ON A CONJECTURE OF BROUGHAN

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ABSTRACT. In this paper, we confirm a conjecture of Broughan, concerning the closure of the set of Fibonacci numbers in the full topology over $\mathbb{Z}$.

1. INTRODUCTION

Let $\mathbb{Z}$ be the ring of integers equipped with the topology τ in which the base of neighborhoods for a point $a \in \mathbb{Z}$ is given by the sets

$$(1) \quad N_{a,b} = \{a + nb : n \in \mathbb{Z}\} \quad \text{for } b \in \mathbb{Z}, b \geq 1.$$

This topology was proposed by H. Fürstenberg in [3], and it enables a very elegant proof of the fact that the set of prime numbers is infinite (see also [1]). It is called the *full topology*. This topology was studied in detail in the recent paper [2], where the following conjecture was proposed.

Let $F = \{F_n\}_{n \geq 0}$ denote the Fibonacci sequence given by $F_0 = 0$, $F_1 = 1$ and

$$(2) \quad F_{n+2} = F_{n+1} + F_n \quad \text{for all } n \geq 0.$$

Let F^- denote the set $\{(-1)^{n+1}F_n : n \in \mathbb{N}\}$.

Conjecture 1. *The closure of $F \subset \mathbb{Z}$ in the topology τ is $F \cup F^-$.*

Some numerical evidence supporting the above conjecture was given in the last section of [2]. In this note, we confirm the above conjecture.

2. COMMON FACTORS OF SHIFTED FIBONACCI NUMBERS

The following result is part of the main theorem in [4].

Proposition 1. *Let a and b be arbitrary integers. Then there exist two effectively computable positive constants C_1 and C_2 depending only on a and b such that the inequality*

$$(3) \quad \gcd(F_m + a, F_n + b) < \exp(C_1 \sqrt{n})$$

holds for all pairs of positive integers $n > m > C_2$ unless either both $|a|$ and $|b|$ are members of the Fibonacci sequence, or there exists $\zeta \in \{\pm 1\}$ such that

$$(4) \quad (5a^2 + 4\zeta)(5b^2 + 4\zeta) = c^2 \quad \text{for some integer } c.$$

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3. THE PROOF OF CONJECTURE 1

Throughout this section we shall denote by ∂F the set of cluster points of F and by $\overline{F}$ the closure of F in the full topology of $\mathbb{Z}$. It is well known that $\overline{F} = F \cup \partial F$.

We proceed in three steps. We first show that if a is a cluster point of F , then $|a| \in F$. Secondly, we show that integers of the form $-F_\ell$ for some odd integer $\ell > 1$ are not in the closure of F . Finally, we show that integers in $F \cup F^-$ are indeed cluster points of F .

Step 1. *If a is a cluster point of F , then $|a| \in F$.* Assume that $a \in \partial F$. Thus, $(N_{a,b} - \{a\}) \cap F \neq \emptyset$ for any positive integer b . We choose $b = F_m$, where m is a large positive integer that is not a multiple of 3. It is well known that

$$(5) \quad F_m \mid F_n \text{ if and only if } m \mid n.$$

Since $2 = F_3$, we get that $2 \mid F_n$ if and only if $3 \mid n$. Hence, F_m is odd. We deduce that for each such m there exists a nonnegative integer n such that $F_n \equiv a \pmod{F_m}$. We next notice that the sequence $\{F_n\}_{n \geq 0}$ is periodic modulo F_m with period $4m$. To see why, let us first recall the well-known formulas

$$(6) \quad 2F_{s+t} = F_s L_t + F_t L_s \quad \text{and} \quad L_{4u} = 5F_{2u}^2 + 2,$$

where $\{L_n\}_{n \geq 0}$ denotes the Lucas sequence given by $L_0 = 2$, $L_1 = 1$ and $L_{n+2} = L_{n+1} + L_n$ for all $n \geq 0$. Taking $t = 4m$ in the first relation (6) above we get

$$(7) \quad 2F_{s+4m} = F_s L_{4m} + F_{4m} L_s.$$

Furthermore, taking $u = m$ in the second relation (6) and inserting the resulting relation in (7) we get

$$(8) \quad 2F_{s+4m} = 2F_s + (F_{4m} L_s + 5F_s F_{2m}^2).$$

From (5), we have that $F_m \mid F_{2m}$ and $F_m \mid F_{4m}$; thus $F_m \mid F_{4m} L_s + 5F_s F_{2m}^2$. Hence, reducing the relation (8) modulo F_m we get

$$2F_{s+4m} \equiv 2F_s \pmod{F_m},$$

and since F_m is odd we get that $F_{s+4m} \equiv F_s \pmod{F_m}$. Thus, $\{F_n\}_{n \geq 0}$ is periodic with period $4m$ modulo F_m .

