STRONG CONVERGENCE RESULTS FOR NONSELF MULTIMAPS IN BANACH SPACES

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ABSTRACT. We prove strong convergence theorems for multimaps under mild conditions, which include Browder's convergence theorem as well as Reich's convergence theorem. We thus provide a partial answer to Jung's question.

1. INTRODUCTION

Let D be a nonempty closed convex subset of a Banach space E and let $CB(E)$, $K(E)$ and $KC(E)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets and nonempty compact convex subsets of E , respectively. Let H be the Hausdorff metric on $CB(E)$ and $T : D \rightarrow CB(E)$ a multimap. Then T is said to be a *contraction* if there exists $0 \leq k < 1$ such that $H(Tx, Ty) \leq k\|x - y\|$ for $x, y \in D$. If $k = 1$, then T is called *nonexpansive*. A point x^* is a *fixed point of T* , if $x^* \in Tx^*$. The set of fixed points is denoted by $F(T)$.

Let $T : D \rightarrow K(D)$. Given a $u \in D$ and a $t \in (0, 1)$, let $G_t : D \rightarrow K(D)$ be defined by

$$G_tx = tTx + (1 - t)u, \quad x \in D.$$

Then G_t is a contraction and admits a fixed point $x_t \in D$ by the Nadler contraction principle [16], that is,

$$x_t \in tTx_t + (1 - t)u.$$

If T is single-valued, then

$$x_t = tTx_t + (1 - t)u.$$

In this case, $\{x_t\}$ is an approximating fixed point of T since it has the property that if $\{x_t\}$ is bounded, then $\lim_{t \rightarrow 1} \|x_t - Tx_t\| = 0$.

The strong convergence of the net $\{x_t\}$ for a self or nonself nonexpansive single-valued map T has been investigated by Browder [3], Halpern [7], Jung and Kim [11], Kim and Takahashi [12], Reich [19], Singh and Watson [22], Takahashi and Kim [24], Xu [27], Xu and Yin [29], etc. For details on iterative methods, we refer to [2].

In 1967, Browder [3] established the following strong convergence theorem.

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Theorem B ([3]). *Let D be a nonempty closed bounded convex subset of a Hilbert space E and T a nonexpansive self-mapping of D . Let $\{t_n\}$ be a sequence in $(0, 1)$ converging to 1. Fix $u \in D$ and define a sequence $\{x_n\}$ in D by*

$$x_n = t_n T x_n + (1 - t_n)u, \quad n \in \mathbf{N}.$$

Then $\{x_n\}$ converges strongly to the element of $F(T)$ nearest to u .

Reich [19] extended Browder's theorem.

Theorem R ([19]). *Let D be a nonempty closed bounded convex subset of a uniformly smooth Banach space E and T a nonexpansive self-mapping of D . Fix $u \in D$ and define a net $\{x_t\}$ in D by*

$$x_t = t T x_t + (1 - t)u, \quad t \in (0, 1).$$

Then $\{x_t\}$ converges strongly as $t \rightarrow 1$ to the element of $F(T)$ nearest to u .

Pietramala [17] (see also Jung [9]) gave an example showing that Browder's theorem cannot be extended to the multivalued case without adding an extra assumption. López Acedo and Xu [15] established the strong convergence of $\{x_t\}$ in a Hilbert space for a subclass of nonexpansive multimaps, that is, for maps T satisfying $Tz = \{z\}$. Later on, Kim and Jung [13] extended López Acedo and Xu's result to a Banach space having a sequentially continuous duality map. Sahu [20] discussed this problem in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm. It was noted by Jung [9] that the condition that $Tz = \{z\}$ should be added in the main results of Sahu [20]. Very recently, Jung [9] gave the strong convergence of $\{x_t\}$ defined by $x_t \in t T x_t + (1 - t)u$, $u \in D$ for the nonexpansive nonself multimap T in a uniformly convex or reflexive Banach space having a uniformly Gâteaux differentiable norm. More precisely, he obtained the following extensions of Browder's theorem.

