AND NOTATION

We denote the norm in $L_p([0, 1])$ by $\|\cdot\|_p$. Suppose that $\Lambda = (\lambda_i)_{i=0}^\infty$ is a sequence of distinct positive real numbers satisfying $\sum_{i=1}^\infty 1/\lambda_i < \infty$ and consider the closed span $[M(\Lambda)]_{L_1([0,1])}$ of the monomials x^λ , $\lambda \in \Lambda$. By the Müntz theorem [1, p. 172], $[M(\Lambda)]_{L_1([0,1])}$ is not all of $L_1([0, 1])$, and by a later theorem of Clarkson-Erdős [5, p. 81] and Schwartz [9], $[M(\Lambda)]_{L_1([0,1])}$ consists only of “power” series $\sum_{\lambda \in \Lambda} a_\lambda x^\lambda$ convergent on $[0, 1]$ if Λ satisfies the gap condition $\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{N}\} > 0$.

Newman [7] gave an example of Müntz space with no complement in $C([0, 1])$. In the present paper, we prove that there is a Müntz space not complemented in $L_1([0, 1])$ (see Theorem 3.6). In Theorem 3.2, we get an l_1 -Schauder decomposition of $[M(\Lambda_\epsilon)]_{L_1}$, where $\Lambda_\epsilon = \bigcup_{k=1}^\infty \{N_k, 2N_k, \dots, kN_k\}$ is not quasilacunary. But the sequence $\{N_1, N_2, \dots\}$ of the initial terms in the blocks composing Λ_ϵ is of course lacunary, and if $\Lambda'_\epsilon = \{N_1, N_2, \dots\}$, the sequence $(x^{N_k})_k$ is a Schauder basis of $[M(\Lambda'_\epsilon)]_{L_1}$, $(1 + \epsilon)$ -equivalent to the canonical basis of l_1 , so in a sense our result contains a result due to Gurariy and Macaev [6].

To know more on the geometry of Müntz spaces, see the recent monograph of Gurariy and Lusky [5], where they prove a similar result to our Theorem 3.2 but only when Λ is a quasilacunary sequence (cf. [5, Theorem 9.3.3]).

2. PRELIMINARY RESULTS

In this section we recall some properties of Müntz spaces that we list as propositions.

Proposition 2.1 ([1, p. 177, E.3.c]). *Suppose $\Lambda = (\lambda_i)_{i=0}^\infty$ is a sequence of positive real numbers satisfying the gap condition*

$$\inf\{\lambda_i - \lambda_{i-1} : i \in \mathbb{N}\} > 0 \quad \text{and} \quad \sum_{i=1}^\infty \frac{1}{\lambda_i} < \infty.$$

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Then, for every $\epsilon > 0$, there is a constant $c(\epsilon, \Lambda)$ depending only on ϵ and $(\lambda_i)_{i=0}^\infty$ such that

$$|a_i|(1 - \epsilon)^{\lambda_i} \leq c(\epsilon, \Lambda)\|p\|_2$$

for every $p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ of the form $p(x) = \sum_{i=0}^n a_i x^{\lambda_i}$.

Proposition 2.2 ([1, p. 185, E.8.a]). Assume that $0 \leq \lambda_0 < \lambda_1 < \dots$ with $\sum_{i=1}^\infty 1/\lambda_i < \infty$. Then, for every $\epsilon \in (0, 1)$, there is a constant $\gamma(\epsilon, \Lambda)$ depending only on ϵ and $(\lambda_i)_{i=0}^\infty$ such that

$$\|p\|_{[0, 1-\epsilon]} \leq \gamma(\epsilon, \Lambda) \int_{1-\epsilon}^1 |p(x)| dx$$

for every $p \in \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$.

Proposition 2.3 (Newman's inequality in L_p , [1, p. 279]). Let $p \in [1, \infty)$. If $\Lambda = (\lambda_i)_{i=0}^\infty$ is a sequence of distinct positive real numbers greater than $-1/p$, then

$$\|xP'\|_p \leq \left(\frac{1}{p} + 12 \sum_{j=0}^n \left(\lambda_j + \frac{1}{p} \right) \right) \|P\|_p$$

for every $P \in M_n(\Lambda) = \text{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots, x^{\lambda_n}\}$.