Since there exists a nonnegative integer n such that $F_n \equiv a \pmod{F_m}$, it follows from the above arguments that such an integer n can be found in the interval $[4m, 8m)$. Thus, $n > m > n/8$ and

$$(9) \quad \gcd(F_m + 0, F_n - a) = F_m \geq \frac{\alpha^m}{3} > \frac{\alpha^{n/8}}{3},$$

where $\alpha = (1 + \sqrt{5})/2$. Here, we used the fact that the inequality $F_k \geq \alpha^k/3$ holds for all positive integers k , the proof of which is an easy exercise using induction. From Proposition 1, we get that either $|a|$ is a member of the Fibonacci sequence, or there exist integers $\zeta \in \{\pm 1\}$ and c such that

$$(10) \quad 4\zeta(5a^2 + 4\zeta) = c^2,$$

or there exist positive constants C_1 and C_2 depending on a such that if $n > 8C_2$, then $m > n/8 > C_2$ and

$$(11) \quad \gcd(F_m, F_n - a) < \exp(C_1 \sqrt{n}).$$

The first alternative gives us the desired conclusion. For the second alternative, note that unless $a = 0$ (which is in F), we have that $5a^2 + 4\zeta > 0$. Hence, equation

(10) forces $\zeta = 1$, $c = 2c_1$ to be even, and $5a^2 + 4 = c_1^2$. Thus, $c_1^2 - 5a^2 = 4$. However, it is well known (see, for example, Theorems 27.1 and 27.2 on page 338 of [5]) and follows easily from the continued fraction expansion of the golden section $(1 + \sqrt{5})/2$, that if (x, y) is an integer solution of the Pell-type equation $x^2 - 5y^2 = \pm 4$, then there exists a nonnegative integer t such that $x = \pm L_t$ and $y = \pm F_t$. Hence, $|a| \in F$ in this case also. Finally, the third alternative leads to the conclusion that both inequalities (11) and (9) hold, therefore that

$$\exp(C_1 \sqrt{n}) > \frac{\alpha^{n/8}}{3} \quad \text{provided that } n > 8C_2,$$

which leads to the conclusion that $n < C_3$ is bounded. This is however impossible since $n \geq 4m$ and m can be chosen to be arbitrarily large. This concludes the proof of the fact that if $a \in \partial F$, then $|a| \in F$.

Step 2. If $a = -F_\ell$ where $\ell > 1$ is odd, then $a \notin \partial F$. In order to show this, it is enough to display for each such ℓ a positive integer b such that the congruence $F_n \equiv -F_\ell \pmod{b}$ has no nonnegative integer solution n .

We now fix the number ℓ and first show that if $16 \mid b$, then n cannot be odd. Indeed, assume that n is odd. Since the Fibonacci sequence is periodic modulo b , there must exist infinitely many positive integers n such that $F_n \equiv -F_\ell \pmod{b}$, and we choose one such that $n > \ell$. Let $\delta \in \{\pm 1\}$ be such that $n + \delta\ell \equiv 2 \pmod{4}$. Note that $n \equiv \delta\ell \pmod{4}$. The formula

$$(12) \quad F_n + F_\ell = F_{(n+\delta\ell)/2} L_{(n-\delta\ell)/2} \quad \text{whenever} \quad n \equiv \delta\ell \pmod{4}$$

can be easily checked, for example, by a direct calculation using the known formulas

$$(13) \quad F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n, \quad \text{where} \quad \beta = -\alpha^{-1}.$$

The sequence $\{F_n\}_{n \geq 0}$ is periodic modulo 4 with period 6 and the sequence $\{L_n\}_{n \geq 0}$ is periodic modulo 8 with period 12. By looking at the first six values of the Fibonacci sequence and at the first twelve values of the Lucas sequence one notices that F_k is never a multiple of 4 if k is odd and that L_k is never a multiple of 8. Since the index $(n + \delta\ell)/2$ of the Fibonacci number appearing in the right-hand side of formula (12) is odd, we conclude from relation (12) that $F_n + F_\ell$ is never a multiple of 16. Hence, it is impossible that $b \mid F_n + F_\ell$ if $16 \mid b$ and n is odd.