Theorem J1 ([9], Theorem 1). *Let E be uniformly convex Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E , and $T : D \rightarrow K(E)$ a nonexpansive nonself multimap. Suppose that D is a nonexpansive retract of E . Suppose that $Ty = \{y\}$ for any fixed point y of T and that for each $u \in D$ and $t \in (0, 1)$, the contraction G_t defined by $G_t x = t T x + (1 - t)u$ has a fixed point $x_t \in D$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \rightarrow 1$ and in this case, $\{x_t\}$ converges strongly as $t \rightarrow 1$ to a fixed point of T .*

Theorem J2 ([9], Theorem 2). *Let E be a reflexive Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E , and $T : D \rightarrow KC(E)$ a nonexpansive nonself multimap satisfying the inwardness condition, where $KC(E)$ denotes the family of compact convex subsets of E . Suppose that every closed bounded convex subset of D is compact. If $F(T) \neq \emptyset$ and $Ty = \{y\}$ for any $y \in F(T)$, then $\{x_t\}$, defined by $x_t \in t T x_t + (1 - t)u$, converges strongly as $t \rightarrow 1$ to a fixed point of T .*

It is still an open question whether the assumption $Ty = \{y\}$ in Theorems J1 and J2 can be omitted (see, Jung [9]).

It seems that, in view of Pietramala's example [17] (see also Jung [9]), it is almost impossible to completely omit this assumption for nonexpansive multimaps.

Our concern now is the following:

Is it possible to extend Browder's theorem to a class of multimaps under mild conditions? In other words: Can an approximating sequence be constructed which converges to a fixed point of a multimap under some mild conditions?

Let D be nonempty closed convex subset of a Banach space E . Let $T : D \rightarrow K(E)$ be a multimap and

$$P_T(x) = \{u_x \in Tx : \|x - u_x\| = d(x, Tx)\}.$$

Then $P_T : D \rightarrow K(E)$ is nonempty and compact for every $x \in D$.

Instead of

$$(1.1) \quad G_t x = tTx + (1-t)u, \quad u \in D,$$

we consider, for $t \in (0, 1)$,

$$(1.2) \quad S_t x := tP_T(x) + (1-t)u, \quad u \in D.$$

Notice that $S_t x \subseteq G_t x$ for all $x \in D$ and if P_T is nonexpansive and T is weakly inward, then S_t is a weakly inward contraction. Now Theorem 1 of Lim [14] guarantees that x_t is a fixed point of S_t , that is,

$$(1.3) \quad x_t \in tP_T(x_t) + (1-t)u \subseteq tT(x_t) + (1-t)u.$$

If T is single-valued, then (1.3) is reduced to

$$(1.4) \quad x_t = tT(x_t) + (1-t)u.$$

It is our purpose in this paper to prove the strong convergence of $\{x_t\}$ defined by (1.3) (i) for a nonself multimap in a uniformly convex Banach space having a uniformly Gâteaux differentiable norm; (ii) for an inward multimap in a reflexive Banach space having a uniformly Gâteaux differentiable norm. This provides an affirmative answer to the above Question as well as gives a partial answer to Jung's Question. Our results unify, extend and complement several known results including those of Browder [3], Jung [9], Jung and Kim [10, 11], Sahu [20], Xu [27] and Xu and Yin [29].

2. PRELIMINARIES

Let $S := \{x \in E : \|x\| = 1\}$ denote the unit sphere of the Banach space E . The space E is said to have a *Gâteaux differentiable* norm if the limit

$$(*) \quad \lim_{n \rightarrow \infty} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in S$, and we call E *smooth*; E is said to have a *uniformly Gâteaux differentiable* norm if for each $y \in S$ the limit $(*)$ is attained uniformly for $x \in S$. Furthermore, E is said to be *uniformly smooth* if the limit $(*)$ exists uniformly for $(x, y) \in S \times S$. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx := \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known [4] that if E is smooth, then any duality mapping on E is single-valued, and if E has a uniformly Gâteaux differentiable norm, then the duality mapping is norm-to-weak* uniformly continuous on bounded sets.

The inward set of $D \subseteq E$ at x is defined by

$$I_D(x) = \{z \in E : z = x + \lambda(y - x) : y \in D, \lambda \geq 0\}.$$

It is known that if D is convex, then the closure of $I_D(x)$ is $\overline{I_D(x)} = x + T_D(x)$ for any $x \in D$, where

$$T_D(x) = \{y \in E : \liminf_{\lambda \rightarrow 0^+} \frac{d(x + \lambda y, D)}{\lambda} = 0\}.$$

A multimap $T : D \rightarrow CB(E)$ is said to satisfy the (i) *weak inwardness condition* if $Tx \subset \overline{I_D(x)}$ for all $x \in D$; (ii) *inwardness condition* if $Tx \subset I_D(x)$ for all $x \in D$. For details, we refer to [5].