In Müntz spaces, we obtain better estimates.

Remark 2.4. Let $\Lambda = (\lambda_i)_{i=0}^\infty$ be a sequence of distinct positive integers satisfying $\sum_{i=1}^\infty 1/\lambda_i < \infty$. Then by Proposition 2.2 there exists a constant $C(\Lambda) > 0$ (depending only on Λ) with

$$\|p'\|_1 \leq C(\Lambda) \left(\sum_{j=m}^n \lambda_j \right) \|p\|_1$$

whenever $p(x) = \sum_{j=m}^n \alpha_j x^{\lambda_j}$ (for any m and n).

The next corollary is a direct consequence of the two previous propositions.

Corollary 2.5. Let $\Lambda = (\lambda_i)_{i=0}^\infty$ be a sequence of distinct positive integers satisfying $\sum_{i=1}^\infty 1/\lambda_i < \infty$. Then there is a constant $K(\Lambda) > 0$ (depending only on Λ) such that

$$\|p\|_\infty \leq K(\Lambda) \left(1 + \sum_{j=m}^n \lambda_j \right) \|p\|_1$$

whenever $p(x) = \sum_{j=m}^n \alpha_j x^{\lambda_j}$ (for any m and n).

Proof. Writing $p(x) = p(0) + \int_0^x p'(t) dt$, we have $|p(x)| \leq |p(0)| + \int_0^1 |p'(t)| dt$. By Proposition 2.2 there is $\gamma(\Lambda)$ such that $|p(0)| \leq \gamma(\Lambda) \int_0^1 |p(t)| dt$. Newman's

inequality applied to p yields $\|p'\|_1 \leq C(\Lambda) \left(\sum_{j=m}^n \lambda_j \right) \|p\|_1$. Then

$$\begin{aligned} \|p\|_\infty &\leq \gamma(\Lambda) \|p\|_1 + C(\Lambda) \left(\sum_{j=m}^n \lambda_j \right) \|p\|_1 \\ &\leq \max(\gamma(\Lambda), C(\Lambda)) \left(1 + \sum_{j=m}^n \lambda_j \right) \|p\|_1. \end{aligned}$$

We take $K(\Lambda) = \max(\gamma(\Lambda), C(\Lambda))$. □

3. MAIN RESULTS

The next lemma is the key point of our construction.

Lemma 3.1. *For every $\epsilon > 0$ and every $m \in \mathbb{N}$, there is a positive integer $N = N(\epsilon, m)$ depending only on ϵ and m (but not on p or q) such that*

$$\|p(x)\|_1 + \|q(x^N)\|_1 \leq (1 + \epsilon) \|p(x) + q(x^N)\|_1$$

for every $p, q \in \text{span}\{1, x, x^2, \dots, x^m\}$ with $q(0) = 0$.

Proof. Let $(N_k)_{k=1}^\infty$ be an increasing sequence of positive integers satisfying $N_1 > m$ and $N_{k+1} > kN_k$. Put $\Lambda_m = \{1, 2, \dots, m\} \cup \bigcup_{k=1}^\infty \{N_k, 2N_k, \dots, kN_k\}$.

Let $p, q \in \text{span}\{1, x, x^2, \dots, x^m\}$ with $q(x) = \sum_{j=1}^m q_j x^j$. Then for any $\alpha \in (0, 1)$ and any $N \in \{N_m, N_{m+1}, \dots\}$, we have

$$\begin{aligned} \int_0^{1-\alpha} |q(x^N)| dx &\leq \sum_{j=1}^m |q_j| \int_0^{1-\alpha} x^{jN} dx \\ &\leq \frac{m}{N} \sup_{1 \leq j \leq m} |q_j| (1 - \alpha)^{jN} \\ &\leq c(\alpha, m) \frac{m}{N} \|p(x) + q(x^N)\|_2. \end{aligned}$$

The last inequality follows from the fact that q_j is the coefficient of x^{jN} in $p(x) + q(x^N) \in M(\Lambda_m)$ and from Proposition 2.1. Therefore

$$\int_0^{1-\alpha} |q(x^N)| dx \leq c(\alpha, m) \frac{m}{N} \|p(x) + q(x^N)\|_\infty^{\frac{1}{2}} \|p(x) + q(x^N)\|_1^{\frac{1}{2}}.$$