From now on, we suppose that n is even. We take $b = 16F_m$, where again m is a large odd positive integer that is not a multiple of 3. Since $F_n \equiv a \pmod{F_m}$ holds with some even n , it follows, by the fact that $\{F_n\}_{n \geq 0}$ is periodic modulo F_m with period $4m$, that we may assume that n is still even in $[4m, 8m)$.

We now follow again the method from [4]. Let $\mathbb{K} = \mathbb{Q}[\sqrt{5}]$. In $\mathbb{K}$, we have that

$$(14) \quad F_m = \frac{1}{\sqrt{5}}(\alpha^m - \beta^m) = \frac{1}{\sqrt{5}}(\alpha^m - (-\alpha^{-1})^m) = \frac{\alpha^{-m}}{\sqrt{5}}(\alpha^{2m} + 1).$$

On the other hand,

$$(15) \quad \begin{aligned} F_n - a &= \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) - a = \frac{1}{\sqrt{5}}(\alpha^n - (-\alpha^{-1})^n) - a \\ &= \frac{\alpha^{-n}}{\sqrt{5}}(\alpha^{2n} - \sqrt{5}a\alpha^n - 1) = \frac{\alpha^{-n}}{\sqrt{5}}(\alpha^n - z_1)(\alpha^n - z_2), \end{aligned}$$

where

$$(16) \quad z_{1,2} = \frac{\sqrt{5}a \pm \sqrt{5a^2 + 4}}{2}.$$

Note that since $a = -F_\ell$ and ℓ is odd, we have that $L_\ell^2 - 5F_\ell^2 = (-1)^\ell 4 = -4$; therefore $5a^2 + 4 = L_\ell^2 + 8$. Note that $5a^2 + 4$ is not a multiple of 5, and it is not a square since otherwise with $x = L_\ell$, we would get that $x^2 + 8 = y^2$ for some positive integer y , or $8 = (y - x)(y + x)$. The only solution in positive integers x, y of the above equation is $x = 1, y = 3$, but this is not convenient since for us $\ell > 1$; therefore $x = L_\ell > 1$. Thus, we deduce that $5a^2 + 4 = dy^2$, where $d > 1$ is a square-free integer that is not a multiple of 5.

Let $\mathbb{L} = \mathbb{K}[z_1]$. The above remarks show that $[\mathbb{L} : \mathbb{Q}] = 4$. Since $F_m \mid F_n - a$, it follows that in $\mathbb{L}$ we have that $(\alpha^{2m} + 1)$ divides $(\alpha^n - z_1)(\alpha^n - z_2)$. Here, for an algebraic integer γ in $\mathbb{L}$ we write (γ) for the principal ideal (in the ring of algebraic integers $\mathcal{O}_{\mathbb{L}}$ of $\mathbb{L}$) generated by γ . Let $I_i = \gcd((\alpha^{2m} + 1), (\alpha^n - z_i))$ for $i = 1, 2$. Then $(F_m) \mid I_1 I_2$ in $\mathbb{L}$.

We shall next bound the norm of I_i in $\mathbb{L}$ for $i = 1, 2$. Let

$$\mathcal{S} = \{\lambda n + 2\mu m : \lambda, \mu \in \{1, \dots, \lfloor m^{1/2} \rfloor\}.$$

If $s = \lambda n + 2\mu m$, then $1 \leq s \leq (n + 2m)m^{1/2} < 10m^{3/2}$. Since there are $(\lfloor m^{1/2} \rfloor)^2$ pairs of positive integers (λ, μ) with $\lambda, \mu \in \{1, \dots, \lfloor m^{1/2} \rfloor\}$, it follows, by the *Pigeon-Hole Principle*, that there exist two distinct pairs $(\lambda_1, \mu_1) \neq (\lambda_2, \mu_2)$ such that

$$|(\lambda_1 - \lambda_2)n + 2(\mu_1 - \mu_2)m| < \frac{10m^{3/2}}{\lfloor m^{1/2} \rfloor^2 - 1} < 11m^{1/2} \quad \text{for } m \text{ large enough.}$$