A mapping $T : D \rightarrow CB(E)$ is **-nonexpansive* ([8]) if for all $x, y \in D$ and $u_x \in T(x)$ with $d(x, u_x) = \inf\{d(x, z) : z \in T(x)\}$, there exists $u_y \in T(y)$ with $d(y, u_y) = \inf\{d(y, w) : w \in T(y)\}$ such that

$$d(u_x, u_y) \leq d(x, y).$$

It is known that *-nonexpansiveness is different from nonexpansiveness for multimaps. There are some *-nonexpansive multimaps which are not nonexpansive and some nonexpansive multimaps which are not *-nonexpansive [28].

Let $D \subseteq E$ be closed and convex. A mapping Q of E into E is said to be a *retraction* if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the *range* of Q . A subset K of E is said to be a *nonexpansive retract* of E if there exists a nonexpansive retraction of E onto K . If E is a Hilbert space, the proximity map is a sunny nonexpansive retraction from H to any closed convex subset of H .

We shall let LIM be a Banach limit. Recall that $\text{LIM} \in (\ell^\infty)^*$ such that $\|\text{LIM}\| = 1$, $\liminf_{n \rightarrow \infty} a_n \leq \text{LIM}_n a_n \leq \limsup_{n \rightarrow \infty} a_n$, and $\text{LIM}_n a_n = \text{LIM}_n a_{n+1}$ for all $\{a_n\}_n \in \ell^\infty$.

In what follows, we shall make use of the following lemmas.

Lemma 2.1 (see, e.g., [6, 25]). *Let E be a Banach space having a uniformly Gâteaux differentiable norm and D a nonempty closed convex subset of E . Let $\{x_n\}$ be a bounded sequence in E , LIM be a Banach limit, and $u \in D$. Then the following are equivalent:*

- (i) $\text{LIM}_n \|x_n - u\|^2 = \min_{y \in D} \text{LIM}_n \|x_n - y\|^2$;
- (ii) $\text{LIM}_n \langle x - u, J(x_n - u) \rangle \leq 0$ for all $y \in D$.

Lemma 2.2 (see, e.g., [18, 23]). *Let E be a uniformly convex Banach space, D a nonempty closed convex subset of E , and $\{x_n\}$ a bounded sequence in E . Then the set*

$$C = \{u \in D : \text{LIM}_n \|x_n - u\|^2 = \min_{y \in D} \text{LIM}_n \|x_n - y\|^2\}$$

consists of one point.

Lemma 2.3 (see, e.g., [26]). *Let D be a nonempty compact convex subset of a Banach space E and $T : D \rightarrow KC(E)$ a nonexpansive multimap satisfying the boundary condition:*

$$Tx \cap \overline{I_D(x)} \neq \emptyset, \quad x \in D.$$

Then T has a fixed point.

3. MAIN RESULTS

Theorem 3.1. *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E , and $T : D \rightarrow K(E)$ a nonself multimap such that P_T is nonexpansive. Suppose that D is a*

nonexpansive retract of E and that for each $u \in D$ and $t \in (0, 1)$, the contraction S_t defined by $S_t x = tP_T x + (1-t)u$ has a fixed point $x_t \in D$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \rightarrow 1$; in this case, $\{x_t\}$ converges strongly as $t \rightarrow 1$ to a fixed point of T .

Proof. Given any $x_t \in D$, we can find some $y_t \in P_T(x_t)$ such that

$$x_t = ty_t + (1-t)u.$$

Notice that $P_T(y) = \{y\}$ whenever y is a fixed point of T . Let $p \in F(T)$. Then $p \in P_T(p)$, and thus we obtain for all $t \in (0, 1)$ that

$$\begin{aligned} \|y_t - p\| &= d(y_t, P_T(p)) \\ &\leq H(P_T(x_t), P_T(p)) \\ (3.1) \quad &\leq \|x_t - p\|. \end{aligned}$$

We thus have

$$\begin{aligned} \|x_t - p\| &= \|ty_t + (1-t)u - p\| \\ &\leq t\|y_t - p\| + (1-t)\|u - p\| \\ &\leq t\|x_t - p\| + (1-t)\|u - p\| \end{aligned}$$

and so

$$\|x_t - p\| \leq \|u - p\|$$

for $t \in (0, 1)$. Hence $\{x_t\}$ is uniformly bounded.