On the other hand, writing $Q(x) = p(x) + q(x^N)$, $Q(x) \in M(\Lambda_m)$. Then we have by Corollary 2.5

$$\|Q\|_\infty \leq K(\Lambda_m) \left(1 + \sum_{j=1}^m j + N \sum_{j=1}^m j \right) \|Q\|_1 \leq K(\Lambda_m) m^2 N \|Q\|_1$$

and hence, if we put $\delta(\alpha, m) = m^2 c(\alpha, m) \sqrt{K(\Lambda_m)}$, we have

$$(*) \quad \int_0^{1-\alpha} |q(x^N)| dx \leq \frac{\delta(\alpha, m)}{\sqrt{N}} \|p(x) + q(x^N)\|_1$$

In the same way, by Corollary 2.5 we have

$$\|p\|_\infty \leq K(\Lambda_m) \left(1 + \sum_{j=0}^m j\right) \|p\|_1 \leq m^2 K(\Lambda_m) \|p\|_1,$$

and so we have

$$(**) \quad \int_{1-\alpha}^1 |p(x)| dx \leq \alpha \|p\|_\infty \leq \alpha m^2 K(\Lambda_m) \|p\|_1.$$

Now we return to the proof of our inequality. We can write

$$\begin{aligned} & \int_0^1 |p(x) + q(x^N)| dx \\ & \geq \left(\int_0^{1-\alpha} |p(x)| dx - \int_0^{1-\alpha} |q(x^N)| dx \right) + \left(\int_{1-\alpha}^1 |q(x^N)| dx - \int_{1-\alpha}^1 |p(x)| dx \right) \\ & \geq \int_0^1 |p(x)| dx + \int_0^1 |q(x^N)| dx - 2 \int_{1-\alpha}^1 |p(x)| dx - 2 \int_0^{1-\alpha} |q(x^N)| dx. \end{aligned}$$

Using(*) and (**), we obtain that

$$\begin{aligned} & \int_0^1 |p(x) + q(x^N)| dx \\ & \geq (1 - 2\alpha m^2 K(\Lambda_m)) (\|p\|_1 + \|q(x^N)\|_1) - 2 \frac{\delta(\alpha, m)}{\sqrt{N}} \|p(x) + q(x^N)\|_1, \end{aligned}$$

and hence

$$\|p(x)\|_1 + \|q(x^N)\|_1 \leq \left(\frac{1 + 2 \frac{\delta(\alpha, m)}{\sqrt{N}}}{1 - 2\alpha m^2 K(\Lambda_m)} \right) \|p(x) + q(x^N)\|_1.$$

Suppose α is such that $2\alpha m^2 K(\Lambda_m) \leq \frac{\epsilon}{2+\epsilon}$ and $N \in \{N_m, N_{m+1}, \dots\}$ such that $2 \frac{\delta(\alpha, m)}{\sqrt{N}} \leq \frac{\epsilon}{2+\epsilon}$. We have $\|p(x)\|_1 + \|q(x^N)\|_1 \leq (1 + \epsilon) \|p(x) + q(x^N)\|_1$. $\square$

In the following theorem, we get an l_1 -Schauder decomposition of $[M(\Lambda_\epsilon)]_{L_1}$, where $\Lambda_\epsilon = \bigcup_{k=1}^\infty \{N_k, 2N_k, \dots, kN_k\}$ is not quasilacunary $\left(\frac{kN_k}{(k-1)N_k} \rightarrow 1\right)$. But, for every set with shape $\Lambda'_\epsilon = \bigcup_{k=1}^\infty A_k$, where $A_k \subset \{N_k, 2N_k, \dots, kN_k\}$ with $\max_k \text{card} A_k < \infty$, Λ'_ϵ is quasilacunary and $[M(\Lambda'_\epsilon)]_{L_1}$ is isomorphic to l_1 . Obviously, the sequence $\{N_1, N_2, \dots\}$ of the initial terms in the blocks composing Λ_ϵ is lacunary ($N_{k+1} > kN_k$). Writing $\Lambda''_\epsilon = \{N_1, N_2, \dots\}$, the sequence $(x^{N_k})_k$ is a Schauder basis of $[M(\Lambda''_\epsilon)]_{L_1}$, $(1 + \epsilon)$ -equivalent to the canonical basis of l_1 , so in a sense our result contains the theorem of Gurariy and Lusky [5, Theorem 9.3.3].