Writing $x = \lambda_1 - \lambda_2$ and $y = \mu_1 - \mu_2$, we get that $(x, y) \neq (0, 0)$, that $x, y \in (-m^{1/2}, m^{1/2})$ and that if we write $s = nx + 2my$, then $|s| < 11m^{1/2}$. Note now that

$$\alpha^{2m} \equiv -1 \pmod{I_i} \quad \text{and} \quad \alpha^n \equiv z_i \pmod{I_i}.$$

Raising the first congruence to the power y and the second to the power x (notice that such operations are justified even if x and y are negative since α is a unit in $\mathbb{K}$, therefore also in $\mathbb{L}$), and multiplying the resulting congruences we get

$$\alpha^s \equiv (-1)^y z_i^x \pmod{I_i}.$$

Thus, I_i divides $(\alpha^s - (-1)^y z_i^x)$. Note that this last ideal is not zero. Indeed, for if not, then we would get that $\alpha^{2s} = z_i^{2x}$. By replacing the pair (x, y) by the pair $(-x, -y)$, we may assume that $x \geq 0$. Since it is easy to see that $z_i^2 \in \mathbb{Q}[\sqrt{5d}]$ and it is not a rational number, it follows that $z_i^{2x} \in \mathbb{Q}[\sqrt{5d}]$ also and it is not a rational number whenever $x > 0$, while $\alpha^{2s} \in \mathbb{K}$. Thus, the only possibility that $z_i^{2x} = \alpha^{2s}$ is $x = s = 0$, and since $s = nx + 2my$, we also get that $y = 0$, which contradicts the fact that $(x, y) \neq (0, 0)$. Hence, I_i divides the nonzero ideal $(\alpha^s - (-1)^y z_i^x)$. Thus,

$$(F_m) \mid I_1 I_2 \mid (\alpha^s - (-1)^y z_1^x)(\alpha^s - (-1)^y z_2^x),$$

and the ideal appearing in the right-hand side above is not zero. Taking norms in $\mathbb{L}$, we get that

$$(17) \quad \begin{aligned} F_m^4 &= N_{\mathbb{L}}(F_m) \leq N_{\mathbb{L}}((\alpha^s - (-1)^y z_1^x)(\alpha^s - (-1)^y z_2^x)) \\ &< (2^s + (5a^2 + 4)^{x/2})^8 < 2^8 (5a^2 + 4)^{\max\{8s, 4x\}}, \end{aligned}$$

where in the above inequalities we have used the fact that α and its conjugate are less than 2 in absolute value and that z_1, z_2 and all their conjugates (in $\mathbb{L}$) are less than $\sqrt{5a^2 + 4}$ in absolute values (see (16)). Since $|s| < 11m^{1/2}$ and $x < m^{1/2}$, we get that

$$\left(\frac{\alpha^m}{3}\right)^4 \leq F_m^4 \leq 2^8(5a^2 + 2)^{88m^{1/2}},$$

which leads to the conclusion that m is bounded, which is of course a contradiction. This shows that indeed if $a = -F_\ell$ and ℓ is odd, then $a \notin \overline{F}$.

Step 3. *The elements of F^- are cluster points of F .* Since the Fibonacci sequence $\{F_n\}_{n \geq 0}$ is periodic modulo b for any positive integer b , it follows that if we write e_b for this period, then $F_{n+te_b} \equiv F_n \pmod{b}$ for any $t \geq 0$. Hence, $N_{F_n, b}$ contains infinitely many members of F . Finally, it is a well-known fact that we may use recurrence (2) to define F_n for *negative values* of n , and in this case the formula for F_n appearing in (13) holds for negative values of n also. Furthermore, using that formula (or by induction using the recurrence (2)) one checks that $(-1)^{n+1}F_n = F_{-n}$. Since this extended sequence $\{F_n\}_{n \in \mathbb{Z}}$ remains periodic modulo b with the same period e_b , we get that $F_{-n+te_b} \equiv F_{-n} \pmod{b}$ for all $n \geq 0$ and all integers t . Taking t to be a sufficiently large positive integer (i.e., such that $-n + te_b > 0$), we see that $N_{F_{-n}, b}$ contains members of F ; therefore $F_{-n} \in \partial F$.

Recall that $\overline{F} = F \cup \partial F$. Step 3 shows that $F \cup F^- \subseteq \overline{F}$. Steps 1 and 2 show that $\overline{F} \subseteq F \cup F^-$. Thus, $\overline{F} = F \cup F^-$.

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