Suppose that $\{x_t\}$ remains bounded as $t \rightarrow 1$. Now we show that $F(T) \neq \emptyset$ and x_t converges to a fixed point of T as $t \rightarrow 1$. Let $t_n \rightarrow 1$ and set $x_n := x_{t_n}$. Define the mapping $\phi : E \rightarrow \mathbb{R}$ by

$$\phi(x) := \text{LIM}_n \|x_n - x\|^2.$$

Since E is reflexive, $\phi(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$, ϕ is continuous and convex, it follows that ϕ attains its infimum over D at p (say) ([1], p. 79) and so the set $C := \{y \in D : \phi(y) = \inf_{x \in D} \phi(x)\} \neq \emptyset$. C is also a closed, bounded and convex subset of D . Let Q be a nonexpansive retraction of E onto D . Then

$$\begin{aligned} \phi(y) &= \text{LIM}_n \|x_n - y\|^2 \\ &\geq \text{LIM}_n \|Qx_n - Qy\|^2 \\ &= \text{LIM}_n \|x_n - Qy\|^2 \\ &\geq \text{LIM}_n \|x_n - p\|^2 = \phi(p) \end{aligned}$$

for any $y \in E$. This implies that p is the global minimum point over all of E . Furthermore, p is unique by Lemma 2.2. Notice that $x_n = t_n y_n + (1-t_n)u$ for some $y_n \in P_T(x_n)$ and $\|x_n - y_n\| \rightarrow 0$. We can find $w_n \in P_T(p)$, $n \geq 1$, by compactness of $P_T(p)$ such that

$$\begin{aligned} \|y_n - w_n\| &= d(y_n, P_T(p)) \\ &\leq H(P_T(x_n), P_T(p)) \\ &\leq \|x_n - p\|. \end{aligned}$$

The sequence $\{w_n\}$ has a convergent subsequence, which we denote again by $\{w_n\}$, with $w_n \rightarrow w \in P_T(p)$. So

$$\begin{aligned} \text{LIM}_n \|x_n - w\|^2 &\leq \text{LIM}_n \|y_n - w_n\|^2 \\ &\leq \text{LIM}_n \|x_n - p\|^2. \end{aligned}$$

But p is the unique global minimum. Therefore $p = w \in P_T(p) \subseteq Tp$ and hence $F(T) \neq \emptyset$. We have also that $P_T(p) = \{p\} \subseteq C$. Moreover, we have from (3.1) that

$$\begin{aligned} \langle x_n - y_n, j(x_n - p) \rangle &= \langle x_n - p + p - y_n, j(x_n - p) \rangle \\ &\geq \|x_n - p\|^2 - \|y_n - p\| \|x_n - p\| \geq 0 \end{aligned}$$

and so

$$0 \leq \langle x_n - y_n, j(x_n - p) \rangle = (1 - t_n) \langle u - y_n, j(x_n - p) \rangle.$$

This together with $\|x_n - y_n\| \rightarrow 0$ imply that

$$(3.2) \quad \text{LIM}_n \langle x_n - u, j(x_n - p) \rangle \leq 0.$$

Also, by Lemma 2.1, we have

$$\text{LIM}_n \langle x - p, j(x_n - p) \rangle \leq 0$$

for all $x \in D$. In particular,

$$(3.3) \quad \text{LIM}_n \langle u - p, j(x_n - p) \rangle \leq 0.$$

Thus using (3.2) and (3.3) we find that $\text{LIM}_n \|x_n - p\| = 0$. Therefore, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightarrow p$ as $k \rightarrow \infty$. Assume that there is another subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l} \rightarrow q$ as $l \rightarrow \infty$. Since $d(x_{n_l}, P_T(x_{n_l})) \leq (1 - t_{n_l}) \|u - y_{n_l}\| \rightarrow 0$ as $l \rightarrow \infty$, it follows that $d(q, P_T(q)) = 0$ and so $q \in P_T(q) \subseteq T(q)$. Notice that $P_T(q) = \{q\}$. With $x_{n_k} \rightarrow p$, it follows from (3.2) that

$$(3.4) \quad \langle p - u, j(p - q) \rangle \leq 0.$$

Also, with $x_{n_l} \rightarrow q$, we have that

$$(3.5) \quad \langle q - u, j(q - p) \rangle \leq 0.$$

Inequalities (3.4) and (3.5) yield that

$$\|p - q\|^2 \leq 0,$$

which implies that $p = q$. Thus $x_n \rightarrow p$ as $n \rightarrow \infty$. This completes the proof. $\square$

It is known that a nonempty closed convex subset of a Hilbert space is a nonexpansive retract with the proximity map as a nonexpansive retraction. Thus, we have the following corollary.