Theorem 3.2. *For every $\epsilon > 0$, there exists a sequence of increasing positive integers, $(N_k)_{k=1}^\infty$ with $N_{k+1} > kN_k$, depending only on ϵ such that if $\Lambda_\epsilon = \bigcup_{k=1}^\infty \{N_k, 2N_k, \dots, kN_k\}$, we have $[M(\Lambda_\epsilon)]_{L_1} \equiv (\sum_{k=1}^\infty F_k)_{(l_1)}$ where $F_k = \text{span}\{x^{N_k}, x^{2N_k}, \dots, x^{kN_k}\}$. More precisely, we have $\sum_{k=1}^\infty \|p_k(x^{N_k})\|_1 \leq (1 + \epsilon) \|f\|_1$ whenever $f \in [M(\Lambda_\epsilon)]_{L_1}$ with an Erdős decomposition:*

$$f(x) = \sum_{k=1}^\infty p_k(x^{N_k}), \quad x \in [0, 1), \quad p_k \in \text{span}\{1, x, x^2, \dots, x^{N_k}\}.$$

Proof. Let $\epsilon_k > 0$, $k = 1, 2, \dots$, be such that $\prod_{k=1}^{\infty} (1 + \epsilon_k) = (1 + \epsilon)$. Now we fix $N_1 \geq 0$ and define $N_{k+1} = N(\epsilon_{k+1}, kN_k)$ as in the preceding lemma.

We shall use induction on n to prove

$$(1) \quad \sum_{k=1}^n \|p_k(x^{N_k})\|_1 \leq \prod_{k=1}^n (1 + \epsilon_k) \left\| \sum_{k=1}^n p_k(x^{N_k}) \right\|_1$$

for every n and every $p_k \in \text{span}\{1, x, x^2, \dots, x^k\}$. The result is trivial for $n = 1$. Now we may assume that it holds for $n - 1$, that is to say

$$\sum_{k=1}^{n-1} \|p_k(x^{N_k})\|_1 \leq \prod_{k=1}^{n-1} (1 + \epsilon_k) \left\| \sum_{k=1}^{n-1} p_k(x^{N_k}) \right\|_1.$$

We now apply Lemma 3.1 with

$$p(x) = \sum_{k=1}^{n-1} p_k(x^{N_k}), \quad q(x) = p_n(x) \quad \text{and} \quad m = (n-1)N_{n-1}.$$

The degrees of p and q are clearly less than m , so the hypotheses of Lemma 3.1 are satisfied. Then

$$\begin{aligned} \sum_{k=1}^n \|p_k(x^{N_k})\|_1 &\leq \prod_{k=1}^{n-1} (1 + \epsilon_k) \left\| \sum_{k=1}^{n-1} p_k(x^{N_k}) \right\|_1 + \|p_n(x^{N_n})\|_1 \\ &\leq \prod_{k=1}^n (1 + \epsilon_k) \left\| \sum_{k=1}^n p_k(x^{N_k}) \right\|_1 \end{aligned}$$

and the induction is complete. Now, let $f(x) = \sum_{k=1}^{\infty} p_k(x^{N_k}) \in [M(\Lambda_\epsilon)]_{L_1}$, where $p_k \in \text{span}\{1, x, x^2, \dots, x^k\}$. Fix $\rho \in (0, 1)$, and let $f_\rho(x) = f(\rho x) \in [M(\Lambda_\epsilon)]_{L_1}$. By (1), we have for every $n \geq 1$

$$\sum_{k=1}^n \|p_k((\rho x)^{N_k})\|_1 \leq (1 + \epsilon) \left\| \sum_{k=1}^n p_k((\rho x)^{N_k}) \right\|_1.$$