Corollary 3.2. *Let E be a Hilbert space, D a nonempty closed convex subset of E , and $T : D \rightarrow K(E)$ a nonself multimap such that P_T is nonexpansive. Suppose that for each $u \in D$ and $t \in (0, 1)$, the contraction S_t defined by $S_t x = tP_T x + (1 - t)u$ has a fixed point $x_t \in D$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \rightarrow 1$ and in this case $\{x_t\}$ converges strongly as $t \rightarrow 1$ to a fixed point of T .*

By definition of the Hausdorff metric, we obtain that if T is $*$ -nonexpansive, then P_T is nonexpansive. Hence, we have the following corollary.

Corollary 3.3. *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E , and $T : D \rightarrow K(E)$ a nonself $*$ -nonexpansive multimap. Suppose that D is a nonexpansive retract of E and that for each $u \in D$ and $t \in (0, 1)$, the contraction S_t defined by $S_t x = tP_T x + (1 - t)u$ has a fixed point $x_t \in D$. Then T has a fixed point if and*

only if $\{x_t\}$ remains bounded as $t \rightarrow 1$ and in this case, $\{x_t\}$ converges strongly as $t \rightarrow 1$ to a fixed point of T .

It is well known that every nonempty closed convex subset D of a uniformly convex Banach space E is Chebyshev, i.e., for every $x \in E$ there is a unique element $u \in D$ such that $\|x - u\| = \inf\{\|x - v\| : v \in D\}$. Thus, we have the following corollary.

Corollary 3.4. *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E , and $T : D \rightarrow KC(E)$ a nonself multimap such that P_T is nonexpansive. Suppose that D is a nonexpansive retract of E and that for each $u \in D$ and $t \in (0, 1)$, the contraction S_t defined by $S_t x = tP_T x + (1 - t)u$ has a fixed point $x_t \in D$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \rightarrow 1$ and in this case, $\{x_t\}$ converges strongly as $t \rightarrow 1$ to a fixed point of T .*

Proof. In this case, Tx is Chebyshev for each $x \in D$ and so P_T is a selector of T and P_T is a single-valued nonexpansive map. The result now follows from Theorem 3.1. $\square$

Corollary 3.5. *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E , and $T : D \rightarrow KC(E)$ a nonself*-nonexpansive multimap. Suppose that D is a nonexpansive retract of E and that for each $u \in D$ and $t \in (0, 1)$, the contraction S_t defined by $S_t x = tP_T x + (1 - t)u$ has a fixed point $x_t \in D$. Then T has a fixed point if and only if $\{x_t\}$ remains bounded as $t \rightarrow 1$ and in this case, $\{x_t\}$ converges strongly as $t \rightarrow 1$ to a fixed point of T .*

We know that if D is a nonempty closed bounded convex subset of a uniformly convex Banach space E and $T : D \rightarrow K(E)$ is a nonexpansive mapping satisfying the weak inwardness condition, then T has a fixed point (see [26]). As a consequence, under this condition, the contraction S_t defined by $S_t x = tP_T x + (1 - t)u$ has a fixed point $x_t \in D$. Thus, we have the following results.