We may now let $n \rightarrow \infty$ to get $\sum_{k=1}^{\infty} \|p_k((\rho x)^{N_k})\|_1 \leq (1 + \epsilon) \|f_\rho\|_1$. Then letting $\rho \rightarrow 1$, gives $\sum_{k=1}^{\infty} \|p_k(x^{N_k})\|_1 \leq (1 + \epsilon) \|f\|_1$. This ends the proof of Theorem 3.2. $\square$

Remark 3.3. For $\epsilon > 0$, let $\Lambda_\epsilon = (N_k)_k$ given by the preceding theorem. We have that $[M(\Lambda_\epsilon)]_{L_1}$ is $(1 + \epsilon)$ -isomorphic to l_1 so by choosing ϵ such that $(1 + \epsilon) \leq \sqrt{2}$, we have, by [2, Theorem A], that $[M(\Lambda_\epsilon)]_{L_1}$ is complemented in $L_1([0, 1])$.

We will need the following:

Theorem 3.4 ([3, Theorem 1.1]). *There is a $c > 0$ such that any projection from $L_1([-1, 1])$ onto the span of $1, x, x^2, \dots, x^n$ has norm $> c \log n$.*

Note that this theorem remains true if one replaces $L_1([-1, 1])$ by $L_1([0, 1])$, because the isometry $\theta: L_1([-1, 1]) \rightarrow L_1([0, 1])$ taking $f(x)$ into $2f(2x-1)$ transforms $\text{sp}\{1, x, x^2, \dots, x^n\}$ onto itself.

Lemma 3.5. *There is a $c > 0$ such that for any $N \geq 1$ and any projection P from $L_1([0, 1])$ onto $\text{sp}\{x^{N-1}, x^{2N-1}, \dots, x^{kN-1}\}$, we have $\|P\| > c \log k$.*

Proof. Let P be such a projection and let T_N denote the map

$$\begin{aligned} T_N : L_1([0, 1]) &\mapsto L_1([0, 1]) \\ f(x) &\mapsto Nx^{N-1}f(x^N). \end{aligned}$$

Here T_N is a surjective isometry with $T_N^{-1}(f)(x) = \frac{1}{N}x^{\frac{1}{N}-1}f(x^{\frac{1}{N}})$ for $x \in (0, 1]$, and $T_N(sp\{1, x, \dots, x^{k-1}\}) = sp\{x^{N-1}, x^{2N-1}, \dots, x^{kN-1}\}$. Then $T_N^{-1}PT_N$ is a projection from $L_1([0, 1])$ onto the span of $\{1, x, x^2, \dots, x^{k-1}\}$. Theorem 3.4 implies that $\|T_N^{-1}PT_N\| \geq c \log(k-1)$. Finally, since T_N is an isometry, we see that $\|P\| = \|T_N^{-1}PT_N\|$ and the lemma is established. $\square$

The following theorem is the main result of the paper.

Theorem 3.6. *There is an infinite dimensional Müntz space that is isomorphic to $(\sum_{k=0}^{\infty} R_k)_{(l_1)}$ and not complemented in $L_1[0, 1]$.*

Proof. Let $\Lambda_{\epsilon} = \bigcup_{k=1}^{\infty} \{N_k, 2N_k, \dots, kN_k\}$ be given by Theorem 3.2 and $\Lambda' = \bigcup_{k=1}^{\infty} \{N_k - 1, 2N_k - 1, \dots, kN_k - 1\}$. By Proposition 2.2 the map

$$\begin{aligned} T : [M(\Lambda')]_{L_1} &\mapsto [M(\Lambda_{\epsilon})]_{L_1} \\ f(x) &\mapsto xf(x) \end{aligned}$$

is a surjective isomorphism. So by Theorem 3.2 and the proof of Lemma 3.5 we have that $[M(\Lambda')]_{L_1}$ is isomorphic to $(\sum_{k=0}^{\infty} R_k)_{(l_1)}$ ($R_k = span\{1, x, x^2, \dots, x^k\}$), and then, if Π_k is the map defined by $\Pi_k(f) = x^{-1}p_k(x^{N_k})$, where $f(x) = \sum_{j=1}^{\infty} x^{-1}p_j(x^{N_j}) \in [M(\Lambda')]_{L_1}$ (its decomposition of Erdős), we have $\|\Pi_k\| \leq c$ independently of k . The map Π_k , however, is a projection from $[M(\Lambda')]_{L_1}$ onto $F_k = sp\{x^{N_k-1}, x^{2N_k-1}, \dots, x^{kN_k-1}\}$. Thus if we had a bounded projection P from $L_1([0, 1])$ onto $[M(\Lambda')]_{L_1}$, then $\Pi_k P$ would be a projection from $L_1[0, 1]$ onto $sp\{x^{N_k-1}, x^{2N_k-1}, \dots, x^{kN_k-1}\}$. The bound $\|\Pi_k P\| \leq c\|P\|$ would therefore give a contradiction to Lemma 3.5! $\square$

3.1. Sequences equivalent to canonical l_1 -bases. Here we give another application of Lemma 3.5. We construct a Müntz space that has a basis equivalent to the natural basis in l_1 . First we need the following important technical lemma.

Lemma 3.7. *Let k be a positive integer, and let $X_k = sp\{x^{k-1}, x^k\} \subset L_1([0, 1])$. Then the constant of the basis $(x^{k-1}, x^k - x^{k-1})$ is bounded independently of k , and so $(x^{k-1}, x^k - x^{k-1})_k$ is a basis of the space $(\sum_{k=1}^{\infty} sp\{x^{k-1}, x^k\})_{(l_1)}$ equivalent to the natural basis in l_1 .*

Proof. It is sufficient to prove that for some $C > 0$, independent of k , and for an arbitrary $a, b \in \mathbb{C}$, we have the following inequality:

$$\|ax^{k-1}\|_1 \leq C \|ax^{k-1} + b(x^k - x^{k-1})\|_1.$$

Without loss of generality we may assume that $a = 1$ and $b > 0$. Let $f(x) = x^{k-1} + b(x^k - x^{k-1}) = (1-b)x^{k-1} + bx^k$. To estimate $\|f\|_1$, we consider two cases:

(1) $b \in [0, 1]$, which implies that $f(x) > 0$, if $x \in [0, 1]$. Then

$$\|f\|_1 = \int_0^1 f(x)dx = \frac{1-b}{k} + \frac{b}{k+1} \geq \frac{1}{k} - \frac{1}{k^2} \geq \frac{1}{2} \|x^{k-1}\|_1.$$

(2) $b > 1$, $f(x) = 0$ implies $x = \frac{b-1}{b} \in]0, 1[$. Then

$$\begin{aligned} \|f\|_1 &= - \int_0^{\frac{b-1}{b}} f(x) dx + \int_{\frac{b-1}{b}}^1 f(x) dx \\ &= \frac{1}{k(k+1)} \left(2b \left(1 - \frac{1}{b}\right)^{k+1} - b + k + 1 \right). \end{aligned}$$

It is sufficient to prove that there is a constant $C \in]0, 1[$ such that

$$(***) \quad 2b \left(1 - \frac{1}{b}\right)^{k+1} - b + k \geq Ck$$

(the constant C will be chosen below).

Let $X = 1 - \frac{1}{b} \in [0, 1]$. Then $(***)$ is equivalent to

$$p(X) = 2X^{k+1} + (C-1)kX + (1-C)k - 1 \geq 0 \quad \text{if } X \in [0, 1],$$

$$p'(X) = 2(k+1)X^k + (C-1)k = 0 \text{ implies } X = \left(\frac{1-C}{2} \frac{k}{k+1} \right)^{\frac{1}{k}} \in]0, 1[.$$

Notice that $p(0) = (1-C)k > 0$ and $p(1) = 1 > 0$. Therefore it is sufficient to prove that

$$\begin{aligned} p \left(\left(\frac{1-C}{2} \frac{k}{k+1} \right)^{\frac{1}{k}} \right) &> 0, \\ p \left(\left(\frac{1-C}{2} \frac{k}{k+1} \right)^{\frac{1}{k}} \right) &= (1-C) \left[\left(\frac{1-C}{2} \right)^{\frac{1}{k}} \left(\frac{k}{k+1} \right)^{\frac{k+1}{k}} - k \left(\frac{1-C}{2} \right)^{\frac{1}{k}} \left(\frac{k}{k+1} \right)^{\frac{1}{k}} + k \right] - 1. \end{aligned}$$