Corollary 3.6. *Let E be a uniformly convex Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed bounded convex subset of E , and $T : D \rightarrow K(E)$ a nonself multimap satisfying the weak inwardness condition such that P_T is nonexpansive. Suppose that D is a nonexpansive retract of E . Then $\{x_t\}$ defined by (1.3) converges strongly as $t \rightarrow 1$ to a fixed point of T .*

Corollary 3.7. *Let E be a Hilbert space, D a nonempty closed bounded convex subset of E , and $T : D \rightarrow KC(E)$ a nonself multimap satisfying the weak inwardness condition such that P_T is nonexpansive. Then $\{x_t\}$ defined by (1.3) converges strongly as $t \rightarrow 1$ to a fixed point of T .*

- Remark 3.8.*
- (1) We now give an example of a multimap T with P_T nonexpansive. Let $D = [0, \infty)$ and T be defined by $Tx = [x, 2x]$ for $x \in D$ ([28]). Then $P_T(x) = \{x\}$ for $x \in D$. Observe that T is *-nonexpansive but *not* nonexpansive (see [28]).
  - (2) Theorem 3.1 extends Theorem 1 and Corollary 1 of Jung and Kim [10], Corollary 2 of Jung and Kim [11] and Theorem 1 and Corollary 2 of Xu and Yin [29] to multimaps. It also improves and complements Theorem J1, Theorem 4.1 of Kim and Jung [13] and Theorem 1 of Sahu [20].
  - (3) Our results apply to L^p spaces and l^p spaces for $1 < p < \infty$.

Theorem 3.9. *Let E be a reflexive Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E , and $T : D \rightarrow KC(E)$ a nonself multimap satisfying the inwardness condition such that P_T is nonexpansive. Suppose that every closed bounded convex subset of D is compact and P_T has a fixed point. Then $\{x_t\}$ defined by (1.3) converges strongly as $t \rightarrow 1$ to a fixed point of T .*

Proof. Let $p \in P_T(p)$. Then, as in the proof of Theorem 3.1, $\{x_t\}$ remains bounded as $t \rightarrow 1$. Now let $t_n \rightarrow 1$ and set $x_n := x_{t_n}$. Define the mapping $\phi : E \rightarrow \mathbb{R}$ by

$$\phi(x) := \liminf_n \|x_n - x\|^2, \forall x \in D$$

and set

$$C := \{y \in D : \phi(y) = \inf_{x \in D} \phi(x)\} \neq \emptyset.$$

C is also a closed, bounded and convex subset of D . By assumption C is compact. Clearly, P_T satisfies the inwardness condition. Essentially the same arguments as in Theorem 2 of Jung [9] yield that

$$P_T(x) \cap I_C(x) \neq \emptyset, \quad x \in C.$$

Now Lemma 2.3 guarantees that there exists $p \in C$ such that $p \in P_T(p) \subseteq Tp$ and so $P_T(p) = \{p\}$. As in the proof of Theorem 3.1, $\{x_t\}$ converges strongly as $t \rightarrow 1$ to p . $\square$

Corollary 3.10. *Let E be a reflexive Banach space having a uniformly Gâteaux differentiable norm, D a nonempty closed convex subset of E , and $T : D \rightarrow KC(E)$ a nonself $*$ -nonexpansive multimap satisfying the inwardness condition. Suppose that every closed bounded convex subset of D is compact and P_T has a fixed point. Then $\{x_t\}$ defined by (1.3) converges strongly as $t \rightarrow 1$ to a fixed point of T .*

Corollary 3.11. *Let E be a uniformly smooth Banach space, D a nonempty closed convex subset of E , and $T : D \rightarrow KC(E)$ a nonself multimap satisfying the inwardness condition such that P_T is nonexpansive. Suppose that every closed bounded convex subset of D is compact and P_T has a fixed point. Then $\{x_t\}$ defined by (1.3) converges strongly as $t \rightarrow 1$ to a fixed point of T .*

Corollary 3.12. *Let E be a reflexive Banach space having a uniformly Gâteaux differentiable norm, D a nonempty compact convex subset of E , and $T : D \rightarrow KC(E)$ a nonself multimap satisfying the inwardness condition such that P_T is nonexpansive. Suppose that P_T has a fixed point. Then $\{x_t\}$ defined by (1.3) converges strongly as $t \rightarrow 1$ to a fixed point of T .*

Remark 3.13. (1) Theorem 3.9 extends Theorem 1 and Corollary 1 of Jung and Kim [11] and Theorem 1 of Xu [27] to multimaps.

(2) Recently it has been shown by Shahzad and Lone [21] that if D is a nonempty closed bounded convex subset of a Banach space E such that $\epsilon_\alpha(E) < 1$ and $T : D \rightarrow KC(E)$ is a $*$ -nonexpansive, $1-\chi$ -contractive multimap satisfying the inwardness condition, then T has a fixed point. So one can relax the assumption that $F(T) \neq \emptyset$ in certain situations.

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