Writing $\left(\frac{1-C}{2} \right)^{\frac{1}{k}} = 1 + \frac{1}{k} \ln \frac{1-C}{2} + o\left(\frac{1}{k}\right)$, we have for large enough k ,

$$p \left(\left(\frac{1-C}{2} \frac{k}{k+1} \right)^{\frac{1}{k}} \right) \approx_k (1-C) \left[1 + \ln \frac{2}{1-C} \right] - 1.$$

Now choose $C \in]0, 1[$ such that $(1-C) \left[1 + \ln \frac{2}{1-C} \right] > 1$, for example $C = \frac{1}{2}$ and the lemma is established. $\square$

Gurariy and Macaeu [6] proved that $(x^\lambda)_{\lambda \in \Lambda}$ is a basis of $[M(\Lambda)]_{L_1}$, equivalent to the natural basis in l_1 if and only if Λ is lacunary. In the next proposition we give a basis equivalent to the natural basis in l_1 for some Müntz spaces $[M(\Lambda)]_{L_1}$, where Λ is not lacunary.

Proposition 3.8. *Let $(N_k)_{k=1}^\infty$ be defined by $N_{k+1} = N(\epsilon_{k+1}, kN_k)$ (as in the proof of Theorem 3.2), where $\prod_{k=1}^\infty (1+\epsilon_k) = (1+\epsilon)$ and $\Lambda = N_1 \cup \bigcup_{k=2}^\infty \{(k-1)N_k, kN_k\}$. Then $[M(\Lambda)]_{L_1}$ is isomorphic to l_1 and the sequence*

$$x^{N_1}, \left(x^{(k-1)N_k}, x^{kN_k} - x^{(k-1)N_k} \right)_{k=2}^\infty$$

is a basis of $[M(\Lambda)]_{L_1}$, equivalent to the natural basis in l_1 .

Proof. Let T_{N_k} denote the map

$$\begin{aligned} T_{N_k} : L_1([0, 1]) &\mapsto L_1([0, 1]) \\ f(x) &\mapsto N_k x^{N_k-1} f(x^{N_k}). \end{aligned}$$

T_{N_k} is a surjective isometry with

$$T_{N_k} \left(\text{span}\{x^{k-1}, x^k\} \right) = \text{span}\left\{ x^{kN_k-1}, x^{(k+1)N_k-1} \right\}.$$

Let

$$X = \left(\text{span}\{1\} + \sum_{k=1}^{\infty} \text{span}\{x^{k-1}, x^k\} \right)_{(l_1)}$$

and

$$Y = \overline{\text{span}}^{\|\cdot\|_1} \left\{ x^{N_1-1}, x^{(k-1)N_k-1}, x^{kN_k-1}; k \geq 2 \right\}.$$

Define T by

$$T : X \mapsto Y$$

$$(p_k)_{k=0}^{\infty} \mapsto \sum_{k=1}^{\infty} T_{N_k}(p_{k-1}).$$

By Theorem 3.2, T is an isomorphism, and by Proposition 2.2, xT is an isomorphism from $(\text{sp}\{1\} + \sum_{k=1}^{\infty} \text{sp}\{x^{k-1}, x^k\})_{(l_1)}$ onto $[M(\Lambda)]_{L_1}$ and hence by Lemma 3.7, the sequence $x^{N_1}, (x^{(k-1)N_k}, x^{kN_k} - x^{(k-1)N_k})_{k=2}^{\infty}$ is a basis of $[M(\Lambda)]_{L_1}$ equivalent to the natural basis in l_1 . $\square$

4. COMPLEMENTED MÜNTZ SPACES ISOMORPHIC TO l_1

A family $F = \{f_j : j \in \mathbb{N}\}$ of $L_1([0, 1])$ functions is called relatively disjoint if $\sup_{j \in \mathbb{N}} \|f_j\|_1 < \infty$ and there exist constants $0 < \epsilon < \delta$ and disjoint measurable sets $E_j \subset [0, 1]$ such that

$$\int_{E_j} |f_j(x)| dx > \delta \quad \text{and} \quad \sum_{i=1, i \neq j}^{\infty} \int_{E_i} |f_j(x)| dx < \epsilon, \quad j \in \mathbb{N}.$$

Lemma 4.1 ([8, Prop.3.1]). *A relatively disjoint family F of $L_1([0, 1])$ functions is equivalent to the natural basis in l_1 , and $\overline{\text{span}}(F)$ is complemented in $L_1([0, 1])$.*

The following result was already proved when $\lambda_j = m^j$, $m \geq 8$ (see [4, Lemma 11]). We actually have a slightly better result.

Theorem 4.2. *Suppose $\Lambda = (\lambda_j)_{j=0}^{\infty}$ is a sequence of positive real numbers satisfying $\lambda_{j+1}/\lambda_j \geq r \geq 4.5$. Then there is $j_0 > 0$ such that $\{(\lambda_j + 1)x^{\lambda_j} : j > j_0\}$ is a relatively disjoint family of $L_1([0, 1])$ and hence $[M(\Lambda)]_{L_1}$ is complemented in $L_1([0, 1])$.*

Proof. Let $E_j = \left(e^{-\frac{2}{\lambda_j}}, e^{-\frac{2}{r\lambda_j}} \right)$. Then

$$\int_{E_j} (\lambda_j + 1)x^{\lambda_j} dx = e^{-\frac{2(\lambda_j+1)}{r\lambda_j}} - e^{-\frac{2(\lambda_j+1)}{\lambda_j}} \geq e^{\frac{2(\lambda_j+1)}{r\lambda_j}} - e^{-2} \xrightarrow{j \rightarrow +\infty} e^{-\frac{2}{r}} - e^{-2}.$$

Moreover $e^{-\frac{2}{r}} - e^{-2} \in]\frac{1}{2}, 1[$; therefore there exist $\delta \in]\frac{1}{2}, 1[$ and $j_0 > 0$ such that for any $j > j_0$, we have

$$\int_{E_j} (\lambda_j + 1)x^{\lambda_j} dx > \delta \quad \text{and} \quad \sum_{i=1, i \neq j}^{\infty} \int_{E_i} (\lambda_j + 1)x^{\lambda_j} dx < 1 - \delta.$$

Choose $\epsilon = 1 - \delta$ so $\{(\lambda_j + 1)x^{\lambda_j} : j > j_0\}$ is (δ, ϵ) -relatively disjoint and then using Lemma 4.1, $[M(\Lambda)]_{L_1}$ is complemented in $L_1([0, 1])$. $\square$

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REFERENCES

1. P. Borwein and T. Erdélyi, *Polynomials and polynomial inequalities*, Springer, Berlin, Heidelberg, New York, 1995. MR1367960 (97e:41001)
2. L. E. Dor, *On projection in L_1* , Ann. of Math. (2) **102** (1975), no. 3, 463–474. MR0420244 (54:8258)
3. E. Görlich and A.P. Rohs, *Asymptotic bounds for projection operators on lebesgue spaces*, Integral Transforms Spec. Funct. **3** (1995), no. 2, 85–98. MR1410597 (97h:41011)
4. E. Görlich and A.P. Rohs, *Bounds for relative projection constants in $L^1(-1, 1)$* , Constr. Approx. **14** (1998), no. 4, 589–597. MR1646559 (99h:41009)
5. V. I. Gurariy and W. Lusky, *Geometry of Müntz spaces and related questions*, Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, New York, 2005. MR2190706
6. V. I. Gurariy and V. Macaev, *Lacunary power sequences in C and L_p* , Izvestiya Acad. Nauk SSSR, **30** (1966), 3–14. MR0190703 (32:8115)
7. D. J. Newman, *A Müntz space having no complement*, J. Approx. Theory **40** (1984), 351–354. MR740647 (85g:41015)
8. H.P. Rosenthal, *On relatively disjoint families of measures, with some applications to Banach space theory*, Studia Math. **37** (1973), 13–36. MR0270122 (42:5015)
9. L. Schwartz, *Étude des sommes d'exponentielles réelles*, Act. Sci. Ind., no. 959, Hermann et Cie, Paris, 1943, 89 pp. MR0014502 (7:294c)

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